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PSEUDODIFFERENTIAL PARAMETRICES OF INFINITE ORDER FOR SG-HYPERBOLIC PROBLEMS

Abstract. In this paper we consider a class of symbols of infinite order and develop a global calculus for the related pseudodifferential operators in the functional frame of the Gelfand-Shilov spaces of type S. As an application, we construct a parametrix for the Cauchy problem associated to an operator with principal part D_t^m and lower order terms given by SG-operators, cf. Introduction. We do not assume here Levi conditions on the lower order terms. Giving initial data in Gelfand-Shilov spaces, we are able to prove the well-posedness for the problem and to give an explicit expression of the solution.

1. Introduction

In this work, we study a class of pseudodifferential operators of infinite order, namely with symbol $p(x,\xi)$ satisfying, for every $\varepsilon > 0$, exponential estimates of the form

$$\sup_{\alpha,\beta\in\mathbb{N}^n}\ \sup_{(x,\xi)\in\mathbb{R}^{2n}} C^{-|\alpha|-|\beta|}(\alpha!)^{-\mu}(\beta!)^{-\nu}\langle\xi\rangle^{|\alpha|}\langle x\rangle^{|\beta|}\cdot \tag{1}$$

$$\cdot\exp\left[-\varepsilon(|x|^{\frac{1}{\theta}}+|\xi|^{\frac{1}{\theta}})\right]\left|D_\xi^\alpha D_x^\beta p(x,\xi)\right| < +\infty$$

where $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$, $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$, for some $\mu, \nu, \theta \in \mathbb{R}$ such that $\mu > 1, \nu > 1, \theta \geq \mu+\nu-1$ and C positive constant independent of α, β. Operators of infinite order were studied by L. Boutet de Monvel [2] in the analytic class and by L. Zanghirati [32] in the Gevrey classes $G^\theta(\Omega)$, $\Omega \subset \mathbb{R}^n$, $\theta > 1$. In our work we develop a global calculus for the symbols defined in (1). The functional frame is given by the Gelfand-Shilov space $S_\theta(\mathbb{R}^n)$, $\theta > 1$ (denoted by $S_\theta^\theta(\mathbb{R}^n)$ in [10]). This space makes part of a larger class of spaces of functions denoted by $S_\mu^\nu(\mathbb{R}^n)$, $\mu > 0$, $\nu > 0$, $\mu+\nu \geq 1$. More precisely, $S_\mu^\nu(\mathbb{R}^n)$ is defined as the space of all functions $u \in C^\infty(\mathbb{R}^n)$ satisfying the following condition: there exist positive constants A, B such that

$$\sup_{\alpha,\beta\in\mathbb{N}^n}\ \sup_{x\in\mathbb{R}^n} A^{-|\alpha|}B^{-|\beta|}(\alpha!)^{-\mu}(\beta!)^{-\nu}\left|x^\alpha u^{(\beta)}(x)\right| < +\infty.$$

Such spaces and the corresponding spaces of ultradistributions have been recently studied in different contexts by A. Avantaggiati [1], by S. Pilipovic [24] following the approach applied by H. Komatsu [17], [18] to the theory of ultradistributions and by S.

Pilipovic and N. Teofanov [25], [26] in the theory of modulation spaces. The space $S_\theta(\mathbb{R}^n)$ which we will consider in the paper corresponds to the case $\mu = \nu = \theta$ and it can be regarded as a global version of the Gevrey classes $G^\theta(\mathbb{R}^n), \theta > 1$. Sections 2,3 are devoted to the presentation of the calculus. In Section 4, as an application we construct a parametrix for the Cauchy problem

$$\begin{cases} P(t,x,D_t,D_x)u = f(t,x) & (t,x) \in [0,T] \times \mathbb{R}^n \\ D_t^k u(s,x) = g_k(x) & x \in \mathbb{R}^n, k = 0,...,m-1 \end{cases} \tag{2}$$

$T > 0, s \in [0,T]$, where $P(t,x,D_t,D_x)$ is a weakly hyperbolic operator with one constant multiple characteristic of the form

$$P(t,x,D_t,D_x) = D_t^m + \sum_{j=1}^{m} a_j(t,x,D_x) D_t^{m-j}. \tag{3}$$

For every fixed $t \in [0,T]$, we assume $a_j(t,x,D_x)$, $j = 1,...,m$ are SG-pseudodifferential operators of order (pj,qj), with $p,q \in [0,1[$, $p+q<1$ i.e. their symbols $a_j(t,x,\xi)$ satisfy estimates of the form

