THE CYCLOTOMIC IDENTITY AND THE CYCLIC GROUP

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Metropolis and Rota [1] discuss the cyclotomic identity

$$\frac{1}{1-\alpha t} = \prod_{n \in \mathbb{N}} \left(\frac{1}{1-t^n} \right)^{M(\alpha,n)} .$$

Thereby they define the necklace-algebra $\mathrm{Nr}(A)$ for a commutative ring A with unit element 1. We show that $\mathrm{Nr}(A)$ is the Burnside-ring $\hat{\Omega}(\mathbb{Z})$ of almost finite $\mathbb{Z}$-sets, where an almost finite $\mathbb{Z}$-set is a set with an operation of the infinite cyclic group $\mathbb{Z}$, such that every element lies in a finite orbit and every orbit type $\mathbb{Z}/n\mathbb{Z}$ occurs only with finite multiplicity. For every $n \in \mathbb{N}$ we have a homomorphism

$$\varphi_n : \hat{\Omega}(\mathbb{Z}) \longrightarrow \mathbb{Z},$$

which assigns to a $\mathbb{Z}$-set the number of elements invariant under the subgroup $n\mathbb{Z}$. The family φ_n with $n > 0$ defines an injective homomorphism

$$\varphi : \hat{\Omega}(\mathbb{Z}) \longrightarrow \mathbb{Z}^J$$

of $\hat{\Omega}(\mathbb{Z})$ into the product ring $\mathbb{Z}^J$, where J denotes the set of positive integers.

By defining symmetric powers of almost finite $\mathbb{Z}$-sets, we get a homomorphism

$$s_t : \hat{\Omega}(\mathbb{Z}) \longrightarrow 1 + t\Omega(\mathbb{Z})[[t]]$$

of the additive group $\hat{\Omega}(\mathbb{Z})$ into the multiplicative group of formal power-series with constant term 1 and coefficients in the ring $\hat{\Omega}(\mathbb{Z})$. Combining this homomorphism with φ_1, we get the isomorphism

$$\varphi_1 \circ s_t : \hat{\Omega}(\mathbb{Z}) \longrightarrow 1 + t\mathbb{Z}[[t]].$$

Finally we show that $\hat{\Omega}(\mathbb{Z})$ is isomorphic to the (generalized) ring of Witt-vectors $W(\mathbb{Z})$.

The Burnside-ring construction applies to every profinite group providing thereby a generalized cyclotomic(?) identity. The construction allows also to define for an arbitrary profinite group and given commutative ring A a generalized ring of Witt-vectors over A.

References

1. N. Metropolis and C.-C. Rota, *Witt vectors and the Algebra of Necklaces*, Adv. in Math. **50** (1983), 95–125.