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EVALUATIONS OF SOME DETERMINANTS OF MATRICES RELATED TO THE PASCAL TRIANGLE

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Abstract. We prove several evaluations of determinants of matrices, the entries of which are given by the recurrence $a_{i,j} = a_{i-1,j} + a_{i,j-1}$, or variations thereof. These evaluations were either conjectured or extend conjectures by Roland Bacher [J. Théorie Nombres Bordeaux **14** (2002), to appear].

1. Introduction. In the recent preprint [2] (see [3] for the final version), Bacher considers determinants of matrices, the entries of which are given by the Pascal triangle recurrence $a_{i,j} = a_{i-1,j} + a_{i,j-1}$ or by similar recurrences, or are related in some other way to such bivariate recurrent sequences. He predicts (in form of precise conjectures) that such determinants also obey recurrence relations. This is most intriguing, because it points towards the possibility of *automatizing determinant evaluations*[1], something that several authors (cf. e.g. [1, 5, 7]) have been aiming at (albeit, with only limited success up to now). In addition, the preprint [2] contains several conjectures on closed form evaluations of such determinants in some instances. These are also interesting since they do not fit into the known families of determinant evaluations (see the recent survey article [5] for a long list of known determinant evaluations).

While we are not able to address Bacher's conjectured recurrence relations, we are able to provide proofs for all the conjectured determinant evaluations in [2], and, in some cases, to prove in fact generalizations of conjectures in [2]. (Some of these results are stated without proof in [3].)

In Section 2 we provide the proofs for the three conjectures in [2] on determinants of matrices, the entries of which are given by the recurrence $a_{i,j} = a_{i-1,j} + a_{i,j-1} +$

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[1]The reader should recall that the successful automatization [8, 11, 12, 13, 14] of the evaluation of binomial and hypergeometric sums is fundamentally based on producing reccurrences by the computer.

$x\, a_{i-1,j-1}$. Section 3 contains three theorems on binomial determinants which generalize (conjectural) determinant evaluations in Section 5 of [2]. All of the methods that we use to prove our theorems are already contained in the review [5] of the "state of the art" of determinant evaluations, namely, on the one hand, the "LU-factorization method" (see [5, Sec. 2.6]) and, on the other hand, the "identification of factors method" (see [5, Sec. 2.4]). There is one case where elementary row and column operations suffice.

2. Some determinants of matrices with entries given by an extension of the Pascal triangle recurrence. Our first theorem proves Conjecture 1.8 in [2]. It is stated (without proof) as Theorem 1.5 in [3].

Theorem 1. *Let $(a_{i,j})_{i,j\ge0}$ be the sequence given by the recurrence*

$$a_{i,j}=a_{i-1,j}+a_{i,j-1}+x\,a_{i-1,j-1},\qquad i,j\ge1 \tag{2.1}$$

and the initial conditions $a_{i,0}=\rho^i$ and $a_{0,i}=\sigma^i$, $i\ge0$. Then

$$\det_{0\le i,j\le n-1}(a_{i,j})=(1+x)^{\binom{n-1}{2}}(x+\rho+\sigma-\rho\sigma)^{n-1}.$$

Proof. Let $F(u,v)=\sum_{i,j\ge0}a_{i,j}u^iv^j$ be the bivariate generating function of the sequence $(a_{i,j})$. By multiplying both sides of (2.1) by u^iv^j and summing over all i and j, we get an equation for $F(u,v)$ with solution

$$F(u,v)=\frac{\frac{1-u}{1-\rho u}+\frac{1-v}{1-\sigma v}-1}{1-u-v-uvx}. \tag{2.2}$$

Now we use the *LU-factorization method* (see [5, Sec. 2.6]). Let M denote the matrix $(a_{i,j})_{0\le i,j\le n-1}$. I claim that

$$M\cdot U=L,$$

where $U=(U_{i,j})_{0\le i,j\le n-1}$ with

$$U_{i,j}=\begin{cases}1 & i=j\\ (-1)^{j-i}\left(\binom{j-1}{i}\sigma+\binom{j-1}{i-1}\right) & i<j,\\ 0 & i>j,\end{cases}$$

and $L=(L_{i,j})_{0\le i,j\le n-1}$ with

$$L_{i,j}=\begin{cases}\rho^i & j=0,\\ (x+\rho+\sigma-\rho\sigma)(1+x)^{j-1}\sum_{\ell=0}^{i-1}\binom{\ell}{j-1}\rho^{i-\ell-1} & j>0.\end{cases}$$

The matrix U is an upper triangular matrix with 1's on the diagonal, whereas L is a lower triangular matrix with diagonal entries

$$1,\ (x+\rho+\sigma-\rho\sigma),\ (x+\rho+\sigma-\rho\sigma)(1+x),\ \dots,\ (x+\rho+\sigma-\rho\sigma)(1+x)^{n-2}.$$

It is obvious that the claimed factorization of M immediately implies the validity of the theorem.

For the proof of the claim we compute the (i,j)-entry of $M\cdot U$. By definition, it is $\sum_{k=0}^{j} a_{i,k}U_{k,j}$. Let first $j=0$. Then the sum is equal to $a_{i,0}=\rho^i$, in accordance with the definition of $L_{i,0}$. If $j>0$, then (here, and in the following, $\langle z^N\rangle f(z)$ denotes the coefficient of z^N in the formal power series $f(z)$)

$$\begin{aligned}(M\cdot U)_{i,j}&=\sum_{k=0}^{j} a_{i,k}U_{k,j}\\&=M_{i,j}+\sum_{k=0}^{j-1} a_{i,k}U_{k,j}\\&=\langle u^iv^j\rangle F(u,v)\left(1+\sum_{k=0}^{j-1}(-1)^{j-k}v^{j-k}\left(\binom{j-1}{k}\sigma+\binom{j-1}{k-1}\right)\right)\\&=\langle u^iv^j\rangle F(u,v)\left((1-v)^{j-1}-\sigma v(1-v)^{j-1}\right)\\&=\langle u^iv^j\rangle F(u,v)(1-v)^{j-1}(1-\sigma v).\end{aligned}\tag{2.3}$$

Now we rewrite $F(u,v)$, as given by (2.2), slightly,

$$\begin{aligned}F(u,v)&=\frac{1-u-v+uv(x+\rho+\sigma-\rho\sigma)-xuv}{(1-\rho u)(1-\sigma v)(1-u-v-uvx)}\\&=\frac{1}{(1-\rho u)(1-\sigma v)}+\frac{uv(x+\rho+\sigma-\rho\sigma)}{(1-\rho u)(1-\sigma v)(1-u-v-uvx)}\\&=\frac{1}{(1-\rho u)(1-\sigma v)}+\frac{uv(x+\rho+\sigma-\rho\sigma)}{(1-\rho u)(1-\sigma v)(1-u)(1-v)\left(1-\frac{uv(x+1)}{(1-u)(1-v)}\right)}.\end{aligned}$$

This is substituted in (2.3):

$$\begin{aligned}&(M\cdot U)_{i,j}\\&\quad=\langle u^iv^j\rangle(1-v)^{j-1}\left(\frac{1}{(1-\rho u)}+\frac{uv(x+\rho+\sigma-\rho\sigma)}{(1-\rho u)(1-u)(1-v)\left(1-\frac{uv(x+1)}{(1-u)(1-v)}\right)}\right).\end{aligned}$$