$$\sup_{t\in[0,T]} \left| D_\xi^\alpha D_x^\beta a_j(t,x,\xi) \right| \leq C^{|\alpha|+|\beta|+1} (\alpha!)^\mu (\beta!)^\nu \langle \xi \rangle^{pj-|\alpha|} \langle x \rangle^{qj-|\beta|} \tag{4}$$

for all $(x,\xi) \in \mathbb{R}^{2n}$, with μ, ν, C as in (1). We also assume continuity of $a_j(t,x,\xi)$ with respect to $t \in [0,T]$. SG-operators were studied by H.O. Cordes [7], C. Parenti [23], E. Schrohe [29] and applied in different contexts to PDEs. Recently, S. Coriasco and L. Rodino [9] treated their application to the solution of a global Cauchy problem for hyperbolic systems or equations with constant multiplicities; under assumptions of Levi type, namely $p = 0, q = 0$ for (3), (4), they obtained well-posedness in the Schwartz spaces $\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)$. In our paper, arguing under the weaker assumption $0 \leq p+q < 1$, we follow a different approach based on the construction of a parametrix of infinite order. This method has been applied by L. Cattabriga and D. Mari [4], L. Cattabriga and L. Zanghirati [6] to the solution of a similar problem in the local context of the Gevrey spaces $G^\theta(\Omega), \Omega \subset \mathbb{R}^n$. In Section 5 of our work we start from initial data in $S_\theta(\mathbb{R}^n)$, and find a global solution in $C^m([0,T], S_\theta(\mathbb{R}^n))$, with $p+q < \frac{1}{\theta} \leq \frac{1}{\mu+\nu-1}$. Analogous results are obtained replacing $S_\theta(\mathbb{R}^n)$ with its dual. We emphasize that our pseudodifferential approach, beside giving well-posedness, provides an explicit expression for the solution. Moreover, it seems possible to extend the present techniques to global Fourier integral operators, which would allow to treat general SG-hyperbolic equations with constant multiplicities. Let us give an example representative of our results in the Cauchy problem, showing the sharpness of the bound $\frac{1}{\theta} > p+q$ in the frame of the Gelfand-Shilov spaces.

EXAMPLE 1. Let $p,q \in [0,1[$ such that $p+q<1$ and consider the problem

$$\begin{cases} D_t^m u - x^{qm} D_x^{pm} u = 0 & (t,x) \in [0,T] \times \mathbb{R} \\ u(0,x) = c_0(x) & x \in \mathbb{R} \\ D_t^j u(0,x) = 0 & j = 1,...,m-1 \end{cases} \tag{5}$$

where pm, qm are assumed to be positive integers, $c_0(x) \in C^\infty(\mathbb{R})$ and it satisfies the estimate

$$\sup_{x\in\mathbb{R}} \left|x^\alpha D_x^\beta c_0(x)\right| \le C^{\alpha+\beta+1}(\alpha!\beta!)^\theta, \qquad \frac{1}{\theta} > p+q, \tag{6}$$

i.e. $c_0(x) \in S_\theta(\mathbb{R})$.
Under these hypotheses, it is easy to verify that the solution of the problem (5) is given by

$$u(t,x) = \sum_{j=0}^{\infty} \frac{(x^{qm}D_x^{pm})^j c_0(x)}{(jm)!} t^{jm}$$

which is well defined thanks to the condition (6) and belongs to $S_\theta(\mathbb{R})$ for every fixed t. We remark that in the critical case $\frac{1}{\theta} = p+q$ the solution is defined only for t belonging to a bounded interval depending on the initial datum $c_0 \in S_{\frac{1}{(p+q)}}(\mathbb{R})$. We also emphasize that from the expression of the solution we have that the solvability of the problem is guaranteed when $c_0(x)$ satisfies the weaker condition

$$\sup_{x\in\mathbb{R}} \left|(x^{qm}D_x^{pm})^j c_0(x)\right| \le C^{j+1}(j!)^{(p+q)m},$$

which would characterize a function space larger than $S_\theta(\mathbb{R})$, $\frac{1}{\theta} \ge p+q$. In the sequel we shall prefer to keep data in the Gelfand spaces $S_\theta(\mathbb{R}^n)$, because well established in literature and particularly suitable to construct a global pseudo-differential calculus. Let us recall some basic results concerning the space $S_\theta(\mathbb{R}^n)$. We refer to [10],[11],[20] for proofs and details.

