Some Conjectures About Invariant Theory and their Applications

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Abstract

It turns out that various algebraic computations can be reduced to the same type of computations: one has to study the series of integrals $\int_K f^n(k)g(k)\,dk$, where f, g are complex valued K-finite functions on a compact Lie group K. So it is tempting to state a general conjecture about the behavior of such integrals, and to investigate the consequences of the conjecture.

MAIN CONJECTURE: *Let K be a compact connected Lie group and let f be a complex-valued K-finite function on K such that $\int_K f^n(k)\,dk = 0$ for any $n > 0$. Then for any K-finite function g, we have $\int_K f^n(k)g(k)\,dk = 0$ for n large enough.*

Especially, we prove that the main conjecture implies the jacobian conjecture. Another very optimistic conjecture is proposed, and its connection to isospectrality problems is explained.

Résumé

Il se trouve que divers calculs algébriques se réduisent à un même type de calcul : il s'agit d'étudier des intégrales $\int_K f^n(k)g(k)\,dk$, où f, g sont des fonctions K-finies et à valeurs complexes sur un groupe de Lie compact K. Il est alors tentant de formuler une conjecture générale sur de telles intégrales et en explorer les conséquences.

CONJECTURE PRINCIPALE : *Soit K un groupe de Lie compact connexe et soit f une fonction K-finie et à valeurs complexes sur K telle que $\int_K f^n(k)\,dk = 0$ pour tout $n > 0$. Alors pour toute fonction K-finie g, on a $\int_K f^n(k)g(k)\,dk = 0$ pour n assez grand.*

En particulier, nous montrons que la conjecture principale implique la conjecture jacobienne. Nous proposons une autre conjecture optimiste et expliquons ses liens avec les problèmes d'isospectralité.

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Introduction

It turns out that various algebraic computations can be reduced to the same type of computations: one has to study the series of integrals $\int_K f^n(k)g(k)\,dk$, where f, g are complex valued K-finite function on a compact Lie group K. So it is tempting to state general conjectures about the behavior of such integrals, and to investigate the consequences of these conjectures. Here we will state the following two conjectures:

Main Conjecture — *Let K be a compact connected Lie group and let f be a complex-valued K-finite function on K such that $\int_K f^n(k)\,dk = 0$ for any $n > 0$. Then for any K-finite function g, we have $\int_K f^n(k)g(k)\,dk = 0$ for n large enough.*

Second Conjecture — *Let $G \supset L$ be a reductive spherical pair, let $f \in \mathbb{C}[G/L]$, and let $C^\#$ be the G-complement of $\mathbb{C}$ in $\mathbb{C}[G/L]$. If $f^n \in C^\#$ for any $n \geq 1$, then 0 belongs to $\overline{G.f}$.*

These two conjectures are closely related. Indeed the second conjecture implies the main one. In this paper we show various examples of questions which can be treated (or partially solved) by the Conjectures above. The main two examples are as follows:

First example: recall that the Jacobian Conjecture states that a volume-preserving polynomial map $F\colon \mathbb{C}^n \to \mathbb{C}^n$ is invertible. In the paper we show that the Jacobian Conjecture follows the main conjecture (see Sections 2, 3, 4 and 5).

Second example: recall that two smooth real-valued functions f, g defined on a compact riemannian manifold are called isospectral if $\Delta + f$ and $\Delta + g$ have the same spectrum. We will see that some results of isospectral rigidity for $\mathbb{RP}^2$ follows from the second conjecture. It should be noted that the second conjecture and the section 7 has been motivated by Guillemin's paper [G].

In order to give some support to the main conjecture, we will see that the integrals $\int_K f^n(k)g(k)\,dk$ are closely related. Indeed we prove that all formal series $\chi_g = \sum_{n\geq 0}(\int_K f^n(k)g(k)\,dk)\,z^n$ can be deduced from one of them by applying a differential operator, see Section 6. To give some motivation for the second conjecture, we will see that a conjecture about invariant theory due to Guillemin implies a special case of the second conjecture.

At the end of the paper, we will investigate the conjecture when the group is a torus. In this case, the integrals considered appear naturally in the computation of the Hasse invariant and in the computation of number of points modulo p of plane algebraic curves.

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1 Equivalent forms of the conjecture.

In the section, we use the classical correspondence between compact Lie groups and algebraic reductive groups to state three equivalent forms of the main conjecture (see (1.1), (1.3), (1.7)).

Let K be a compact group. A continous complex-valued function defined on K will be called *K-finite* if the K-module generated by f is finite dimensional. Equivalently, f is a matrix coefficient of a finite dimensional representation. Denote by dk the Haar measure of K. The main conjecture of the paper is as follows:

Main Conjecture 1.1 — *Let K be a compact connected Lie group and let f, g be complex-valued K-finite functions. Assume that $\int_K f^n(k)\,dk = 0$ for any $n > 0$. Then $\int_K f^n(k)g(k)\,dk = 0$ for n large.*

Let G be a connected reductive algebraic group over an algebraically closed field F of characteristic zero. Denote by $\hat{G}$ the space of isomorphism classes of simple rational representations of G (for simplicity, the elements in $\hat{G}$ will be called the *types* of G). For any type $\tau \in \hat{G}$, denote by τ^* the dual type. For any G-module M, set $M = \bigoplus_{\tau\in\hat{G}} M_\tau$, where M_τ is the τ-isotypical component of M. Similarly, for any $m \in M$, set $m = \sum_{\tau\in\hat{G}} m_\tau$, where m_τ is the τ-isotypical component of m. In particular, denote by M_{triv} and m_{triv} the trivial components. Also set $X(m) = \{\tau \in \hat{G} | m_\tau \neq 0\}$. We have $F[G]_{\mathrm{triv}} = F$, hence we can define a linear form $L\colon F[G] \to F$ by $L(f) = f_{\mathrm{triv}}$.

Lemma 1.2 — (i) *Assume $F = \mathbb{C}$. Let K be a maximal compact subgroup of G. Then we have $L(f) = \int_K f(k)\,dk$.*

(ii) *The bilinear form $b\colon F[G] \times F[G] \to F$, $f, g \mapsto L(fg)$ is non degenerate.*

Proof. (i) Since K is Zariski dense in G, the map $L'\colon f \in F[G] \mapsto \int_K f(k)\,dk$ is G-invariant. Since $F[G]_{\mathrm{triv}} = F$ and $L'(1) = L(1) = 1$, L' and L are equal.