Since the degree of $(1-v)^{j-1}$ in v is smaller than j, it is obvious that the first expression in parentheses does not contribute anything to the coefficient of u^iv^j. Hence we obtain

$$\begin{aligned}(M\cdot U)_{i,j}&=\langle u^iv^j\rangle(1-v)^{j-1}\frac{uv(x+\rho+\sigma-\rho\sigma)}{(1-u)(1-v)}\sum_{h=0}^{\infty}\frac{u^hv^h(x+1)^h}{(1-u)^h(1-v)^h}\sum_{\ell=0}^{\infty}\rho^\ell u^\ell\\&=(x+\rho+\sigma-\rho\sigma)\sum_{h,\ell\ge0}\rho^\ell(1+x)^h\binom{i-\ell-1}{h}\binom{0}{h+1-j}.\end{aligned}$$

Because of the second binomial coefficient, the summand is nonvanishing only for $h = j - 1$. This yields exactly the claimed expression for $L_{i,j}$. $\square$

The next theorem proves the conjecture about the evaluation of $\det(A(2n))$ on page 4 of [2]. It is stated (without proof) in [3, equation for $\det(B(2n))$ after Theorem 1.5].

Theorem 2. *Let $(a_{i,j})_{i,j\ge 0}$ be the sequence given by the recurrence*

$$a_{i,j} = a_{i-1,j} + a_{i,j-1} + x\, a_{i-1,j-1}, \qquad i,j \ge 1$$

and the initial conditions $a_{i,i} = 0$, $i \ge 0$, $a_{i,0} = \rho^{i-1}$ and $a_{0,i} = -\rho^{i-1}$, $i \ge 1$. Then

$$\det_{0\le i,j\le 2n-1}(a_{i,j}) = (1+x)^{2(n-1)^2}(x+\rho)^{2n-2}.$$

Proof. We compute again the generating function $F(u,v) = \sum_{i,j\ge 0} a_{i,j} u^i v^j$. This time we have

$$F(u,v) = \frac{(1-u-v+\rho uv)}{(1-u-v-xuv)} \frac{(u-v)}{(1-\rho u)(1-\rho v)}.$$

Also here, we apply the LU-factorization method. However, we cannot apply LU-factorization directly to the original matrix since it is skew-symmetric and, hence, all odd-dimensional principal minors vanish. Instead, we apply LU-factorization to the matrix that results from the original matrix by first reversing the order of rows and then transposing the resulting matrix. This is the matrix $M = (a_{2n-1-j,i})_{0\le i,j\le 2n-1}$. I claim that

$$M \cdot U = L,$$

where $U = (U_{i,j})_{0\le i,j\le n-1}$ with

$$U_{i,j} = \begin{cases} 0 & i > j,\\ \frac{(-1)^{j-i}}{\rho}\left(\binom{j-1}{i} + \binom{j-1}{i-1}\rho\right) & i \le j < 2n-1,\\ 1 & i = j = 2n-1,\\ -\frac{1}{(\rho+x)(1+x)^{n-2}}\Big(\sum_{s=0}^{n-i/2-2} 2^{2s+1}\binom{n-\frac{i}{2}-1+s}{2s+1}\sum_{t=0}^{i/2-s-1}\binom{n-1}{t}x^{\frac{i}{2}-s-t-1} & \\ \qquad +\rho\sum_{s=0}^{n-i/2-1} 2^{2s}\binom{n-\frac{i}{2}-1+s}{2s}\sum_{t=0}^{i/2-s-1}\binom{n-1}{t}x^{\frac{i}{2}-s-t-1}\Big) & \\ & i \text{ even, } i < j = 2n-1,\\ \frac{1}{(\rho+x)(1+x)^{n-2}}\Big(\sum_{s=0}^{n-i/2-3/2} 2^{2s}\binom{n-\frac{i}{2}-\frac{3}{2}+s}{2s}\sum_{t=0}^{i/2-s-1/2}\binom{n-1}{t}x^{\frac{i}{2}-s-t-\frac{1}{2}} & \\ \qquad +\rho\sum_{s=0}^{n-i/2-3/2} 2^{2s+1}\binom{n-\frac{i}{2}-\frac{1}{2}+s}{2s+1}\sum_{t=0}^{i/2-s-3/2}\binom{n-1}{t}x^{\frac{i}{2}-s-t-\frac{3}{2}}\Big) & \\ & i \text{ odd, } i < j = 2n-1, \end{cases}$$

and where L is a lower triangular matrix with diagonal entries

$$\rho^{2n-2},\ -\frac{\rho+x}{\rho},\ \frac{(\rho+x)(1+x)}{\rho},\ -\frac{(\rho+x)(1+x)^2}{\rho},\ \dots,\ \frac{(\rho+x)(1+x)^{2n-3}}{\rho},\ -(1+x)^{n-1} \tag{2.4}$$

Again, it is immediately obvious that the claimed factorization of M implies the validity of the theorem.

The proof of the claim requires again some calculations, which turn out to be more tedious here. The (i,j)-entry of $M\cdot U$ is $\sum_{k=0}^{j} a_{2n-1-k,i}U_{k,j}$. It suffices to consider the case where $0\le i\le j\le 2n-1$.

Case 1: $i=j=0$. Then $\sum_{k=0}^{j} a_{2n-1-k,i}U_{k,j}=a_{2n-1,0}=\rho^{2n-2}$, in accordance with (2.4).

Case 2: $0=i<j<2n-1$. We calculate the sum in question:

$$\begin{aligned}(M\cdot U)_{i,j}&=\sum_{k=0}^{j} a_{2n-1-k,i}U_{k,j}\\&=\langle u^{2n-1}v^i\rangle\sum_{k=0}^{j}\frac{(-1)^{j-k}}{\rho}\left(\binom{j-1}{k}+\binom{j-1}{k-1}\rho\right)u^kF(u,v)\\&=\langle u^{2n-1}v^i\rangle\frac{(-1)^j}{\rho}\left((1-u)^{j-1}-\rho u(1-u)^{j-1}\right)F(u,v)\\&=-\langle u^{2n-1}v^i\rangle\frac{(u-1)^{j-1}}{\rho}\frac{(1-u-v+\rho uv)(u-v)}{(1-u-v-xuv)(1-\rho v)}.\end{aligned}\tag{2.5}$$

If $i=0$ then the expression in the last row reduces to

$$-\langle u^{2n-1}\rangle\frac{(u-1)^{j-1}}{\rho}\cdot u.$$

The degree of the polynomial in u of which the coefficient of u^{2n-1} has to be read off is j. Since $j<2n-1$, this coefficient is 0.