Let $\theta > 1$ and A, B be positive integers and denote by $S_{\theta,A,B}(\mathbb{R}^n)$ the space of all functions u in $C^\infty(\mathbb{R}^n)$ such that

$$\sup_{\alpha,\beta\in\mathbb{N}^n} \sup_{x\in\mathbb{R}^n} A^{-|\alpha|}B^{-|\beta|}(\alpha!\beta!)^{-\theta}\left|x^\alpha u^{(\beta)}(x)\right| < +\infty.$$

We may write

$$S_\theta(\mathbb{R}^n) = \bigcup_{A,B\in\mathbb{Z}_+} S_{\theta,A,B}(\mathbb{R}^n).$$

PROPOSITION 1. *$S_{\theta,A,B}(\mathbb{R}^n)$ is a Banach space endowed with the norm*

$$\|u\|_{A,B} = \sup_{\alpha,\beta\in\mathbb{N}^n} \sup_{x\in\mathbb{R}^n} A^{-|\alpha|}B^{-|\beta|}(\alpha!\beta!)^{-\theta}|x^\alpha u^{(\beta)}(x)|. \tag{7}$$

By Proposition 1, we can give to $S_\theta(\mathbb{R}^n)$ the topology of inductive limit of an increasing sequence of Banach spaces. We remark that this topology is equivalent to the one given in [10] and that all the statements of this section hold in both the frames. Let us give a characterization of the space $S_\theta(\mathbb{R}^n)$, providing another equivalent topology to $S_\theta(\mathbb{R}^n)$, cf. the proof of Theorem 2 below.

PROPOSITION 2. *$S_\theta(\mathbb{R}^n)$ is the space of all functions $u \in C^\infty(\mathbb{R}^n)$ such that*

$$\sup_{\beta\in\mathbb{N}^n} \sup_{x\in\mathbb{R}^n} B^{-|\beta|}(\beta!)^{-\theta} e^{a|x|^{\frac{1}{\theta}}} |D_x^\beta u(x)| < +\infty$$

for some positive a, B.

PROPOSITION 3. *The following statements hold:*
(i) $S_\theta(\mathbb{R}^n)$ is closed under the differentiation;
(ii) $G_0^\theta(\mathbb{R}^n) \subset S_\theta(\mathbb{R}^n) \subset G^\theta(\mathbb{R}^n)$,
where $G^\theta(\mathbb{R}^n)$ is the space of the Gevrey functions of order θ and $G_0^\theta(\mathbb{R}^n)$ is the space of all functions of $G^\theta(\mathbb{R}^n)$ with compact support.

We shall denote by $S'_\theta(\mathbb{R}^n)$ the dual space, i.e. the space of all linear continuous forms on $S_\theta(\mathbb{R}^n)$. From (ii) of Proposition 3, we deduce the following important result.

THEOREM 1. *There exists an isomorphism between $\mathcal{L}(S_\theta(\mathbb{R}^n), S'_\theta(\mathbb{R}^n))$, space of all linear continuous maps from $S_\theta(\mathbb{R}^n)$ to $S'_\theta(\mathbb{R}^n)$, and $S'_\theta(\mathbb{R}^{2n})$, which associates to every $T \in \mathcal{L}(S_\theta(\mathbb{R}^n), S'_\theta(\mathbb{R}^n))$ a distribution $K_T \in S'_\theta(\mathbb{R}^{2n})$ such that*

$$\langle Tu, v\rangle = \langle K_T, v \otimes u\rangle$$

for every $u, v \in S_\theta(\mathbb{R}^n)$. The distribution K_T is called the kernel of T.

Finally we give a result concerning the action of the Fourier transformation on $S_\theta(\mathbb{R}^n)$.

PROPOSITION 4. *The Fourier transformation is an automorphism of $S_\theta(\mathbb{R}^n)$ and it extends to an automorphism of $S'_\theta(\mathbb{R}^n)$.*

2. Symbol classes and operators.

Let μ, ν, θ be real numbers such that $\mu > 1, \nu > 1, \theta \geq \max\{\mu, \nu\}$.

DEFINITION 1. *For every $C > 0$ we denote by $\Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n}; C)$ the Fréchet space of all functions $p(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ satisfying the following condition: for every $\varepsilon > 0$*

$$\|p\|_{\varepsilon,C} = \sup_{\alpha,\beta\in\mathbb{N}^n} \sup_{(x,\xi)\in\mathbb{R}^{2n}} C^{-|\alpha|-|\beta|}(\alpha!)^{-\mu}(\beta!)^{-\nu}\langle\xi\rangle^{|\alpha|}\langle x\rangle^{|\beta|}\cdot$$

$$\cdot \exp\left[-\varepsilon(|x|^{\frac{1}{\theta}} + |\xi|^{\frac{1}{\theta}})\right]\left|D_\xi^\alpha D_x^\beta p(x,\xi)\right| < +\infty$$

endowed with the topology defined by the seminorms $\|\cdot\|_{\varepsilon,C}$, for $\varepsilon > 0$. We set

$$\Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n}) = \varinjlim_{C\to+\infty} \Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n}; C)$$

with the topology of inductive limit of an increasing sequence of Fréchet spaces.