(ii) Clearly, the kernel of b is a G-invariant ideal of $F[G]$. Hence its zero set in G is G-stable and so the kernel of b is zero. ☐

Let us call *G-algebra* any commutative algebra endowed with a rational action of G by algebra automorphisms. For a G-algebra A, denote by $C(A)$ the conjecture:

> $C(A)$: *Let $f \in A$ and $\tau \in \hat{G}$. Assume that $(f^n)_{\mathrm{triv}} = 0$ for all $n > 0$. Then $(f^n)_\tau = 0$ for n large.*

Corollary 1.3 — *Assume that the main conjecture holds. Then the conjecture $C(F[G])$ holds.*

Proof. Let $f \in F[G]$. Note that f is defined over a finitely generated subfield E of F and such a field can be embeded in $\mathbb{C}$. Hence we can assume that $F = \mathbb{C}$. Let $\tau \in \hat{G}$. The τ^*-component of $\mathbb{C}[G]$ is finite dimensional. Let K be a maximal compact subgroup of G. By hypothesis, we have $\int_K f^n(k)\,dk = 0$ for any $n > 0$. By the main conjecture 1.1, there exist $N = N(\tau)$ such that $\int_K f^n(k)g(k)dk = 0$ for all $g \in \mathbb{C}[G]_{\tau^*}$ and $n \geq N(\tau)$. By Lemma 1.2, we have $(f^n)_\tau = 0$ for any $n \geq N(\tau)$. ☐

For X, Y two subsets of $\hat{G}$, denote by $X.Y$ the set of all types occuring in the tensor product $x \otimes y$ for some $x \in X$ and some $y \in Y$.

Lemma 1.4 — *Let A be a G-algebra, let I be a G-invariant nilpotent ideal, let $f \in A$ and let $\overline{f} \in A/I$ be its residue modulo I. There exists an integer $d \geq 0$ and some finite subsets $X_0, X_1, ..., X_d$ in $\hat{G}$ such that $X(f^n) \subset \cup_{0 \leq i \leq d} X_i.X(\overline{f}^{n-i})$, for any $n \geq d$.*

Proof. Denote by A_0 the algebra A with a trivial action of G. The structure map $\Delta\colon A \to F[G] \otimes A_0$ is an injective morphism of G-algebras. Hence we can assume that A is of the form $F[G] \otimes R$ for some algebra R, and I is of the form $F[G] \otimes J$ for some ideal J of R. Witout loss of generality, we can assume that R is finitely generated and J is the radical of R.

It follows from the existence of a primary decomposition for R that R embeds in a finite sum of primary algebras (apply Theorem 11 of [Ms] to the R-module R). Hence we can assume that $A \simeq F[G] \otimes R$, where R is primary and noetherian, and $I \simeq F[G] \otimes J$, where J is the radical of R. As R embeds in its quotient field, we can assume that R is already a quotient field. By Cohen's structure theorem (Theorem 60 of [Ms]), we have $R \simeq L \oplus J$, where $L \simeq R/J$ is a field. Thus we have $R \otimes F[G] \simeq L \otimes F[G] \oplus J \otimes F[G]$ and accordingly, we have $f = \overline{f} + h$, where $h \in F[G] \otimes J$.

Let d such that $J^{d+1} = 0$, and set $X_i = X(h^i)$. We have $f^n = \sum_{0 \leq i \leq d} \binom{n}{i} h^i.\overline{f}^{n-i}$. Hence we have $X(f^n) \subset \cup_{0 \leq i \leq d} X_i.X(g^{n-i})$, for any $n \geq d$. ☐

For any $X \subset \hat{G}$, denote by X^* the set of all types dual to those of X. For any $X, Y \subset \hat{G}$, denote by $X : Y$ the set of all types $\mu \in \hat{G}$ such that τ occurs in $\mu \otimes \sigma$ for some $\tau \in X$ and $\sigma \in Y$. For a sequence of subsets X_n in $\hat{G}$, we denote by $\lim X_n$ the set of all $\tau \in \hat{G}$ which belongs to infinitely many X_n. With these notations, the conclusion of conjecture $C(A)$ can be written as $\lim X(f^n) = \emptyset$.

Lemma 1.5 —

(i) *Let $X, Y \subset \hat{G}$. We have $X : Y = X.Y^*$.*

(ii) *Let X_n be a sequence of subsets in $\hat{G}$ and let $X \subset \hat{G}$ be finite. Then* $\lim(X_n.X) = (\lim X_n).X$.

Proof. (i) We have $Hom_G(\sigma \otimes \mu, \tau) \simeq Hom_G(\mu, \tau \otimes \sigma^*)$. Hence $X : Y = X.Y^*$.

(ii) Let $\tau \in \lim(X_n.X)$. Hence we have $X_n \cap (\{\tau\} : X) \neq \emptyset$ for infinitely many n. As X is finite, $\{\tau\} : X$ is finite. Hence there exists some $\mu \in \{\tau\} : X$ such that μ belongs to infinitely many X_n. Hence τ belongs to $(\lim X_n).X$, and we have $\lim(X_n.X) \subset (\lim X_n).X$. As the opposite inclusion is obvious, (ii) follows. $\square$

Lemma 1.6 — *Let A be a commutative G-algebra and let I be a G-invariant nilpotent ideal. Then conjecture $C(A/I)$ implies conjecture $C(A)$.*

Proof. Assume $C(A/I)$. Let $f \in A$, let $\overline{f} \in A/I$ be its residue modulo I and let $d \geq 0$, $X_0, ..., X_d \subset \hat{G}$ as in lemma 1.4. Assume that $(f^n)_{\mathrm{triv}} = 0$ for any $n > 0$. We have $X(f^n) \subset \cup_{0 \leq i \leq d} X_i.X(\overline{f}^{n-i})$, for any $n \geq d$. Hence by lemma 1.5, we have $\lim X(f^n) = \emptyset$. So conjecture $C(A)$ holds. $\square$

Corollary 1.7 — *Assume the main conjecture. Then for any G-algebra A, the conjecture $C(A)$ holds.*

Proof. Using lemma 1.6, we reduce the conjecture $C(A)$ for a general G-algebra A to the case where A is prime. So we will assume that A is prime. Let Φ be its fraction field. The structure map $\Delta\colon A \to F[G] \otimes A_0$ (where A_0 is the algebra A with a trivial action of G) induces a G-equivariant embeding $A \to \Phi[G]$. Hence the conjecture $C(A)$ follows from corollary 1.3. $\square$

It is possible to prove a very special case of the main conjecture, namely:

Proposition 1.8 — *Let V be a G-module, let $f \in V$ and let $\tau \in \hat{G}$. Consider f as an element of the G-algebra SV and assume that $(f^n)_{\mathrm{triv}} = 0$ for any $n > 0$. Then $(f^n)_\tau = 0$ for n large.*

Proof. There is a natural comultiplication map $\Delta\colon SV \to SV \otimes SV$ which is dual of the algebra structure on SV^*. For $n \geq 0$, let $B_n \subset S^n V$ be the G-module generated by f^n. We have $\Delta(f^n) = n! \sum_{p+q=n} f^p/p! \otimes f^q/q!$. Thus $\bigoplus_{n \geq 0} B_n$ is a sub-coalgebra of SV. Hence $R = \bigoplus_{n \geq 0} B_n^*$ is a quotient algebra of SV^*. Let $\tau \in \hat{G}$. By Hilbert's Theorem, R_{τ^*} is finitely generated as a R_{triv}-module. As $R_{\mathrm{triv}} = \mathbb{C}$, R_{τ^*} is finite dimensional, i.e. $(f^n)_\tau = 0$ for n large. $\square$

Remark. Let $f \in V$ as in Proposition 1.8. Indeed we have $(f^n)_{\mathrm{triv}} = 0$ for any $n > 0$ if and only if f is in the nilcone of V, i.e. 0 is belongs to the closure of the G-orbit of f.