Case 3: $0<i\le j<2n-1$. We start with the expression (2.5) and apply the partial fraction expansion (with respect to v)

$$\begin{aligned}\frac{(1-u-v+\rho uv)(u-v)}{(1-u-v-xuv)(1-\rho v)}=\frac{1-\rho u}{\rho(1+xu)}&-\frac{(1-\rho)(1-\rho u)}{\rho(1-\rho v)(1-\rho+\rho u+xu)}\\&+\frac{u(\rho+x)(u^2x+2u-1)}{(1+xu)(1-\rho+\rho u+xu)\left(1-\frac{v(1+xu)}{1-u}\right)}.\end{aligned}\tag{2.6}$$

After having substituted this in (2.5), we can read off the coefficient of v^i. In particular, the first term on the right-hand side of (2.6) does not contribute anything, since we have $i>0$ by assumption. We obtain

$$\begin{aligned}(M\cdot U)_{i,j}=-\langle u^{2n-1}\rangle\frac{(u-1)^{j-1}}{\rho(1-\rho+\rho u+xu)}\bigg(&-(1-\rho)(1-\rho u)\rho^{i-1}\\&+u(\rho+x)(u^2x+2u-1)\frac{(1+xu)^{i-1}}{(1-u)^i}\bigg).\end{aligned}\tag{2.7}$$

Let first $i < j$. The term $(1-u)^i$ in the denominator cancels with the term $(u-1)^{j-1}$. But the term $(1-\rho+\rho u+xu)$ in the denominator cancels as well. because the expression in parentheses contains this term as a factor. This can be seen by observing that the expression in parentheses vanishes for $u=(\rho-1)/(\rho+x)$ (which is the zero of the term $(1-\rho+\rho u+xu)$). Hence, the task in (2.7) is to read off the coefficient of u^{2n-1} from a *polynomial* in u of degree j. Since $j<2n-1$, this coefficient is 0.

If on the other hand we have $i=j$, then (2.7) reduces to

$$\begin{aligned}(M\cdot U)_{i,j} &= (-1)^i \langle u^{2n-1}\rangle \frac{1}{1-u}\\ &\quad\times\left(\frac{1}{\rho(1-\rho+\rho u+xu)}\bigg(-(1-\rho)(1-\rho u)(1-u)^i\rho^{i-1}\right.\\ &\quad\left.\qquad\qquad+u(\rho+x)(u^2x+2u-1)(1+xu)^{i-1}\bigg)\right).\end{aligned}$$

For the same reasons as before, the term $(1-\rho+\rho u+xu)$ in the expression in parentheses cancels, whence the expression in parentheses is in fact a polynomial in u of degree $i+1$. Now we use the (easily verified) fact that the coefficient of u^N in a formal power series of the form $P(u)/(1-qu)$, where $P(u)$ is a polynomial in u of degree $\le N$, is equal to $q^N P(1/q)$. Thus we obtain

$$(M\cdot U)_{i,j}=\frac{(-1)^i}{\rho}(\rho+x)(1+x)^{i-1},$$

in accordance with (2.4).

Case 4: $j=2n-1$. We must evaluate the following sum:

$$\begin{aligned}(M\cdot U)_{i,2n-1} &= \sum_{k=0}^{2n-1} a_{2n-1-k,i}U_{k,2n-1}\\ &= a_{0,i}+\sum_{k=0}^{n-1} a_{2n-1-2k,i}U_{2k,2n-1}+\sum_{k=1}^{n-1} a_{2n-2k,i}U_{2k-1,2n-1}.\end{aligned}\tag{2.8}$$

Since $U_{0,2n-1}=0$, we may restrict the first sum on the right-hand side of (2.8) to

$1 \le k \le n-1$. Consequently, it is equal to

$$
\begin{aligned}
&-\langle u^{2n-1}v^i\rangle \sum_{k=1}^{n-1} \frac{1}{(\rho+x)(1+x)^{n-2}}\\
&\quad\cdot\Bigg(\sum_{s=0}^{n-k-2} 2^{2s+1}\binom{n-k-1+s}{2s+1}\sum_{t=0}^{k-s-1}\binom{n-1}{t}x^{k-s-t-1}\\
&\qquad+\rho\sum_{s=0}^{n-k-1} 2^{2s}\binom{n-k-1+s}{2s}\sum_{t=0}^{k-s-1}\binom{n-1}{t}x^{k-s-t-1}\Bigg)u^{2k}F(u,v)\\
&=-\langle u^{2n-1}v^i\rangle \frac{1}{(\rho+x)(1+x)^{n-2}}\\
&\quad\cdot\Bigg(\sum_{s,k} 2^{2s+1}\binom{n-k-1}{2s+1}u^{2k+2s}\sum_{t=0}^{k-1}\binom{n-1}{t}x^{k-t-1}\\
&\qquad+\rho\sum_{s,k} 2^{2s}\binom{n-k-1}{2s}u^{2k+2s}\sum_{t=0}^{k-1}\binom{n-1}{t}x^{k-t-1}\Bigg)F(u,v).
\end{aligned}
$$

Because of completely analogous arguments, the second sum on the right-hand side of (2.8) is equal to

$$
\begin{aligned}
&\langle u^{2n-1}v^i\rangle \sum_{k=1}^{n-1} \frac{1}{(\rho+x)(1+x)^{n-2}}\\
&\quad\cdot\Bigg(\sum_{s=0}^{n-k-1} 2^{2s}\binom{n-k-1+s}{2s}\sum_{t=0}^{k-s-1}\binom{n-1}{t}x^{k-s-t-1}\\
&\qquad+\rho\sum_{s=0}^{n-k-1} 2^{2s+1}\binom{n-k+s}{2s+1}\sum_{t=0}^{k-s-2}\binom{n-1}{t}x^{k-s-t-2}\Bigg)u^{2k-1}F(u,v)\\
&=\langle u^{2n-1}v^i\rangle \frac{1}{(\rho+x)(1+x)^{n-2}}\\
&\quad\cdot\Bigg(\sum_{s,k} 2^{2s}\binom{n-k-1}{2s}u^{2k+2s-1}\sum_{t=0}^{k-1}\binom{n-1}{t}x^{k-t-1}\\
&\qquad+\rho\sum_{s,k} 2^{2s+1}\binom{n-k-1}{2s+1}u^{2k+2s+1}\sum_{t=0}^{k-1}\binom{n-1}{t}x^{k-t-1}\Bigg)F(u,v).
\end{aligned}
$$

If this is substituted in (2.8) and the sums are put together, then we obtain

$$\begin{aligned}
(M\cdot U)_{i,2n-1} &= a_{0,i} + \frac{1}{(\rho+x)(1+x)^{n-2}} \langle u^{2n-1}v^i\rangle (1-\rho u) F(u,v)\\
&\qquad \times \sum_{s,k} (-2)^s \binom{n-k-1}{s} u^{2k+s-1} \sum_{t=0}^{k-1} \binom{n-1}{t} x^{k-t-1}\\
&= a_{0,i} + \frac{1}{(\rho+x)(1+x)^{n-2}} \langle u^{2n-1}v^i\rangle (1-\rho u) F(u,v)\\
&\qquad \times \sum_{k} (1-2u)^{n-k-1} u^{2k-1} \sum_{t=0}^{k-1} \binom{n-1}{t} x^{k-t-1}\\
&= a_{0,i} + \frac{1}{(\rho+x)(1+x)^{n-2}} \langle u^{2n-1}v^i\rangle (1-\rho u) F(u,v)\\
&\qquad \times \sum_{t=0}^{n-1} \binom{n-1}{t} \frac{1}{u} x^{-t-1} \sum_{k=t+1}^{n-1} (1-2u)^{n-k-1} (xu^2)^k\\
&= a_{0,i} + \frac{1}{(\rho+x)(1+x)^{n-2}} \langle u^{2n-1}v^i\rangle (1-\rho u) F(u,v)\\
&\qquad \times \sum_{t=0}^{n-1} \binom{n-1}{t} \frac{1}{u} x^{-t-1} (1-2u)^{n-1} \frac{\left(\frac{xu^2}{1-2u}\right)^{t+1} - \left(\frac{xu^2}{1-2u}\right)^{n}}{1-\frac{xu^2}{1-2u}}.
\end{aligned}$$