It is easy to verify that $\Gamma^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$ is closed under the differentiation and the sum and the product of its elements. In the sequel, we will also consider SG-symbols of finite order which are defined as follows, cf. Introduction.
Let $m_1, m_2 \in \mathbb{R}$ and let μ, ν be positive real numbers such that $\mu > 1, \nu > 1$.

DEFINITION 2. *For* $C > 0$, *we denote by* $\Gamma^{m_1,m_2}_{\mu\nu}(\mathbb{R}^{2n}; C)$ *the Banach space of all functions* $p \in C^{\infty}(\mathbb{R}^{2n})$ *such that*

$$\|p\|_C = \sup_{\alpha,\beta\in\mathbb{N}^n} \sup_{(x,\xi)\in\mathbb{R}^{2n}} C^{-|\alpha|-|\beta|}(\alpha!)^{-\mu}(\beta!)^{-\nu}\langle\xi\rangle^{-m_1+|\alpha|}\langle x\rangle^{-m_2+|\beta|}\cdot$$

$$\cdot\left|D^{\alpha}_{\xi}D^{\beta}_{x}p(x,\xi)\right| < +\infty$$

endowed with the norm $\|\cdot\|_C$ *and define*

$$\Gamma^{m_1,m_2}_{\mu\nu}(\mathbb{R}^{2n}) = \varinjlim_{C\to+\infty} \Gamma^{m_1,m_2}_{\mu\nu}(\mathbb{R}^{2n}; C).$$

We have obviously

$$\Gamma^{m_1,m_2}_{\mu\nu}(\mathbb{R}^{2n}) \subset \Gamma^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$$

for all $\theta \geq \max\{\mu, \nu\}$ and for all $m_1, m_2 \in \mathbb{R}$.

Given a symbol $p \in \Gamma^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$, we consider the associated pseudodifferential operator

$$Pu(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}p(x,\xi)\hat{u}(\xi)d\xi, \quad u \in S_{\theta}(\mathbb{R}^n). \tag{8}$$

The integral (8) is absolutely convergent in view of Propositions 2 and 4.

LEMMA 1. *Given* $t > 0$, *let*

$$m_t(\eta) = \sum_{j=0}^{\infty}\frac{\eta^j}{(j!)^t}, \quad \eta \geq 0.$$

Then, for every $\epsilon > 0$ *there exists a constant* $C = C(t,\epsilon) > 0$ *such that*

$$C^{-1}e^{(t-\epsilon)\eta^{\frac{1}{t}}} \leq m_t(\eta) \leq Ce^{(t+\epsilon)\eta^{\frac{1}{t}}} \tag{9}$$

for every $\eta \geq 0$.

See [16] for the proof.

In the following we shall denote for $t, \zeta > 0, x \in \mathbb{R}^n$,

$$m_{t,\zeta}(x) = m_t(\zeta\langle x\rangle^2).$$

THEOREM 2. *The map* $(p, u) \to Pu$ *defined by* (8) *is a bilinear and separately continuous map from* $\Gamma^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n}) \times S_{\theta}(\mathbb{R}^n)$ *to* $S_{\theta}(\mathbb{R}^n)$ *and it extends to a bilinear and separately continuous map from* $\Gamma^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n}) \times S'_{\theta}(\mathbb{R}^n)$ *to* $S'_{\theta}(\mathbb{R}^n)$.

Proof. Let us fix $p \in \Gamma^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$ and show that $u \to Pu$ is continuous from $S_\theta(\mathbb{R}^n)$ to itself. Basing on Proposition 2, we fix $B \in \mathbb{Z}_+$, $a > 0$ and consider the bounded set F determined by $C_1 > 0$

$$\sup_{x \in \mathbb{R}^n} e^{a|x|^{\frac{1}{\theta}}} |u^{(\beta)}(x)| \leq C_1 B^{|\beta|} (\beta!)^{\theta}$$

for all $u \in F$, $\beta \in \mathbb{N}^n$. To prove the continuity with respect to u, we need to show that there exist $A_1, B_1 \in \mathbb{N} \setminus \{0\}$ and a positive constant C_2 such that

$$\sup_{x \in \mathbb{R}^n} \left| x^{\alpha} D_x^{\beta} Pu(x) \right| \leq C_2 A_1^{|\alpha|} B_1^{|\beta|} (\alpha! \beta!)^{\theta}$$

for all $\alpha, \beta \in \mathbb{N}^n$ and for all $u \in F$. We observe that for every $\zeta \in \mathbb{R}^+$,

$$\frac{1}{m_{2\theta,\zeta}(x)} \sum_{j=0}^{\infty} \frac{\zeta^j}{(j!)^{2\theta}} (1 - \Delta_\xi)^j e^{i\langle x,\xi \rangle} = e^{i\langle x,\xi \rangle}.$$