2 A technical version of the main conjecture

In order to show that the Jacobian conjecture follows from the main conjecture (Section 5), we state another version of the main conjecture (Proposition 2.2).

Let F be an algebraically closed field of characteristic zero, and let G be a connected reductive group over F. Choose a Borel subgroup $B \in G$ and let P the group of characters of B. We identify $\hat{G}$ with the subset $P^+ \subset P$ of dominant weights, by using the map which associates to each simple representation its highest weight. For $\tau \in P$, we define the τ-isotypical component of a G-module (or of an element in a G-module) as previously if τ is dominant and as zero otherwise. For $\tau \in P^+$, we will denote by $L(\tau)$ one simple module with highest weight τ. Moreover τ^* denotes the highest weight of $L(\tau)^*$.

Lemma 2.1 — *Let $\lambda_1, \lambda_2, \mu_1\mu_2$ be in P^+. Assume $\lambda_1 - \mu_1^* = \lambda_2 - \mu_2^*$ If $\lambda_1 - \lambda_2$ is dominant, then there is a surjective morphism $L(\lambda_1) \otimes L(\mu_1) \to L(\lambda_2) \otimes L(\mu_2)$.*

Proof. Let $\mathfrak{g}$ (resp. $\mathfrak{b}$) be the Lie algebra of G (resp. B). Choose a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{b}$. Let l be the rank of G, let $(\alpha_i)_{1\leq i\leq l}$ be the simple roots of G and let $(h_i)_{1\leq i\leq l}$ be the simple coroots of G. For $1 \leq i \leq l$, denote by e_i, f_i the root vectors of weight $\pm\alpha_i$. Let $\lambda, \mu \in P^+$ and let $\mathfrak{b}^-$ be the opposite Borel algebra.

As a $\mathfrak{b}^-$-module, $L(\lambda)$ is the cyclic $\mathfrak{b}^-$-module generated by its highest weight vector v_λ^+ and defined by the following relations: $h.v_\lambda^+ = \lambda(h).v_\lambda^+$ for all $h \in\mathfrak{h}$ and $f_i^{\lambda(h_i)+1}.v_\lambda^+ = 0$ for all i, $1 \leq i \leq l$ (this follows easily from Theorem **??** of [Hu]). Similarly, $L(\mu)$ is the cyclic $\mathfrak{b}$-module generated by its lowest weight vector v_μ^- and defined by the following relations: $h.v_\mu^- = -\mu^*(h).v_\mu^-$ for all $h \in \mathfrak{h}$ and $e_i^{\mu^*(h_i)+1}.v_\mu^- = 0$ for all i, $1 \leq i \leq l$.

It follows that $L(\lambda) \otimes L(\mu)$ is the cyclic $\mathfrak{g}$-module generated by the vector $v_{\lambda,\mu} = v_\lambda^+ \otimes v_\mu^-$ and defined by the relations $h.v_{\lambda,\mu} = (\lambda - \mu^*)(h).v_{\lambda,\mu}$, $f_i^{\lambda(h_i+1)}.v_{\lambda,\mu} = 0$ and $e_i^{\mu^*(h_i+1)}.v_{\lambda,\mu} = 0$, for all $h \in \mathfrak{h}$ and for all i, $1 \leq i \leq l$.

Note that $\lambda_1(h_i) \geq \lambda_2(h_i)$ and $\mu_1^*(h_i) \geq \mu_2^*(h_i)$ for all i. Hence, there is a surjective morphism ϕ: $L(\lambda_1) \otimes L(\mu_1) \to L(\lambda_2) \otimes L(\mu_2)$ such that $\phi(v_{\lambda_1,\mu_1}) = v_{\lambda_2,\mu_2}$. ☐

Let $\lambda \in P$, let D be the unique B-invariant line in $L(\lambda)$. Set $\Sigma = G.D \cup \{0\}$ and $A(\lambda) = k[\Sigma]$. Recall that Σ is a closed cone of $L(\lambda)$ and by Borel-Weil Theorem, the degree n component of $A(\lambda)$ is the simple module isomorphic to $L(n.\lambda^*)$. Hence the n^{th}-power map $\xi \in L(\lambda^*) \mapsto \xi^n \in L(n.\lambda^*)$ is well defined up to multiplication by a scalar. The algebra $A(\lambda)$ is sometimes called the Cartan algebra.

For two graded algebras A, A', set $A * A' = \sum_{n\geq 0} A_n \otimes A'_n$.

Proposition 2.2 — *Let $\tau \in \hat{G}$. Assume that F is not countable.*

(i) *Let A be a G-algebra and assume that the conjecture $C(A \otimes A(\tau))$ holds. Let $f \in A$ and let $\mu \in P$. Assume that $(f^n)_{n\tau} = 0$ for all $n > 0$. Then $(f^n)_{\mu+n\tau} = 0$ for n large.*

(ii) *Let A be a graded G-algebra and assume that the conjecture $C(A * A(\tau))$*

holds. Let $f \in A_1$ and let $\mu \in P$. Assume that $(f^n)_{n\tau} = 0$ for all $n > 0$. Then $(f^n)_{\mu+n\tau} = 0$ for n large.

Proof. We prove together (i) and (ii). Let $f \in A$ with $(f^n)_{n\tau} = 0$ for all $n > 0$. For the proof of (ii), we assume in addition that A is graded and f belongs to A_1. Without loss of generality, we can assume that $\mu + n.\tau$ is dominant for n large, say $n \geq N$. Let Z be the set of $n \geq N$ such that $(f^n)_{\mu+n\tau}$ is non zero.

By Lemma 2.1, there exists $\nu \in P^+$ and $N \geq 0$ such that $L(\nu)$ occurs in $L(\mu + n\tau) \otimes L(n.\tau^*)$ for any $n \geq N$. Let U_n be the set of elements $\xi \in L(\tau^*)$ such that the ν-component of $\xi^n \otimes f^n$ is non zero. Let $n \in Z$. Since $L(n.\tau^*)$ is spanned by the elements ξ^n for $\xi \in L(\tau^*)$, U_n is a dense open subset of $L(\tau^*)$. As F is not countable, $\cap_{n \in Z} U_n$ is non empty (because Baire's Theorem holds for the Zariski topology over non countable fields).