The sum over t is easily evaluated, since it is only geometric series that have to be summed. After some simplification this leads to

$$\begin{aligned}
(M\cdot U)_{i,2n-1} &= a_{0,i} + \frac{1}{(\rho+x)(1+x)^{n-2}} \langle u^{2n-1}v^i\rangle (1-\rho u)\\
&\qquad \times \frac{u((1-u)^{2n-2} - (u^2(1+x))^{n-1})}{1-2u-xu^2} F(u,v)\\
&= a_{0,i} + \frac{1}{(\rho+x)(1+x)^{n-2}} \langle u^{2n-1}v^i\rangle \frac{(1-u-v+\rho uv)(u-v)}{(1-u-v-xuv)(1-\rho v)}\\
&\qquad \times \frac{u((1-u)^{2n-2} - (u^2(1+x))^{n-1})}{1-2u-xu^2}. \qquad (2.9)
\end{aligned}$$

If we have $i=0$, then it is immediate that the expression (2.9) vanishes.

Hence, from now on we assume $i>0$. We split the expression on the right-hand side of (2.9) into

$$(M\cdot U)_{i,2n-1}=-\rho^{i-1}-\frac{1}{(\rho+x)(1+x)^{n-2}}\langle v^i\rangle\frac{(1-u-v+\rho uv)(u-v)}{(1-u-v-xuv)(1-\rho v)}\\ \times\frac{(1+x)^{n-1}}{(1-2u-xu^2)}\\ +\frac{1}{(\rho+x)(1+x)^{n-2}}\langle u^{2n-1}v^i\rangle\frac{(1-u-v+\rho uv)(u-v)}{(1-u-v-xuv)(1-\rho v)}\frac{u(1-u)^{2n-2}}{(1-2u-xu^2)}\\ =-\rho^{i-1}+\rho^{i-1}\frac{1+x}{\rho+x}\\ +\frac{1}{(\rho+x)(1+x)^{n-2}}\langle u^{2n-1}v^i\rangle\frac{(1-u-v+\rho uv)(u-v)}{(1-u-v-xuv)(1-\rho v)}\frac{u(1-u)^{2n-2}}{(1-2u-xu^2)}. \tag{2.10}$$

To the expression in the last line we apply again the partial fraction expansion (2.6). After having substituted this in (2.10), we may directly read off the coefficient of v^i. Once again, the first term on the right-hand side of (2.6) does not contribute to it since we have $i>0$ by assumption. We obtain

$$(M\cdot U)_{i,2n-1}\\ =\rho^{i-1}\frac{1-\rho}{\rho+x}+\frac{1}{(\rho+x)(1+x)^{n-2}}\langle u^{2n-1}\rangle\frac{u(1-u)^{2n-2}}{(1-2u-xu^2)}\frac{1}{(1-\rho+\rho u+xu)}\\ \times\left(-(1-\rho)(1-\rho u)\rho^{i-1}+u(\rho+x)(u^2x+2u-1)\frac{(1+xu)^{i-1}}{(1-u)^i}\right)\\ =\rho^{i-1}\frac{1-\rho}{\rho+x}-\frac{1}{(\rho+x)(1+x)^{n-2}}\langle u^{2n-1}\rangle\frac{u(1-u)^{2n-2}(1-\rho)(1-\rho u)}{(1-2u-xu^2)(1-\rho+\rho u+xu)}\rho^{i-1}\\ -\frac{1}{(\rho+x)(1+x)^{n-2}}\langle u^{2n-1}\rangle\frac{u^2(1-u)^{2n-2-i}(\rho+x)(1+xu)^{i-1}}{(1-\rho+\rho u+xu)}. \tag{2.11}$$

Let us for the moment assume that $i<2n-1$. The coefficient of u^{2n-1} in the last line can be easily determined by making use of the previously used fact that the coefficient of u^N in a formal power series of the form $P(u)/(1-qu)$, where $P(u)$ is a polynomial in u of degree $\le N$, is equal to $q^NP(1/q)$. Thus we obtain

$$(M\cdot U)_{i,2n-1}\\ =\rho^{i-1}\frac{1-\rho}{\rho+x}-\frac{1}{(\rho+x)(1+x)^{n-2}}\langle u^{2n-1}\rangle\frac{u(1-u)^{2n-2}(1-\rho)(1-\rho u)}{(1-2u-xu^2)(1-\rho+\rho u+xu)}\rho^{i-1}\\ +\frac{1}{(\rho+x)(1+x)^{n-2}}\frac{(\rho+x)(1+x)^{2n-3}}{(\rho-1)^{2n-2}}\rho^{i-1}. \tag{2.12}$$

In order to determine the remaining coefficient of u^{2n-1} on the right-hand side, we apply once again a partial fraction expansion:

$$\frac{(1-\rho u)}{(1-2u-xu^2)(1-\rho+\rho u+xu)}=\frac{\rho+x}{(1+x)(1-\rho+\rho u+xu)}\\ -\frac{1}{2(1+x)}\frac{1}{(u-\omega)}-\frac{1}{2(1+x)}\frac{1}{(u-\overline{\omega})},$$

where $\omega=-\frac{1}{x}(1-\sqrt{1+x})$ and $\overline{\omega}=-\frac{1}{x}(1+\sqrt{1+x})$. This is now substituted in (2.12). By a routine calculation, which uses again the above fact of how to read off the coefficient of u^N in a formal power series of the form $P(u)/(1-qu)$, it is seen that (2.12) vanishes for $i<2n-1$.

On the other hand, if we have $i=2n-1$, then the expression in the last line of (2.11) is equal to

$$-\frac{1}{(\rho+x)(1+x)^{n-2}}\langle u^{2n-1}\rangle\frac{u^2(\rho+x)(1+xu)^{2n-2}}{(1-u)(1-\rho+\rho u+xu)}.$$

In order to read off the coefficient of u^{2n-1} in this expression, we must apply a partial fraction expansion another time:

$$\frac{1}{(1-u)(1-\rho+\rho u+xu)}=\frac{1}{(1+x)}\frac{1}{(1-u)}+\frac{1}{(1+x)}\frac{\rho+x}{(1-\rho+\rho u+xu)}.$$

After having done that, a routine calculation has to be performed that shows that for $i=2n-1$ the expression (2.11) simplifies to $-(1+x)^{n-1}$, in accordance with (2.4).

This completes the proof of the theorem. $\square$

Finally we proof the conjectured evaluation of $\det(\tilde{A}(2n))$ on page 4 of [2]. As it turns out, this is much simpler than the previous evaluations.