Thus, fixed $\alpha, \beta \in \mathbb{N}^n$, we have

$$x^{\alpha} D_x^{\beta} Pu(x) = (2\pi)^{-n} x^{\alpha} \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1! \beta_2!} \int_{\mathbb{R}^n} e^{i\langle x,\xi \rangle} \xi^{\beta_1} D_x^{\beta_2} p(x,\xi) \hat{u}(\xi) d\xi =$$

$$(2\pi)^{-n} \frac{x^{\alpha}}{m_{2\theta,\zeta}(x)} \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1! \beta_2!} \sum_{j=0}^{\infty} \frac{\zeta^j}{(j!)^{2\theta}} \int_{\mathbb{R}^n} e^{i\langle x,\xi \rangle} (1-\Delta_\xi)^j \left[\xi^{\beta_1} D_x^{\beta_2} p(x,\xi) \hat{u}(\xi) \right] d\xi.$$

By Proposition 4, there exist $a, \overline{B}, C > 0$ independent of $u \in F$ and for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that, for $\zeta < \frac{1}{C}$

$$\left| x^{\alpha} D_x^{\beta} Pu(x) \right| \leq C_\varepsilon \frac{|x|^{|\alpha|}}{m_{2\theta,\zeta}(x)} e^{\varepsilon |x|^{\frac{1}{\theta}}} \sum_{j=0}^{\infty} (C\zeta)^j \cdot$$

$$\cdot \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1! \beta_2!} \overline{B}^{|\beta_2|} (\beta_2!)^{\nu} \int_{\mathbb{R}^n} |\xi|^{|\beta_1|} e^{-(a-\varepsilon)|\xi|^{\frac{1}{\theta}}} d\xi.$$

Hence, for ε sufficiently small,using Lemma 1 and standard estimates for binomial and factorial coefficients, we conclude that there exist $C_2, A_1, B_1 > 0$ depending only on $\zeta, \theta, \varepsilon$ such that

$$\sup_{x \in \mathbb{R}^n} \left| x^{\alpha} D_x^{\beta} Pu(x) \right| \leq C_2 A_1^{|\alpha|} B_1^{|\beta|} (\alpha! \beta!)^{\theta}.$$

This concludes the first part of the proof. To prove the second part we observe that, for $u, v \in S_\theta(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} Pu(x) v(x) dx = \int_{\mathbb{R}^n} \hat{u}(\xi) p_v(\xi) d\xi$$

where

$$p_v(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)v(x)dx$$

Furthermore, by the same argument of the first part of the proof, it follows that the map $v \to p_v$ is linear and continuous from $S_\theta(\mathbb{R}^n)$ to itself. Then, by Proposition 4 we can define, for $u \in S'_\theta(\mathbb{R}^n)$,

$$Pu(v) = \hat{u}(p_v), \quad v \in S_\theta(\mathbb{R}^n).$$

This is a linear continuous map from $S'_\theta(\mathbb{R}^n)$ to itself and it extends P. The same argument used before allows to prove the continuity of the map

$$p \to Pu$$

for a fixed u in $S_\theta(\mathbb{R}^n)$ or in its dual.

$\square$

We denote by $OPS^\infty_{\mu\nu\theta}(\mathbb{R}^n)$ the space of all operators of the form (8) defined by a symbol of $\Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})$.

As a consequence of Theorems 1 and 2, there exists a unique distribution K in $S'_\theta(\mathbb{R}^{2n})$ such that

$$\langle K, v \otimes u\rangle = (2\pi)^{-n} \int\int\int e^{i\langle x-y,\xi\rangle} p(x,\xi)u(y)v(x)dyd\xi dx, \quad u, v \in S_\theta(\mathbb{R}^n).$$

We may write formally

$$K(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} p(x,\xi)d\xi. \tag{10}$$

THEOREM 3. *Let* $p \in \Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})$. *For* $k \in (0,1)$, *define:*

$$\Omega_k = \{(x,y) \in \mathbb{R}^{2n} : |x-y| > k\langle x\rangle\}.$$

Then the kernel K *of* P *defined by* (10) *is in* $C^\infty(\Omega_k)$ *and there exist positive constants* C, a *depending on* k *such that*

$$\left|D_x^\beta D_y^\gamma K(x,y)\right| \le C^{|\beta|+|\gamma|+1}(\beta!\gamma!)^\theta \exp\left[-a(|x|^{\frac{1}{\theta}} + |y|^{\frac{1}{\theta}})\right] \tag{11}$$

for every $(x,y) \in \overline{\Omega_k}$ *and for every* $\beta, \gamma \in \mathbb{N}^n$.