Consider the element $L = f \otimes \xi \in A \otimes A(\tau^*)$, with $\xi \in \cap_{n\in Z} U_n$. For the proof of (ii), note that L belongs to $A * A(\tau)$. By our hypothesis, the trivial component of L^n is zero for any $n \geq 0$. However the ν component of F^n is non zero for any $n \in Z$. Thus conjecture $C(A \otimes A(\tau))$ (for the proof of (i)) or the conjecture $C(A * A(\tau))$ (for the proof of (ii)) implies that Z is finite. □

3 A few computations about tensor product decompositions

In this section, we will make explicit computations about the decomposition of $S^m V \otimes S^l V^*$. Let G be a connected reductive group over an algebraically closed field F of characteristic zero.

Lemma 3.1 — *Let $\lambda, \mu \in P^+$.*

(i) *The module $L(\lambda + \mu)$ occurs with multiplicity one in $L(\lambda) \otimes L(\mu)$.*

(ii) *Assume that $\lambda - \mu^*$ is dominant. Then $L(\lambda - \mu^*)$ occurs with multiplicity one in $L(\lambda) \otimes L(\mu)$.*

Proof. Point (i) is obvious. Assume that $\lambda - \mu^*$ is dominant. Then we have:

$$\begin{aligned}
&[L(\lambda) \otimes L(\mu) \colon L(\lambda - \mu^*)] \\
&\qquad = \dim\, Hom_G(L(\lambda) \otimes L(\mu), L(\lambda - \mu^*)) \\
&\qquad = \dim\, Hom_G(L(\lambda), L(\mu^*) \otimes L(\lambda - \mu^*)) \\
&\qquad = 1.
\end{aligned}$$

□

Often the component $L(\lambda+\mu)$ in $L(\lambda)\otimes L(\mu)$ is called the *Cartan component*, and the component $L(\lambda-\mu^*)$ in $L(\lambda)\otimes L(\mu)$ is called the *component* of Parthasaraty, Ranga-Rao and Varadarajan (or PRV component).

A simple G-module is called *weight multiplicity free* if all non-zero weight multiplicities are 1. The tensor product of two modules is called *multiplicity free* if each component has multiplicity one. A dominant weight μ is called minuscule if μ is the unique dominant weight of $L(\mu)$. For μ minuscule, $L(\mu)$ is weight multiplicity free.

Lemma 3.2 — *Let λ, μ, $\nu\in P^+$.*

(i) *Assume that $L(\mu)$ is weight multiplicity free. Then $L(\lambda)\otimes L(\mu)$ is multiplicity free. Moreover, if $L(\nu)$ occurs in $L(\lambda)\otimes L(\mu)$, then $\nu-\lambda$ is a weight of $L(\mu)$.*

(ii) *Assume that μ is minuscule. Then $L(\nu)$ occurs in $L(\lambda)\otimes L(\mu)$ if and only if $\nu-\lambda$ is a weight of $L(\mu)$.*

Proof. Let us prove point (i). We have $[L(\lambda)\otimes L(\mu)\colon L(\nu)]$ =dim $Hom_G(L(\lambda)\otimes L(\nu^*), L(\mu^*))$. The G-module $L(\lambda)\otimes L(\nu^*)$ is genereated by a weight vector of weight $\lambda-\nu$ (see the proof of Lemma 2.1). Hence $[L(\lambda)\otimes L(\mu)\colon L(\nu)]$ is less than or equal to the multiplicity of the weight $\lambda-\nu$ in $L(\nu^*)$. This proves point (i). Point (ii) is well-known: see e.g. [Mh] (Lemma 11). ☐

Let V be a vector space of dimension $n\geq 2$ and let $G=SL(V)$. For any i, $1\leq i\leq n$, $\wedge^i(V)$ is a simple G-module, and denote by ω_i the corresponding highest weight. We will recall a few facts about the decomposition of the G-modules $S^mV\otimes S^lV^*$

3.1 Symmetric powers of V and V^*.

For any m, the G-module S^mV is simple, and it is isomorphic to $L(m.\omega_1)$. Similarly, S^mV^* is isomorphic to $L(m.\omega_{n-1})$. In what follows we will identify SV with the algebra of polynomial functions on V^*, and SV^* with the space of invariant differential operators. A basis of V will be denoted by $x_1,...,x_n$ and the dual basis will be denoted by $(\partial/\partial x_i)_{1\leq i\leq n}$. For any n-tuple $\alpha=(\alpha_1,...,\alpha_n)$, we set $x^\alpha=x_1^{\alpha_1}...x_n^{\alpha_n}$ and $\partial^{(\alpha)}=\prod_{1\leq i\leq n}(1/\alpha_i!)(\partial/\partial x_i)^{\alpha_i}$.

3.2 Decomposition of the Lie algebra of vector fields.

Set $\mathscr{L}=SV\otimes V^*$ and $\mathscr{L}_m=S^{m+1}V\otimes V^*$. Apply Lemma 3.2 to the minuscule representation V^*. We get $\mathscr{L}_m\simeq L(m.\omega_1)\oplus L(m.\omega_1+\theta)$, where $\theta=\omega_1+\omega_{n-1}$ is the highest root. In order to make this decomposition more explicit, identify $\mathscr{L}$ with the Lie algebra of vector fields on V^*. The divergence div: $\sum_{1\leq i\leq n}P_i\partial/\partial x_i\mapsto$

$\sum_{1\leq i\leq n}\partial/\partial x_i.P_i$ defines a map from $\mathscr{L}\to SV$. The component of type $L(m\omega_1+\theta)$ in $\mathscr{L}_m$ is the subspace of divergence-free vector fields. The other component, which is of type $L(m\omega_1)$, is the subspace of vector fields of the form $f.E$, where $f\in S^mV$ and where $E=\sum_{1\leq i\leq m+1}x_i\partial/\partial x_i$ is the Euler vector field.

3.3 The PRV component of $S^mV\otimes S^lV^*$.

By identifying SV^* with the space of invariant differential operators on V^*, we see that SV is a right SV^*-module. The module structure map $SV\otimes SV^*\to SV$ will be called the divergence and denoted by div. Note that the restriction of this map to the subspace $SV\otimes V^*\simeq\mathscr{L}$ is the usual divergence map defined in the previous section. We have $div(\sum_\alpha p_\alpha\otimes\partial^{(\alpha)})=\sum_\alpha\partial^{(\alpha)}p_\alpha$.

Let $m\geq l$. Then the PRV component of $S^mV\otimes S^lV^*$ is isomorphic to $L((m-l)\omega_1)\simeq S^{(m-l)}V$. Hence div is precisely the projection on the PRV factor of $S^mV\otimes S^lV^*$.