Theorem 3. *Let $(a_{i,j})_{i,j\ge0}$ be the sequence given by the recurrence*

$$a_{i,j}=a_{i-1,j}+a_{i,j-1}+x\,a_{i-1,j-1},\qquad i,j\ge1$$

and the initial conditions $a_{i,0}=i$ and $a_{0,i}=-i$, $i\ge0$. Then

$$\det_{0\le i,j\le 2n-1}(a_{i,j})=(1+x)^{2n(n-1)}.$$

Proof. Again, we compute the generating function $F(u,v)=\sum_{i,j\ge0}a_{i,j}u^iv^j$. Here we get

$$F(u,v)=\frac{u-v}{(1-u-v-uvx)(1-u)(1-v)}.$$

This can be written in the following way:

$$\begin{aligned}F(u,v)&=\frac{u}{(1-u)(1-u-v-uvx)}-\frac{v}{(1-v)(1-u-v-uvx)}\\&=\frac{u}{(1-u)^2(1-v)\left(1-\frac{uv(x+1)}{(1-u)(1-v)}\right)}-\frac{v}{(1-u)(1-v)^2\left(1-\frac{uv(x+1)}{(1-u)(1-v)}\right)}\\&=\sum_{\ell=0}^{\infty}\frac{u^\ell v^\ell(x+1)^\ell}{(1-u)^{\ell+1}(1-v)^{\ell+1}}\left(\frac{u}{1-u}-\frac{v}{1-v}\right).\end{aligned}$$

Therefore we have

$$a_{i,j} = \sum_{\ell=0} \left(\binom{i}{\ell+1}\binom{j}{\ell} - \binom{i}{\ell}\binom{j}{\ell+1} \right) (x+1)^{\ell}.$$

Now we apply the following row and column operations: We subtract row i from row $i+1$, $i = 2n-2, 2n-3, \dots, 0$, and subsequently we subtract column j from column $j+1$, $j = 2n-2, 2n-3, \dots, 0$. This is repeated another $2n-2$ times. It is not too difficult to see (using the above expression for $a_{i,j}$) that, step by step, the rows and columns are "emptied" until finally the determinant

$$\det \begin{pmatrix} 0 & -1 & 0 & 0 & \dots\dots\dots\dots & & 0 \\ 1 & 0 & -X & 0 & \dots\dots\dots\dots & & 0 \\ 0 & X & 0 & -X^2 & \dots\dots\dots\dots & & 0 \\ 0 & 0 & X^2 & 0 & \ddots & & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots\dots & & & X^{2n-3} & 0 & -X^{2n-2} \\ 0 & \dots\dots & & & 0 & X^{2n-2} & 0 \end{pmatrix}$$

is obtained, where we have written X for $x+1$. The theorem follows now immediately. □

3. Some binomial determinants. In this section we prove some evaluations of binomial determinants which arose from conjectures in [2] and subsequent private discussion with Roland Bacher.

The first theorem in this section proves a common extension of two conjectures on page 15 in [2]. It is also stated (without proof) in [3, p. 14].

Theorem 4. *We have*

$$\det_{0\le i,j\le n-1} \left(\binom{2i+2j+a}{i} - \binom{2i+2j+a}{i-1} \right) = 2^{\binom{n}{2}}. \tag{3.1}$$

Proof. We apply LU-factorization. Let us denote the matrix in (3.1) by M. I claim that

$$M \cdot U = L,$$

where $U = \left((-1)^{j-i}\binom{j}{i}\right)_{0\le i,j\le n-1}$, and where $L = (L_{i,j})_{0\le i,j\le n-1}$ is a lower triangular matrix with diagonal entries $1, 2, 4, 8, \dots, 2^{n-1}$. It is obvious that the claim immediately implies the theorem.

Again, we verify the claim by a direct calculation. Let $0 \le i \le j \le n-1$. The (i,j)-entry of $M \cdot U$ is

$$(M \cdot U)_{i,j} = \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} \frac{(1+a+2\,k)\,(a+2\,i+2\,k)!}{i!\,(1+a+i+2\,k)!}.$$

In standard hypergeometric notation

$$ {}_rF_s\left[\begin{matrix} a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};z\right]=\sum_{m=0}^{\infty}\frac{(a_1)_m\cdots(a_r)_m}{m!\,(b_1)_m\cdots(b_s)_m}z^m\ , $$

where the Pochhammer symbol $(\alpha)_m$ is defined by $(\alpha)_m:=\alpha(\alpha+1)\cdots(\alpha+m-1)$, $m\ge 1$, $(\alpha)_0:=1$, the above binomial sum can be written as

$$ (-1)^j\frac{(1+a)\ (2+a+i)_{i-1}}{i!}\,{}_4F_3\left[\begin{matrix} \frac{3}{2}+\frac{a}{2},\frac{1}{2}+\frac{a}{2}+i,1+\frac{a}{2}+i,-j\\ \frac{3}{2}+\frac{a}{2}+\frac{i}{2},1+\frac{a}{2}+\frac{i}{2},\frac{1}{2}+\frac{a}{2}\end{matrix};1\right]. $$

To this ${}_4F_3$-series we apply the contiguous relation

$$ {}_4F_3\left[\begin{matrix} a,b,c,d\\ e,f,g\end{matrix};z\right]={}_4F_3\left[\begin{matrix} a-1,b,c,d\\ e,f,g\end{matrix};z\right]+z\frac{bcd}{efg}{}_4F_3\left[\begin{matrix} a,b+1,c+1,d+1\\ e+1,f+1,g+1\end{matrix};z\right]. $$

We obtain

$$ (-1)^j\frac{(1+a)\ (2+a+i)_{i-1}}{i!}\,{}_3F_2\left[\begin{matrix} \frac{1}{2}+\frac{a}{2}+i,1+\frac{a}{2}+i,-j\\ \frac{3}{2}+\frac{a}{2}+\frac{i}{2},1+\frac{a}{2}+\frac{i}{2}\end{matrix};1\right]\\ -(-1)^j\frac{2j\,(4+a+i)_{i-1}}{i!}\,{}_3F_2\left[\begin{matrix} \frac{3}{2}+\frac{a}{2}+i,2+\frac{a}{2}+i,1-j\\ \frac{5}{2}+\frac{a}{2}+\frac{i}{2},2+\frac{a}{2}+\frac{i}{2}\end{matrix};1\right]. $$

Now we apply the transformation formula (cf. [4, Ex. 7, p. 98])

$$ {}_3F_2\left[\begin{matrix} a,b,-n\\ d,e\end{matrix};1\right]=\frac{(-a-b+d+e)_n}{(e)_n}\,{}_3F_2\left[\begin{matrix} -n,-a+d,-b+d\\ d,-a-b+d+e\end{matrix};1\right], \tag{3.2} $$

where n is a nonnegative integer, to both ${}_3F_2$-series. The above expression then becomes

$$ (-1)^j\frac{(1+a)\ (1-i)_j\,(2+a+i)_{i-1}}{i!\,(1+\frac{a}{2}+\frac{i}{2})_j}\,{}_3F_2\left[\begin{matrix} -j,\frac{1}{2}-\frac{i}{2},1-\frac{i}{2}\\ \frac{3}{2}+\frac{a}{2}+\frac{i}{2},1-i\end{matrix};1\right]\\ -(-1)^j\frac{2j\,(1-i)_{j-1}\,(4+a+i)_{i-1}}{i!\,(2+\frac{a}{2}+\frac{i}{2})_{j-1}}\,{}_3F_2\left[\begin{matrix} 1-j,\frac{1}{2}-\frac{i}{2},1-\frac{i}{2}\\ \frac{5}{2}+\frac{a}{2}+\frac{i}{2},1-i\end{matrix};1\right]. $$