LEMMA 2. *For any given* $R > 0$, *we may find a sequence* $\psi_N(\xi) \in C_0^\infty(\mathbb{R}^n)$, $N = 0, 1, 2, \ldots$ *such that* $\sum_{N=0}^{\infty} \psi_N = 1$ *in* $\mathbb{R}^n$,

$$supp\psi_0 \subset \{\xi : \langle\xi\rangle \le 3R\}$$

$$supp\psi_N \subset \{\xi : 2RN^\mu \le \langle\xi\rangle \le 3R(N+1)^\mu\},\ N = 1, 2, \ldots$$

and

$$\left|D_\xi^\alpha \psi_N(\xi)\right| \leq C^{|\alpha|+1}(\alpha!)^\mu \left[R \sup(N^\mu, 1)\right]^{-|\alpha|}$$

for every $\alpha \in \mathbb{N}^n$ *and for every* $\xi \in \mathbb{R}^n$*.*

Proof. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi(\xi) = 1$ if $\langle \xi \rangle \leq 2$, $\phi(\xi) = 0$ if $\langle \xi \rangle \geq 3$ and

$$\left|D_\xi^\alpha \phi(\xi)\right| \leq C^{|\alpha|+1}(\alpha!)^\mu$$

for all $\alpha \in \mathbb{N}^n$ and for all $\xi \in \mathbb{R}^n$. We may then define

$$\psi_0(\xi) = \phi\left(\frac{\xi}{R}\right)$$

$$\psi_N(\xi) = \phi\left(\frac{\xi}{R(N+1)^\mu}\right) - \phi\left(\frac{\xi}{RN^\mu}\right), \quad N \geq 1.$$

□

Proof of Theorem 3. Let us consider a sequence $\{\psi_N\}_{N\geq 0}$ as in Lemma 2. We observe that, by the condition $\theta \geq \mu$,

$$\sum_{N=0}^{\infty} \int_{\mathbb{R}^n} \left|e^{i\langle x,\xi\rangle} \psi_N(\xi) p(x,\xi) \hat{u}(\xi)\right| d\xi < +\infty$$

for every $x \in \mathbb{R}^n$. Then we have, for $u, v \in S_\theta(\mathbb{R}^n)$,

$$\langle K, v \otimes u \rangle = \sum_{N=0}^{\infty} \langle K_N, v \otimes u \rangle$$

with

$$K_N(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} p(x,\xi) \psi_N(\xi) d\xi$$

so we may decompose

$$K = \sum_{N=0}^{\infty} K_N.$$

Let $k \in (0,1)$ and $(x,y) \in \overline{\Omega}_k$. Let $h \in \{1, ..., n\}$ such that $|x_h - y_h| \geq \frac{k}{n}\langle x \rangle$. Then, for every $\alpha, \gamma \in \mathbb{N}^n$,

$$D_x^\alpha D_y^\gamma K_N(x,y) = \frac{(-1)^{|\gamma|}}{(2\pi)^n} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \xi^{\beta+\gamma} \psi_N(\xi) D_x^{\alpha-\beta} p(x,\xi) d\xi =$$

$$\frac{(-1)^{|\gamma|+N}}{(2\pi)^n} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (x_h - y_h)^{-N} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} D_{\xi_h}^N \left[\xi^{\beta+\gamma} \psi_N(\xi) D_x^{\alpha-\beta} p(x,\xi)\right] d\xi =$$

$$\frac{(-1)^{|\gamma|+N}}{(2\pi)^n} \cdot \frac{(x_h - y_h)^{-N}}{m_{2\theta,\zeta}(x-y)} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \sum_{j=0}^{\infty} \frac{\zeta^j}{(j!)^{2\theta}} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \lambda_{hjN\alpha\beta\gamma}(x,\xi)\, d\xi$$

with

$$\lambda_{hjN\alpha\beta\gamma}(x,\xi) = (1-\Delta_\xi)^j D^N_{\xi_h}\left[\xi^{\beta+\gamma}\psi_N(\xi) D_x^{\alpha-\beta} p(x,\xi)\right]. \tag{12}$$

Let e_h be the h-th vector of the canonical basis of $\mathbb{R}^n$ and $\beta_h = \langle \beta, e_h \rangle$, $\gamma_h = \langle \gamma, e_h \rangle$. Developing in the right-hand side of (12) we obtain that

$$\lambda_{hjN\alpha\beta\gamma}(x,\xi) = \sum_{\substack{N_1+N_2+N_3=N \\ N_1 \le \beta_h+\gamma_h}} (-i)^{N_1} \frac{N!}{N_1!N_2!N_3!} \cdot \frac{(\beta_h+\gamma_h)!}{(\beta_h+\gamma_h-N_1)!} \cdot$$

$$\cdot (1-\Delta_\xi)^j \left[\xi^{\beta+\gamma-N_1 e_h} D^{N_2}_{\xi_h} \psi_N(\xi) D^{N_3}_{\xi_h} D_x^{\alpha-\beta} p(x,\xi)\right].$$