3.4 The map Euler: $SV\to V^*\otimes SV$.

The Euler vector field $E\in V\otimes V^*\simeq V^*\otimes V$ is G-invariant. Hence the multiplication by E in the commutative algebra $SV^*\otimes SV$ defines a G-equivariant map Euler: $S^mV\to V^*\otimes S^{m+1}V$, for any $m\geq 0$.

3.5 The $L((m-1)\omega_1+\theta)$-component of $S^mV\otimes S^lV^*$.

Let $m\geq l$.

Consider the maps (defined in subsections 3.3 and 3.4):

$$\text{Euler}\otimes 1\colon S^mV\otimes S^lV^*\to(V^*\otimes S^{m+1}V)\otimes S^lV^*,\ \text{and}$$
$$1\otimes div\colon V^*\otimes(S^{m+1}V\otimes S^lV^*)\to V^*\otimes S^{m+1-l}V.$$

By composing these maps with the natural isomorphism $V^*\otimes S^{m+1-l}V\simeq\mathscr{L}_{m-l}$, one gets a map $\Phi: S^mV\otimes S^lV^*\to\mathscr{L}_{m-l}$.

Let $X\subset\hat{G}$, and let $f\colon M\to N$ be a surjective morphism of G-modules. We say that f is a projection along the types in X if f gives rise to an isomorphism along each type $\tau\in X$ and if f is zero along the other types.

Proposition 3.3 — *Let $m\geq l$. The simple modules $L((m-1)\omega_1)$ and $L((m-1)\omega_1+\theta)$ occur with multiplicity one in $S^mV\otimes S^lV^*$, and Φ is the corresponding projection along these two types.*

Proof. It follows from Lemma 3.2 that $S^mV\otimes S^lV^*$ is multiplicity free. Hence it suffices to show that Φ is onto. Let $f\in S^mV$ and set $D=f(\partial/\partial x_1)^{(l)}$. One gets

$\Phi(D)=\sum_{1\leq i\leq n}(\partial/\partial x_1)^{(l)}(x_i.f).\partial/\partial x_i$

$= ((\partial/\partial x_1)^{(l)} f).E + ((\partial/\partial x_1)^{(l-1)} f).\partial/\partial x_1.$

For a good choice of f, e.g. $f = x_1^m$, the vector field $\Phi(D)$ is not proportional to E, and its divergence is non-zero. It follows from subsection 3.2 that the $L((m-1)\omega_1$-isotypical and $L((m-1)\omega_1 + \theta)$-isotypical components of $\Phi(D)$ are non zero. Hence $\Phi(D)$ generates the G-module $\mathscr{L}_{m-l}$. Therefore Φ is surjective.

□

Let T_l be the the set of all *tuples* $(\alpha_1, ..., \alpha_n)$ with $\sum_{1\leq i\leq n} \alpha_i = l$. An explicit form of the previous proposition is the following:

Proposition 3.4 — *Let $m \geq l$ and let $D = \sum_{\alpha\in T_l} f_\alpha \otimes \partial^{(\alpha)}$ be an element in $S^m V \otimes S^l V^*$, where $f_\alpha \in S^m V$.*

(i) *The $L((m-l).\omega_1)$-component of D is $div(F) = \sum_{\alpha\in T_l} \partial^{(\alpha)} f_\alpha$.*

(ii) *We have $\Phi(D) = \sum_{1\leq i\leq n} \sum_{\alpha\in T_l} (\partial^{(\alpha)} x_i f_\alpha)\partial/\partial x_i$.*

(iii) *If the $L((m-l).\omega_1)$-component of D is zero, then its $L((m-1)\omega_1 + \theta)$-component is the divergence-free vector field $\sum_{1\leq i\leq n} \sum_{\alpha\in T_l} (\partial^{(\alpha)} x_i f_\alpha)\partial/\partial x_i$.*

Proof. Point (i) follows from subsection 3.3. Point (ii) follows from the definition of Φ. Point (iii) follows from Proposition 3.3 and Point (ii). □

Remark. Assume $m \geq l$. In what follows we will only use the $L((m-l).\omega_1)$-component and the $L((m-1)\omega_1 + \theta)$-component of $S^m V \otimes S^l V^*$. However it is well-known and easy to prove:

$S^m V \otimes S^l V^* \simeq \sum_{0\leq i\leq l} L((m-l)\omega_1 + i\theta).$

4 Review of results about the Jacobian conjecture

Let F be an algebraically closed field of characteristic zero.

For any polynomial map $f\colon F^n \to F^n$, denote by $j(f)$ its jacobian. Let us recall the Jacobian Conjecture.

Jacobian Conjecture 4.1 — *Let $n \geq 1$ and let $f\colon F^n \to F^n$ be a polynomial map with $j(f) = 1$. Then f is invertible.*

Let $d \geq 2$. Consider also the following conjecture (implicitely stated in the introduction of [BCW]):

d-Restricted Jacobian Conjecture 4.2 — *Let $n \geq 1$ and let $f = (f_1, ..., f_n)\colon F^n \to F^n$ be a polynomial map with $j(f) = 1$. Assume that $f_i = x_i - h_i$, where h_i is a homogenous polynomial of degree d. Then f is invertible.*

Of course, the restricted Jacobian Conjecture seems a mere particular case of the Jacobian Conjecture. However, they are equivalent, as proved in [BCW], see Theorem 2.1 and Corollary 2.2.

Theorem 4.3 (Bass, Connell, Wright) — *The 3-restricted Jacobian implies the Jacobian conjecture.*

Let $f = (f_1, ..., f_n)$ be a formal automorphism of F^n, where $f_i \in F[[x_1, ..., x_n]]$, and assume that $f_i = x_i - h_i$, where the h_i have no constant or linear terms. Let T the set of all n-tuples. For any $\alpha = (\alpha_1, ..., \alpha_n) \in T$, set $h^\alpha = h_1^{\alpha_1}...h_n^{\alpha_n}$. Let $L = (L_1, ..., L_n)$ be its formal inverse. From [A], we have (see also [BCW], Theorem 2.1).

Inversion Formula 4.4 (*[A], [BCW]*) — *Let $f\colon F^n \to F^n$ such that $j(f) = 1$.*

(i) *We have $1 = \sum_{\alpha\in T} \partial^{(\alpha)} h^\alpha$*

(ii) *We have $L_i = \sum_{\alpha\in T} \partial^{(\alpha)}(h^\alpha.x_i)$.*

5 The main conjecture implies the Jacobian Conjecture

Let F be an algebraically closed field of characteristic zero.