If $j\ge i$, the first summand is always 0 because of the term $(1-i)_j$. The second summand vanishes as long as $j>i$ because of the term $(1-i)_{j-1}$. Thus, the matrix L is indeed a lower triangular matrix. It remains to evaluate the above expression for $i=j$. As we already mentioned, in that case it reduces to the second term, which itself reduces to

$$ -(-1)^i\frac{2i\,(1-i)_{i-1}\,(4+a+i)_{i-1}}{i!\,(2+\frac{a}{2}+\frac{i}{2})_{i-1}}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}-\frac{i}{2},1-\frac{i}{2}\\ \frac{5}{2}+\frac{a}{2}+\frac{i}{2}\end{matrix};1\right]. $$

This ${}_2F_1$-series can be evaluated by means of Gauß' ${}_2F_1$-summation (cf. [9, (1.7.6); Appendix (III.3)])

$$ {}_2F_1\left[\begin{matrix} a,b\\ c\end{matrix};1\right]=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}. \tag{3.3} $$

After some further simplification we obtain 2^i, as desired. □

Our next theorem presents a variation of the previous theorem. It is again a common extension of two conjectures in [2, p. 15]. It is also stated (without proof) in [3, p. 14].

Theorem 5. *We have*

$$\det_{0\le i,j\le n-1}\left(\binom{2i+2j+a}{i+1}-\binom{2i+2j+a}{i}\right)=2^{\binom n2}\frac{\prod_{i=0}^{n-1}(a+2i-1)}{n!}. \tag{3.4}$$

Proof. We apply LU-factorization. Let us denote the matrix in (3.4) by M. I claim that

$$M\cdot U=L,$$

where

$$U=\left((-1)^{j-i}\binom{j}{i}\frac{(a+2j-1)}{(a+2i-1)}\right)_{0\le i,j\le n-1},$$

and where $L=(L_{i,j})_{0\le i,j\le n-1}$ is a lower triangular matrix with diagonal entries

$$a-1,\frac{2}{2}(a+1),\frac{4}{3}(a+3),\frac{8}{4}(a+5),\dots,\frac{2^{n-1}}{n}(a+2n-3).$$

It is obvious that the claim immediately implies the theorem.

Let us do the required calculations. Let $0\le i\le j\le 2n-1$. The (i,j)-entry of $M\cdot U$ is

$$(M\cdot U)_{i,j}=\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}\frac{(a+2\,j-1)\;(a+2\,i+2\,k)!}{(i+1)!\;(a+i+2\,k)!}.$$

We convert this binomial sum into hypergeometric notation. The result is

$$(-1)^j\frac{(a+2\,j-1)\;(a+2\,i)!}{(i+1)!\,(a+i)!}\,{}_3F_2\left[\begin{matrix}-j,\frac12+\frac a2+i,1+\frac a2+i\\ 1+\frac a2+\frac i2,\frac12+\frac a2+\frac i2\end{matrix};1\right].$$

This time it is not even necessary to apply a contiguous relation. We may directly apply the transformation formula (3.2) and obtain

$$(-1)^j\frac{(a+2\,j-1)\;(a+2\,i)!\,(-i)_j}{(i+1)!\,(a+i)!\,(\frac12+\frac a2+\frac i2)_j}\,{}_3F_2\left[\begin{matrix}-j,-\frac i2,\frac12-\frac i2\\ 1+\frac a2+\frac i2,-i\end{matrix};1\right].$$

Again, this expression is 0 if $j>i$ since it contains the term $(-i)_j$. If $j=i$ then it reduces to

$$(-1)^i\frac{(a+2\,i-1)\;(a+2\,i)!\,(-i)_i}{(i+1)!\,(a+i)!\,(\frac12+\frac a2+\frac i2)_i}\,{}_2F_1\left[\begin{matrix}-\frac i2,\frac12-\frac i2\\ 1+\frac a2+\frac i2\end{matrix};1\right].$$

The ${}_2F_1$-series can again be evaluated by means of the Gauß summation formula (3.3). After further simplification we obtain $2^i(a+2i-1)/(i+1)$, as desired. □

The final determinant evaluation that we present here proves a common generalization of determinant evaluations that arose in discussion with Roland Bacher. It is interesting to note that the limiting case where $Y\to\infty$, i.e., the evaluation of the determinant $\det_{0\le i,j\le 2n-1}\left((i-j)\,(X+i+j)!\right)$ is covered by a theorem by Mehta and Wang [6]. In fact, the main theorem of [6] gives the evaluation of the more general determinant $\det_{0\le i,j\le n-1}\left((Z+i-j)\,(X+i+j)!\right)$. It seems difficult to find a common extension of the Mehta and Wang evaluation and the evaluation below, i.e., an evaluation of

$$\det_{0\le i,j\le n-1}\left((Z+i-j)\frac{(X+i+j)!}{(Y+i+j)!}\right).$$

Neither the method used by Mehta and Wang nor the "identification of factors method" used below seem to help.

Theorem 6. *Let X and Y be arbitrary nonnegative integers. Then*

$$\det_{0\le i,j\le 2n-1}\left((i-j)\frac{(X+i+j)!}{(Y+i+j)!}\right)=\prod_{i=0}^{2n-1}\frac{(X+i)!}{(Y+i+2n-1)!}$$
$$\times\prod_{i=0}^{n-1}(2i+1)!^2(X+2i+1)(Y+4n-2i-2)(X-Y-2i)^{2n-2i-2}(X-Y-2i-1)^{2n-2i-2}. \tag{3.5}$$

Remark. The theorem is formulated only for integral X and Y. But in fact, if we replace the factorials by the appropriate gamma functions then the theorem would also make sense and be true for complex X and Y.

Proof. To begin with, we take some factors out of the determinant:

$$\det_{0\le i,j\le 2n-1}\left((i-j)\frac{(X+i+j)!}{(Y+i+j)!}\right)=\prod_{i=0}^{2n-1}\frac{(X+i)!}{(Y+i+2n-1)!}$$
$$\times\det_{0\le i,j\le 2n-1}\big((i-j)\,(X+i+1)_j\,(Y+i+j+1)_{2n-j-1}\big).$$

If we compare with (3.5), then we see that it suffices to establish that

$$\det_{0\le i,j\le 2n-1}\big((i-j)\,(X+i+1)_j\,(Y+i+j+1)_{2n-j-1}\big)$$
$$=\prod_{a=0}^{n-1}(2a+1)!^2(X+2a+1)(Y+4n-2a-2)$$
$$\cdot(X-Y-2a)^{2n-2a-2}(X-Y-2a-1)^{2n-2a-2}. \tag{3.6}$$

This time we apply a different method. We make use of the *"identification of factors" method* as explained in [5, Sec. 2.4].

Let us denote the determinant in (3.6) by $T_n(X,Y)$. Here is a brief outline of how we proceed. The determinant $T_n(X,Y)$ is a polynomial in X and Y. In the first step we shall show that the product on the right-hand side of (3.6) divides the determinant as a polynomial in X and Y. Then we compare the degrees in X and Y of the determinant and the right-hand side of (3.6). As it turns out, the degree of the determinant is at most the degree of the product. Hence, the determinant must be equal to a constant multiple of the right-hand side of (3.6). Finally, in the last step, this constant is computed by setting $X=-2n$ on both sides of (3.6).