Hence, for $\varepsilon > 0$,

$$\left|\lambda_{hjN\alpha\beta\gamma}(x,\xi)\right| \le C_\varepsilon \sum_{\substack{N_1+N_2+N_3=N \\ N_1 \le \beta_h+\gamma_h}} \frac{N!}{N_1!N_2!N_3!} \cdot \frac{(\beta_h+\gamma_h)!}{(\beta_h+\gamma_h-N_1)!} C_1^{|\alpha-\beta|+N_2+N_3} \cdot$$

$$\cdot (N_2!N_3!)^\mu \left[(\alpha-\beta)!\right]^\nu C_2^j (j!)^{2\theta} \left(\frac{1}{RN^\mu}\right)^{N_2} \langle \xi \rangle^{|\beta|+|\gamma|-N_1-N_3} \exp\left[\varepsilon(|x|^{\frac{1}{\theta}} + |\xi|^{\frac{1}{\theta}})\right].$$

We observe that on the support of ψ_N, $2RN^\mu \le \langle \xi \rangle \le 3R(N+1)^\mu$. Thus, from standard factorial inequalities,since $\theta \ge \max\{\mu, \nu\}$, it follows that

$$\left|\lambda_{hjN\alpha\beta\gamma}(x,\xi)\right| \le C_\varepsilon C_1^{|\alpha|+|\gamma|} (\alpha!\gamma!)^\theta C_2^j (j!)^{2\theta} \left(\frac{C_3}{R}\right)^N e^{\varepsilon|x|^{\frac{1}{\theta}}} \exp\left[\varepsilon (3R)^{\frac{1}{\theta}} (N+1)^{\frac{\mu}{\theta}}\right]$$

with C_3 independent of R. From these estimates, choosing $\zeta < \frac{1}{C_2}$, we deduce that

$$\left|D_x^\alpha D_y^\gamma K_N(x,y)\right| \le C'_\varepsilon C_1^{|\alpha|+|\gamma|} (\alpha!\gamma!)^\theta \left(\frac{C_4}{R}\right)^N \exp\left[\varepsilon|x|^{\frac{1}{\theta}} - c\zeta^{\frac{1}{\theta}} |x-y|^{\frac{1}{\theta}}\right]$$

with $C_4 = C_4(k)$ independent of R. Finally, the condition $\theta \ge \nu$ implies that there exists $a_k > 0$ such that

$$\sup_{(x,y)\in\Omega_k} \exp\left[a_k(|x|^{\frac{1}{\theta}} + |y|^{\frac{1}{\theta}}) - c\zeta^{\frac{1}{\nu}} |x-y|^{\frac{1}{\nu}}\right] \le 1.$$

Then, choosing R sufficiently large, we obtain the estimates (11). □

DEFINITION 3. *A linear continuous operator T from $S_\theta(\mathbb{R}^n)$ to itself is said to be $\theta-$regularizing if it extends to a linear continuous map from $S'_\theta(\mathbb{R}^n)$ to $S_\theta(\mathbb{R}^n)$.*

By Theorem 1 it follows that an operator T is $\theta-$regularizing if and only if its kernel belongs to $S_\theta(\mathbb{R}^{2n})$.

3. Symbolic calculus and composition formula

In this section, we develop a symbolic calculus for operators of the form (8) defined by symbols from $\Gamma^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$. From now on we will assume the more restrictive condition

$$\mu > 1, \nu > 1, \theta \geq \mu + \nu - 1. \tag{13}$$

which will be crucial for the composition of our operators.
We emphasize that the condition (13) appears also in the local theory of pseudodifferential operators in Gevrey classes and it is necessary to avoid a loss of Gevrey regularity occurring in the composition formula, see [3], [4], [13], [15], [32] where $\mu = 1, \nu = \theta$ and in the stationary phase method, see [12].
To simplify the notations, we set, for $t \geq 0$

$$Q_t = \left\{(x, \xi) \in \mathbb{R}^{2n} : \langle x\rangle < t, \langle\xi\rangle < t\right\}$$

$$Q^e_t = \mathbb{R}^{2n} \setminus Q_t.$$

DEFINITION 4. *Let $B, C > 0$. We shall denote by $\mathcal{FS}^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n}; B, C)$ the space of all formal sums $\sum_{j\geq 0} p_j(x, \xi)$ such that $p_j(x, \xi) \in C^{\infty}(\mathbb{R}^{2n})$ for all $j \geq 0$ and for every $\varepsilon > 0$*