Lemma 5.1 — *Let A be a finitely generated G-algebra and let J be the radical of A^G.*

(i) *There exists a maximal G-invariant ideal I with $I^G = J$, and I is the kernel of the bilinear form $b\colon A \times A \to A^G/J$ defined by $b(f, g) = (fg)_{\mathrm{triv}}$ modulo J.*

(ii) *If the conjecture $C(A/I)$ holds, then the conjecture $C(A)$ holds.*

Proof. Assertion (i) is obvious. Let us prove (ii). Let $f \in A$ with $(f^n)_{\mathrm{triv}} = 0$ and let $\tau \in \hat{G}$. Set $A'_n = I^n/I^{n+1}$ and $A' = \bigoplus_{n\geq 0} A'_n$. Note that A' is a finitely generated and graded G-algebra. As J is nilpotent, there exists some $d \geq 0$ such that $A'^G = \bigoplus_{0\leq n\leq d} A_n^G$. As A'_τ is a finitely generated A'^G-module, we have $(A'_n)_\tau = \{0\}$ for any $n \geq d'$ for some integer d'. Set $I' = I^{d'}$.

Assume $C(A/I)$. Denote by $\overline{f}$ the residue modulo I' of f. By Lemma 1.6, the conjecture $C(A/I')$ holds. Hence we have $(\overline{f}^n)_\tau = \{0\}$ for $n \geq N(\tau)$ for some integer $N(\tau)$. However, by definition of d', the projection $A_{tau} \to (A/I')_\tau$ is an isomorphism. Hence we have $(f^n)_\tau = \{0\}$ for $n \geq N(\tau)$. □

A G-algebra A is called non degenerate if $A^G = F$ and if the bilinear form $f, g \in A \mapsto (f.g)_{\mathrm{triv}}$ is non degenerate. It is easy to prove that the non degenerate algebras are the algebras $F[G/L]$, where L is a reductive subgroup of G.

Lemma 5.2 — *Assume that F is not countable. Let A be a finitely generated G-algebra. If the conjecture $C(R)$ holds for any non degenerate G-quotient R of A, then the conjecture $C(A)$ holds.*

Proof. Using Lemma 1.6, one can assume that A is prime. Let I be the maximal ideal with $I^G = \{0\}$. Any non degenerate quotient of A is a quotient of A/I. Hence by Lemma 5.1, we can assume that $I = \{0\}$.

Let $f \in A$ with $(f^n)_{\text{triv}} = 0$ and let $\tau \in \hat{G}$. There exists a countable algebraically closed field $E \subset F$ such that the group G, the G-algebra A and f are defined over E. Let G_E, A_E and f_E such E-forms and identify A_E to a subalgebra of A. By countabilities hypotheses, there is an F-algebra morphism $\mu\colon A^G \to F$ whose restriction to $A_E^{G_E}$ is one to one. Let J the kernel of the map $b\colon f, g \in A \mapsto \mu((fg)_{\text{triv}})$.

By assumption the bilinear $f, g \mapsto (fg)_{\text{triv}}$ has no kernel. Hence the restriction ob b to A_E is injective. Hence A_E embeds in the non-degenerate F-algebra A/J. As $C(A/J)$ holds, we have $(f^n)_\tau = 0$ for n large. ☐

Let $n \geq 1$, $d \geq 2$ be two integers. Denote by $A(n,d)$ the $GL(n)$-algebra $\bigoplus_{d\geq 0} S^{dl}V \otimes S^l V^* \otimes S^{(d-1)l}V^*$.

Proposition 5.3 — *Let $d \geq 2$ be an integer. Assume that F is not countable and the conjecture $C(A(n,d)$ holds for any $n \geq 1$. Then the d-restricted jacobian conjecture holds.*

Proof. Let $f = (f_1, ..., f_n)\colon F^n \to F^n$ be a polynomial map with $j(f) = 1$. Moreover assume that $f_i = x_i - h_i$, where h_i is homogenous of degree d for some $d \geq 2$.

Set $V = \mathbb{C}.x_1 \oplus ... \oplus \mathbb{C}.x_n$. Let $L = (L_1, ..., L_n)$ be the formal inverse of F. Set $L_i = \sum_{l\geq 0} L_i^{[l]}$, where $L_i^{[l]}$ is homogenous of degree l. Set $D = \sum_i h_i \otimes \partial/\partial x_i$ and $\xi_l = \sum_{1\leq i\leq n} L_i^{[l]}.\partial/\partial x_i$ for $l \geq 0$. In what follows, we consider D as a degree one element of the commutative algebra $A' = \bigoplus_{l\geq 0} S^{ld}V \otimes S^l V^*$.

Let $m \geq 1$. Take the homogenous component of degree $m(d-1)$ in the identity: $1 = \sum_{\alpha\in T} \partial^{(\alpha)} H^\alpha$ (Formula 4.4 (i)). One obtains $\sum_{\alpha\in T_m} \partial^{(\alpha)} H^\alpha = 0$. Since $D^m = m! \sum_{\alpha\in T_m} H^\alpha \partial^{(\alpha)}$, we have $div(D^m) = 0$. Hence it follows from Proposition 3.4 that the $L(m(d-1).\omega_1)$-component of D^m is zero for any $m \geq 1$.

We have $A(n,d) = A' * A((d-1)\omega_1)$. Assume that the conjecture $C(A(n,d)$ holds. By Proposition 2.2 (ii), the $L(m(d-1).\omega_1 + \theta)$-component of D^m is zero for $m >> 0$. Hence we have $\Phi(D^m) = 0$ for m large (Proposition 3.4).

By taking each homogenous component in Formula 4.4 (ii), one obtains:

$L_i^{[l]} = \sum_{\alpha\in U_m} \partial^{(\alpha)}(H^\alpha.x_i)$, if $l = 1 + m(d-1)$ for some $m \geq 0$,

$L_i^{[l]} = 0$ otherwise.

By Proposition 3.4 (iii), we have $\xi_{1+m(d-1)} = \Phi(D^m/m!)$, hence $\xi_{1+m(d-1)} = 0$ for $m >> 0$, i.e. $L_i^{[l]} = 0$ for $l >> 0$. Therefore the formal inverse L is a polynomial and f is invertible. ☐

Lemma 5.4 — *Beside F, the non degenerate G-quotient of the G-algebra $A(n,d)$ are isomorphic to $F[SL(n)/GL(n-1)]$.*

Proof. Let I be the kernel of the natural morphism $A(n,d) \to \bigoplus_{l\geq 0} S^{dl}V \otimes S^{dl}V^*$. As $S^l V^* \otimes S^{(d-1)l}$ contains the Cartan component $S^{dl}V^*$ with multiplicity one, each homogenous components of $A(n,d)$ contains the trivial representation with multiplicity one. Hence I contains no invariants, and the non degenerate quotients of $A(n,d)$ are those of $A(n,d)/I = A'$. Let $\mathcal{P}$ be the set of rank one endomorphisms of V. Note that the group μ_d of d-roots of 1 acts on $\mathcal{P}$ by multiplication. It is clear that the spectrum of A' equals $\mathcal{P}/\mu_d$. Clearly the non degenerate quotients of $A(n,d)$ correspond with the closed orbits of the spectrum. Beside $\{0\}$, the closed orbits are the orbits of non nilpotent endomorphisms of rank 1. There are all isomorphic to $SL(n)/GL(n-1)$. $\square$