Step 1. The product $\prod_{a=0}^{n-1}(X+2a+1)$ *divides* $T_n(X,Y)$. Let a be fixed with $0\le a\le n-1$. In order to show that $X+2a+1$ divides the determinant, it suffices to show that the determinant vanishes for $X=-2a-1$. The latter would follow immediately if we could show that the rows of the matrix underlying the determinant are linearly dependent for $X=-2a-1$. I claim that in fact we have

$$\sum_{i=0}^{a}(-1)^i\binom{a}{i}\frac{(Y+2n+2a-i)_i}{(\frac{Y}{2}+2a-i)_i}\big(\text{row }(2a-i)\text{ of }T_n(-2a-1,Y)\big)=0.$$

This claim is easily verified. Indeed, if we restrict it to the j-th column, we see that we must check

$$\sum_{i=0}^{a}(-1)^i\binom{a}{i}\frac{(Y+2n+2a-i)_i}{(\frac{Y}{2}+2a-i)_i}(2\,a-i-j)\;(-i)_j\;(Y+2a-i+j+1)_{2n-j-1}=0.$$

Because of the term $(-i)_j$ this equation is certainly true if $a<j$. If $a\ge j$ then, because of the same reason, we may restrict the sum to $j\le i\le a$. We convert the sum into hypergeometric notation, and obtain

$$2\frac{(a-j)_{j+1}\,(1+2\,a+Y)_{2n-1}}{(\frac{Y}{2}+2a-j)_j}\,{}_3F_2\left[\begin{matrix}1-2\,a+2\,j,-a+j,-2\,a-Y\\ -2\,a+2\,j,1-2\,a+j-\frac{Y}{2}\end{matrix};1\right].$$

Next we apply the contiguous relation

$${}_3F_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix};z\right]={}_3F_2\left[\begin{matrix}a-1,b,c\\ d,e\end{matrix};z\right]+z\frac{bc}{de}\,{}_3F_2\left[\begin{matrix}a,b+1,c+1\\ d+1,e+1\end{matrix};z\right].\tag{3.7}$$

This yields

$$2\frac{(a-j)_{j+1}\,(1+2\,a+Y)_{-1+2\,n}}{(2\,a-j+\frac{Y}{2})_j}\,{}_2F_1\left[\begin{matrix}-2\,a-Y,-a+j\\ 1-2\,a+j-\frac{Y}{2}\end{matrix};1\right]\\ +\frac{(a-j)_{j+1}\,(2\,a+Y)_{2\,n}}{(-1+2\,a-j+\frac{Y}{2})_{j+1}}\,{}_2F_1\left[\begin{matrix}1-2\,a-Y,1-a+j\\ 2-2\,a+j-\frac{Y}{2}\end{matrix};1\right].$$

Both ${}_2F_1$-series can be evaluated by means of the Chu-Vandermonde summation formula (cf. [9, (1.7.7); Appendix (III.4)]

$${}_2F_1\left[\begin{matrix}a,-n\\ c\end{matrix};1\right]=\frac{(c-a)_n}{(c)_n},\tag{3.8}$$

where n is a nonnegative integer. After some further simplification it is seen that the two terms cancel each other.

Step 2. The product $\prod_{a=0}^{n-1}(Y+4n-2a-2)$ *divides* $T_n(X,Y)$. We proceed analogously. Let a be fixed with $0\le a\le n-1$. I claim that

$$\sum_{i=0}^{a}(-1)^i\binom{a}{i}\frac{(\frac{X}{2}+2n-a-i+\frac{1}{2})_i}{(X+2n-a-i)_i}\\ \cdot\big(\text{row }(2n-a-i-1)\text{ of }T_n(X,-4n+2a+2)\big)=0.$$

Restricted to the j-th column, this is

$$\sum_{i=0}^{a}(-1)^i\binom{a}{i}\frac{(\frac{X}{2}+2n-a-i+\frac{1}{2})_i}{(X+2n-a-i)_i}\\ \cdot(2n-a-i-j-1)\,(X+2n-a-i)_j(a-2n-i+j+2)_{2n-j-1}=0.$$

In order to establish this equation, we convert the sum again into hypergeometric notation, and obtain

$$-\,(1+a+j-2\,n)_{-j+2\,n}\,(-a+2\,n+X)_j\\ \times{}_3F_2\left[\begin{matrix}2+a+j-2\,n,-1-a-j+2\,n,\frac{1}{2}+a-2\,n-\frac{X}{2}\\ 1+a+j-2\,n,1+a-j-2\,n-X\end{matrix};1\right].$$

Now we apply the contiguous relation (3.7), to get

$$-\,(1+a+j-2\,n)_{2n-j}\,(-a+2\,n+X)_j\\ \times{}_2F_1\left[\begin{matrix}\frac{1}{2}+a-2\,n-\frac{X}{2},-1-a-j+2\,n\\ 1+a-j-2\,n-X\end{matrix};1\right]\\ -\left(\tfrac{1}{2}+a-2\,n-\tfrac{X}{2}\right)\,(1+a+j-2\,n)_{2n-j}\,(-a+2\,n+X)_{j-1}\\ \times{}_2F_1\left[\begin{matrix}\frac{3}{2}+a-2\,n-\frac{X}{2},-a-j+2\,n\\ 2+a-j-2\,n-X\end{matrix};1\right].$$

Finally, we evaluate the ${}_2F_1$-series by means of the Chu-Vandermonde summation formula (3.8).

Step 3. The product $\prod_{a=0}^{n-1}(X-Y-2a)^{2n-2a-2}$ *divides* $T_n(X,Y)$. Let a be fixed with $0\le a\le n-1$. I claim that for $0\le v\le 2n-2a-3$ we have

$$\sum_{i=0}^{2a+2}(-1)^i\binom{2a+2}{i}\frac{(X+2n+v-i+2)_i}{(X+2a+v-i+3)_i}\\ \cdot\big(\text{row }(2a+v+2-i)\text{ of }T_n(X,X-2a)\big)=0.$$

Restricted to the j-th column, this is

$$\sum_{i=0}^{2a+2}(-1)^i\binom{2a+2}{i}\frac{(X+2n+v-i+2)_i}{(X+2a+v-i+3)_i}\\ \cdot(2a+v-i-j+2)\,(X+2\,a+v-i+3)_j\,(X+v-i+j+3)_{2n-j-1}=0.$$

In order to establish this equation, we convert the sum into hypergeometric notation, and obtain

$$(2\,a+v-j+2)\,(X+2\,a+v+3)_j\,(X+v+j+3)_{2n-j-1}\\ \times{}_3F_2\left[\begin{matrix}-1-2a+j-v,-2-2a,-2-j-v-X\\ -2-2a+j-v,-2-2a-j-v-X\end{matrix};1\right].$$

Next we apply the contiguous relation (3.7), to get

$$\begin{aligned}(2a+v-j+2)\;(X+2\,a+v+3)_j\;(X+v+j+3)_{2n-j-1}&\\ \times {}_2F_1\!\left[\begin{matrix}-2-j-v-X,-2-2\,a\\ -2-2\,a-j-v-X\end{matrix};1\right]&\\ +2\,(a+1)\;(X+2a+v+3)_{j-1}\,(X+v+j+2)_{2n-j}&\\ \times {}_2F_1\!\left[\begin{matrix}-1-j-v-X,-1-2\,a\\ -1-2a-j-v-X\end{matrix};1\right].&\end{aligned}$$

Finally, the ${}_2F_1$-series are again evaluated by means of the Chu-Vandermonde summation formula (3.8). The result is

$$(2+2\,a+3\,j+2\,a\,j+v+2\,a\,v+2\,X+2\,a\,X)\\ \times \frac{(-2a)_{2a+1}\,(X+2a+v+3)_{j-1}\,(X+v+j+3)_{2n-j-1}}{(-2a-j-v-X-1)_{2a+1}}.$$

Since this expression contains the term $(-2a)_{2a+1}$, it must vanish.