Theorem 5.5 — *Assume that the conjectures $C(\mathbb{C}[SL(n)/GL(n-1)])$ holds for any $n \geq 1$ (this follows from the main conjecture). Then the Jacobian conjecture holds for any algebraically field of characteristic zero.*

Proof. To prove the jacobian conjecture, we can assume that $F = \mathbb{C}$. By Lemmas 5.2 and 5.4, the conjecture $C(\mathbb{C}[SL(n)/GL(n-1)])$ implies the conjectures $C(A(n,3)$. By Proposition 5.3, the conjectures $C(n,3)$ for any $n \geq 1$ imply the 3-restricted jacobian conjecture. By Theorem 4.3, the later conjecture implies the jacobian conjecture. $\square$

6 Dependance of the integrals $\int_K f^n(k)g(k)\,dk$

In this section we will see that the series of integrals $\int_K f^n(k)g(k)\,dk$ for different g are closely related each other, what supports the main conjecture (see Proposition 6.2 (ii)).

Let K be a connected compact Lie group, let G be its complexification and let $\mathfrak{g}$ be the Lie algebra of G. Fix $f \in \mathbf{C}[G]$. For any $g \in \mathbb{C}[G]$, denote by $\chi_g(z)$ the formal series $\sum_{n\geq 0}(\int_K f^n(k)g(k)\,dk).z^n$. Let $A_1 = \mathbf{C} < z, d/dz >$ be the Weyl algebra. Denote by $M(f)$ the A_1-submodule of $\mathbb{C}[[z]]$ generated by $\chi_g(z)$, when g runs over $\mathbb{C}[G]$.

Lemma 6.1 — *The A_1-module $M(f)$ is holonomic.*

Proof. Let Ω be the complement in $\mathbb{A}^1_{\mathbb{C}} \times G$ of the hypersurface $zf = 1$ and let $p\colon \Omega \to \mathbb{A}^1_{\mathbb{C}}$ be the projection on the first factor. Let $A^*_{\Omega/\mathbb{A}^1_{\mathbb{C}}}$ be the de Rham complex relative to p, and let $H^*(\Omega/\mathbb{A}^1_{\mathbb{C}})$ be its cohomology. Define a map $T\colon \mathbb{C}[\Omega]) \to \mathbb{C}[[z]]$ as follows. Any $\phi \in \mathbb{C}[\Omega]$ admits an expansion at $z = 0$ as $\phi = \sum_{n\geq 0}\phi_n z^n$, where $\phi_n \in \mathbb{C}[G]$. Then, set $T(\phi) = \sum_{n\geq 0}(\int_K \phi_n(k)\,dk).z^n$.

Set $N = \dim G$. We have $A^{N+1}_{\Omega/\mathbb{A}^1_{\mathbb{C}}} = 0$, $A^N_{\Omega/\mathbb{A}^1_{\mathbb{C}}} = \mathbf{C}[\Omega].v$ and $A^{N-1}_{\Omega/\mathbb{A}^1_{\mathbb{C}}} = i(\mathfrak{g}).\mathbf{C}[\Omega].v$, where v is an invariant volume form on G. Thus we have $d.A^{N-1}_{\Omega/\mathbb{A}^1_{\mathbb{C}}} = \mathfrak{g}.\mathbf{C}[\Omega].v$. So we get $H^N(\Omega/\mathbb{C}) \simeq H_0(\mathfrak{g};\mathbb{C}[\Omega])$. Hence $H_0(\mathfrak{g};\mathbb{C}[\Omega])$ is holonomic as a A_1-module. Note that T factorizes through $H_0(\mathfrak{g};\mathbb{C}[\Omega])$. Hence $Im\,T$ is holonomic as a A_1-module (see e.g. [B], ch. 5). Moreover, we have $\chi_g(z) = T(g/(1-zf))$. Therefore, $M(f) \subset Im\,T$ is holonomic. $\square$

Proposition 6.2 — (i) *For any $g \in \mathbb{C}[G]$, the formal series $\chi_g(z)$ is the solution of a differential equation with polynomial coefficients.*

(ii) *There exists $g_0 \in \mathbb{C}[G]$ such that for any $g \in \mathbb{C}[G]$, we have $\chi_g(z) = P.\chi_{g_0}(z)$, for some differential operator $P \in A_1$.*

Proof. Point (i) follows from the holonomicity of $M(f)$. Point (ii) follows from the fact that any holonomic module is cyclic (see [B], ch. 1). $\square$

It is natural to ask when we can choose $g_0 = 1$ in Proposition 6.2. For example, the main conjecture can be stated as follows:

If $\int_K f^n(k)dk = 0$ for any $n \geq 0$, then χ_1 generates $M(f)$.

7 The second Conjecture

In this section, we will state another conjecture. This second conjecture implies the main conjecture. Beside this, it is also connected with a conjecture of Guillemin [G]. Let G be a connected reductive algebraic group. A subgroup L is called spherical if the algebra of regular functions over G/L is multiplicity free.

Second Conjecture 7.1 — *Let L be a reductive spherical subgroup of G, and let $f \in F[G/L]$. If $(f^n)_{\mathrm{triv}} = 0$ for any $n > 0$, then 0 belongs to $\overline{G.f}$.*

Let $\mathfrak{g}$ be the Lie algebra of G. By Hilbert-Mumford stability criterion [MFK], the condition $0 \in \overline{G.f}$ is equivalent to the existence of an element $h \in \mathfrak{g}$ and a finite decomposition $f = \sum_{i\geq 1} f_i$ such that $h.f_i = i.f_i$. If $M \subset F[G/L]$ is the G-module generated by f, this condition also means that f is in the nilcone of M, i.e. the set of all $m \in M$ such that $\phi(m) = \phi(0)$ for all $\phi \in (SM^*)^G$.