Step 4. The product $\prod_{a=0}^{n-1}(X-Y-2a-1)^{2n-2a-2}$ *divides* $T_n(X,Y)$. Let a be fixed with $0\le a\le n-1$. I claim that for $0\le v\le 2n-2a-4$ (sic!) we have

$$\sum_{i=0}^{2a+3}(-1)^i\binom{2a+3}{i}\frac{(X+2n+v-i+2)_i}{(X+2a+v-i+4)_i}\\ \cdot\big(\text{row }(2a+v+3-i)\text{ of }T_n(X,X-2a-1)\big)=0.$$

Restricted to the j-th column, this is

$$\sum_{i=0}^{2a+3}(-1)^i\binom{2a+3}{i}\frac{(X+2n+v-i+2)_i}{(X+2a+v-i+4)_i}\\ \cdot(2a+v-i-j+3)\;(X+2\,a+v-i+4)_j\;(X+v-i+j+3)_{2n-j-1}=0.$$

In order to establish this equation, we convert the sum into hypergeometric notation, and obtain

$$(2\,a+v-j+3)\;(X+2\,a+v+4)_j\;(X+v+j+3)_{2n-j-1}\\ \times {}_3F_2\!\left[\begin{matrix}-2-2a+j-v,-3-2a,-2-j-v-X\\ -3-2a+j-v,-3-2a-j-v-X\end{matrix};1\right].$$

Now we apply the contiguous relation (3.7), to get

$$\begin{aligned}(2a+v-j+3)\;(X+2\,a+v+4)_j\;(X+v+j+3)_{2n-j-1}&\\ \times {}_2F_1\!\left[\begin{matrix}-3-2a,-2-j-v-X\\ -3-2a-j-v-X\end{matrix};1\right]&\\ +(2a+3)\;(X+2a+v+4)_{j-1}\,(X+v+j+2)_{2n-j}&\\ \times {}_2F_1\!\left[\begin{matrix}-2-2a,-1-j-v-X\\ -2-2a-j-v-X\end{matrix};1\right].&\end{aligned}$$

Finally, we evaluate the ${}_2F_1$-series by means of the Chu-Vandermonde summation formula (3.8). The result is

$$(3+2\,a+4\,j+2\,a\,j+2\,v+2\,a\,v+3\,X+2\,a\,X)\\ \times\frac{(-2a-1)_{2a+2}\,(X+2a+v+4)_{j-1}\,(X+v+j+3)_{2n-j-1}}{(-X-2a-j-v-2)_{2a+2}}.$$

Since this expression contains the term $(-2a-1)_{2a+2}$, it must vanish.

The arguments thus far prove that $\prod_{a=0}^{n-1}(X-Y-2a-1)^{2n-2a-3}$ divides the determinant $T_n(X,Y)$. It should be observed that the exponents in this product are by 1 smaller than what we would need. However, the original determinant in (3.5) is the determinant of a skew-symmetric matrix. Hence it is a square (of the corresponding Pfaffian; see e.g. [10, Prop. 2.2]). This implies that, for fixed a, the multiplicity of a factor $(X-Y-2a-1)$ in $T_n(X,Y)$ must be *even*. Phrased differently, if $(X-Y-2a-1)^e$ divides $T_n(X,Y)$ with e maximal, then e must be even. Since we already proved that $(X-Y-2a-1)^{2n-2a-3}$ divides $T_n(X,Y)$, we get for free that in fact $(X-Y-2a-1)^{2n-2a-2}$ divides $T_n(X,Y)$, as required.

Step 5. Comparison of degrees. It is obvious that the degree in X of the determinant $T_n(X,Y)$ is at most $\binom{2n}{2}=2n^2-n$. The degree in X on the right-hand side of (3.6) is $n+4\binom{n}{2}=2n^2-n$, which is exactly the same number. The same is true for Y. It follows that

$$\begin{aligned}T_n(X,Y)&=\det_{0\le i,j\le 2n-1}\left((i-j)\,(X+i+1)_j\,(Y+i+j+1)_{2n-j-1}\right)\\ &=C\prod_{a=0}^{n-1}(X+2a+1)(Y+4n-2a-2)\\ &\qquad\cdot(X-Y-2a-1)^{2n-2a-2}(X-Y-2a)^{2n-2a-2},\end{aligned}\tag{3.9}$$

where C is some constant independent of X and Y.

Step 6. The computation of the constant. In order to compute the constant, we set $X=-2n$ in (3.9). Because of

$$(-2n+1+i)_j=(-2n+i+1)(-2n+i+2)\cdots(-2n+i+j),$$

this implies that all entries in $T_n(-2n,Y)$ in the i-th row and j-th column with $i+j\ge 2n$, vanish. This means that the matrix underlying $T_n(-2n,Y)$ is triangular. Thus its determinant is easily evaluated. We leave it to the reader to check that the result for C is in accordance with the original claim. □

References

1. T. Amdeberhan and D. Zeilberger, *Determinants through the looking glass*, Adv. Appl. Math. **27** (2001), 225–230.
2. R. Bacher, *Matrices related to the Pascal triangle*, preprint, `math/0109013v1`.
3. R. Bacher, *Determinants of matrices related to the Pascal triangle*, J. Théorie Nombres Bordeaux **14** (2002) (to appear).

4. W. N. Bailey, *Generalized hypergeometric series*, Cambridge University Press, Cambridge, 1935.
5. C. Krattenthaler, *Advanced determinant calculus*, Séminaire Lotharingien Combin. **42** ("The Andrews Festschrift") (1999), Article B42q, 67 pp.
6. M. L. Mehta and R. Wang, *Calculation of a certain determinant*, Commun. Math. Phys. **214** (2000), 227–232.
7. M. Petkovšek and H. S. Wilf, *A high-tech proof of the Mills-Robbins-Rumsey determinant formula*, Electron. J. Combin. **3** (no. 2, "The Foata Festschrift") (1996), Art. #R19, 3 pp.
8. M. Petkovšek, H. Wilf and D. Zeilberger, *A=B*, A.K. Peters, Wellesley, 1996.
9. L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, Cambridge, 1966.
10. J. R. Stembridge, *Nonintersecting paths, pfaffians and plane partitions*, Adv. in Math. **83** (1990), 96–131.
11. K. Wegschaider, *Computer generated proofs of binomial multi-sum identities*, diploma thesis, Johannes Kepler University, Linz, Austria, 1997.
12. H. S. Wilf and D. Zeilberger, *An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities*, Invent. Math. **108** (1992), 575–633.
13. D. Zeilberger, *A fast algorithm for proving terminating hypergeometric identities*, Discrete Math. **80** (1990), 207–211.
14. D. Zeilberger, *The method of creative telescoping*, J. Symbolic Comput. **11** (1991), 195–204.

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