Proposition 7.2 — *Under the second conjecture 7.1, the main conjecture holds.*

Proof. Let K be a connected compact Lie group, and let f be a K-finite function over K such that $\int_K f^n(k)\,dk = 0$ for any $n > 0$. Let G be the complexification of K. Denote again by f its extension to G. Define $\phi\colon G \times G \to \mathbb{C}$ by $\phi(g_1, g_2) = f(g_1 g_2^{-1})$ and let $M \subset \mathbb{C}[G \times G]$ be the $G \times G$-module generated by ϕ. Note that $\phi \in \mathbf{C}[(G \times G)/G]$ where $G \subset G \times G$ is the diagonal. One obtains that $0 \in \overline{G \times G.f}$ by applying Conjecture 7.1 to the reductive spherical pair $G \times G \supset G$. Let $\tau \in \hat{G}$. By Proposition 1.8, one obtains that, viewed as elements of SM, we have $(\phi^n)_{\tau\otimes\tau^*} = 0$ for n large. It follows that $\int_K f^n(k)g(k)\,dk = 0$ for all $g \in \mathbb{C}[G]$ of type τ and n large enough. $\square$

Proposition 7.3 — *Assume that F is non countable. If Conjecture 7.1 holds for the spherical pairs $SL(n) \supset GL(n-1)$, then the Jacobian Conjecture holds.*

Proof. This is a reformulation of Theorem 5.5. □

Let X be a compact riemanian manifold. Two smooth functions $f, g \in C^\infty(X)$ are called isospectral if $\Delta + f$ and $\Delta + g$ have the same spectrum. Endow $\mathbb{RP}^2 \simeq SO(3,\mathbb{R})/O(2,\mathbb{R})$ with the standard $SO(3,\mathbb{R})$-invariant metric. Each simple $SO(3,\mathbb{R})$-module of dimension $4n+1$ occurs with multiplicity one in $C^\infty(\mathbb{RP}^2)$. The corresponding subspace $\mathcal{H}_n \subset C^\infty(\mathbb{RP}^2)$ is called the space of n^{th}-order harmonic functions on $\mathbb{RP}^2$. Let N be the normalizer of the subgroup of diagonal matrices in $SL(2)$, and set $\overline{N} = N/\pm 1$.

Proposition 7.4 — *Assume Conjecture 7.1 holds for the spherical pair $PSL(2) \supset \overline{N}$. Let M be a finite dimensional $SO(3,\mathbb{R})$-submodule of $C^\infty(\mathbb{RP}^2)$. For $f \in M$ denote by $I(f)$ be the set of all $g \in M$ which are isospectral to f. Then $I(f)$ contains only finitely many $SO(3,\mathbb{R})$-orbits.*

Proof. Set $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$, $\mathcal{H}_\mathbb{C} = \mathbb{C} \otimes \mathcal{H}$ and $M_\mathbb{C} = \mathbb{C} \otimes M$. Let $\mathcal{N}$ be the nil-cone of $M_\mathbb{C}$. For $f \in M_\mathbb{C}$ and $r \geq 0$, set $p_r(f) = \int_{\mathbb{RP}^2} f^r$. We have $\mathcal{H}_\mathbb{C} \simeq \mathbb{C}[PSL(2)/\overline{N}]$.

Set $A = (SM_\mathbb{C}^*)^{PSL(2)}$, let A' be the subalgebra of A generated by $p_1, p_2, \ldots$ and let A^+, A'^+ be the unique maximal homogenous ideals of A, A'. By definition, the nilcone is (set theorytically) defined as the set of $f \in M_\mathbb{C}$ such that $\phi(f) = 0$ for any $f \in A^+$. By Conjecture 7.1, the set of equations $p_r(f) = 0$ is enough to define $\mathcal{N}$. Hence the radical of the ideal $A'^+.A$ in A is A^+. Hence A is finitely generated as a A'-module, and $Spec(A) \to Spec(A')$ is finite.

Let $\phi \in M$. Because the $SO(3,\mathbb{R})$-orbits in M are closed in the real Zariski topology (see [S]), the set $J(\phi) = \{\psi \in M | p_r(\psi) = p_r(\phi) \text{ for all } r \geq 1\}$ contains only finitely many $SO(3,\mathbb{R})$-orbits.

For each $x \in \mathbb{RP}^2$ denote by γ_x the set of all lines of $\mathbb{R}^3$ which are orthogonal to x. It is clear that γ_x is a closed geodesic. For $f \in C^\infty(\mathbb{RP}^2)$, set $\hat{f}(x) = \int_{\gamma_x} f$ (see [G]). The map $f \mapsto \hat{f}$, called the Radon transform, is an $SO(3,\mathbb{R})$-invariant injective map. Thus the Radon transform induces a linear isomorphism from M to itself. By Weinstein's Theorem (**[W]**; see also [G], Proposition 2.3), $\int_{\mathbb{RP}^2} \hat{\phi}^n$ are spectral invariants of ϕ. Therefore, we have $I(\hat{f}) \subset J(\hat{f})$. Hence $I(f)$ contains only finitely many $SO(3,\mathbb{R})$-orbits. □

Recall Guillemin's Conjecture [G]. Set $K = SO(3,\mathbb{R})$ and denote by $H(4n)$ the real irreducible K-representation of dimension $4n+1$. Identify $H(4n)$ with the space $\mathcal{H}_n$ of all n^{th}-order harmonic on $\mathbb{RP}^2$. For $f \in H(4n)$, set $p_r(f) = \int_{\mathbb{RP}^2} f^r$, where the integral is relative to the standard K-invariant measure of $\mathbb{RP}^2$.

Guillemin's Conjecture 7.5 — *The polynomials p_r, $r = 1, 2, \ldots$, separate the K-orbits in $H(4n)$.*

An element of a rational $PSL(2)$-module is called isotypical if all its types but one are zero.

Proposition 7.6 — *Guillemin's Conjecture implies Conjecture 7.1 for the spherical pair $PSL(2) \supset \overline{N}$ and for any isotypical function $f \in \mathbb{C}[PSL(2)/\overline{N}]$, where $\overline{N} = N/\{\pm 1\}$.*

Proof. The proof of Proposition 7.6 is similar to those of Proposition 7.4. However, one should use Conjecture 7.5 instead of Conjecture 7.1. □

8 The torus case

The computation of the series of integrals $\int_K f^n(k)\,dk$ are connected to difficult questions even for the group $K = S^1$. For example let us consider the elliptic curve C given by the equation $y^2z = x(x+z)(x+\lambda z)$, where λ is an integer. For any prime number p, denote by C_p the reduction of C modulo p. One says that the Hasse invariant of C_p is zero if C has good reduction at p and C_p has no p-torsion points.

It turns out that for odd p the Hasse invariant is zero exactly if we have (see [Ha]):

$\sum_{1\leq i\leq (p-1)/2} \binom{(p-1)/2}{i}^2 \lambda^i = 0$ modulo p.

Set $f = (x+1)(x+\lambda)/x$. It is clear that $\sum_{1\leq i\leq (p-1)/2} \binom{(p-1)/2}{i}^2 \lambda^i = \int_{S^1} f^{(p-1)/2}$. Hence the Hasse invariant can be expressed in terms of reduction modulo p of integrals as considered before. Similar integrals occur when one computes the number of points of a plane algebraic curve. One gets these integrals by using the Chevalley-Warning Lemma. Let us mention the version of the main conjecture$^{(*)}$ for the S^1-case.

Let $f \in \mathbb{C}[t, t^{-1}]$. If $\mathrm{Res} f^n\, dt/t = 0$ for all $n \geq 1$, then f is a polynomial in t or a polynomial in t^{-1}.

$(*)$ Note added on proofs: W. van der Kallen and J. J. Duistermaat proved our conjecture for S^1; see their preprint: *Constant Terms of Powers of a Laurent Polynomial*.

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