$$\sup_{j\geq 0} \sup_{\alpha,\beta\in\mathbb{N}^n} \sup_{(x,\xi)\in Q^e_{Bj^{\mu+\nu-1}}} C^{-|\alpha|-|\beta|-2j} (\alpha!)^{-\mu} (\beta!)^{-\nu} (j!)^{-\mu-\nu+1}. \tag{14}$$

$$\cdot \langle\xi\rangle^{|\alpha|+j} \langle x\rangle^{|\beta|+j} \exp\left[-\varepsilon(|x|^{\frac{1}{\theta}} + |\xi|^{\frac{1}{\theta}})\right] \left|D^{\alpha}_{\xi} D^{\beta}_{x} p_j(x, \xi)\right| < +\infty.$$

Consider the space $FS^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n}; B, C)$ obtained from $\mathcal{FS}^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n}; B, C)$ by quotienting by the subspace

$$E = \left\{\sum_{j\geq 0} p_j(x, \xi) \in \mathcal{FS}^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n}; B, C) : supp(p_j) \subset Q_{Bj^{\mu+\nu-1}} \quad \forall j \geq 0\right\}.$$

By abuse of notation, we shall denote the elements of $FS^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n}; B, C)$ by formal sums of the form $\sum_{j\geq 0} p_j(x, \xi)$. The arguments in the following are independent of the choice of representative. We observe that $FS^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n}; B, C)$ is a Fréchet space endowed with the seminorms given by the left-hand side of (14), for $\varepsilon > 0$. We set

$$FS^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n}) = \varinjlim_{B,C\to+\infty} FS^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n}, B, C).$$

A symbol $p \in \Gamma^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$ can be identified with an element $\sum_{j\geq 0} p_j$ of $FS^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$, where $p_0 = p$, $p_j = 0 \quad \forall j \geq 1$.

DEFINITION 5. *We say that two sums* $\sum_{j\geq 0} p_j(x,\xi)$, $\sum_{j\geq 0} q_j(x,\xi)$ *from* $FS^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$ *are equivalent* $\left(\text{we write } \sum_{j\geq 0} p_j \sim \sum_{j\geq 0} q_j\right)$ *if there exist constants* $B, C > 0$ *such that for all* $\varepsilon > 0$

$$\sup_{N\in\mathbb{Z}_+}\ \sup_{\alpha,\beta\in\mathbb{N}^n}\ \sup_{(x,\xi)\in Q^e_{BN^{\mu+\nu-1}}} C^{-|\alpha|-|\beta|-2N}(\alpha!)^{-\mu}(\beta!)^{-\nu}(j!)^{-\mu-\nu+1}\langle\xi\rangle^{|\alpha|+N}\langle x\rangle^{|\beta|+N}\cdot$$

$$\cdot \exp\left[-\varepsilon(|x|^{\frac{1}{\theta}}+|\xi|^{\frac{1}{\theta}})\right]\left|D^{\alpha}_{\xi}D^{\beta}_{x}\sum_{j<N}(p_j-q_j)\right| < +\infty.$$

THEOREM 4. *Given a sum* $\sum_{j\geq 0} p_j \in FS^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$, *there exists* $p \in \Gamma^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$ *such that*

$$p \sim \sum_{j\geq 0} p_j \quad \text{in} \quad FS^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n}).$$

Proof. Let $\varphi \in C^{\infty}(\mathbb{R}^{2n})$, $0 \leq \varphi \leq 1$ such that $\varphi(x,\xi) = 0$ if $(x,\xi) \in Q_1$, $\varphi(x,\xi) = 1$ if $(x,\xi) \in Q^e_2$ and

$$\left|D^{\delta}_{x}D^{\gamma}_{\xi}\varphi(x,\xi)\right| \leq C^{|\gamma|+|\delta|+1}(\gamma!)^{\mu}(\delta!)^{\nu} \quad \forall(x,\xi)\in\mathbb{R}^{2n}. \tag{15}$$

We define:

$$\varphi_0(x,\xi) = \varphi\left(\frac{2}{R}x, \frac{2}{R}\xi\right)$$

and

$$\varphi_j(x,\xi) = \varphi\left(\frac{1}{Rj^{\mu+\nu-1}}x, \frac{1}{Rj^{\mu+\nu-1}}\xi\right), \quad j \geq 1.$$

We want to prove that if R is sufficiently large,

$$p(x,\xi) = \sum_{j\geq 0}\varphi_j(x,\xi)p_j(x,\xi) \tag{16}$$

is well defined as an element of $\Gamma^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$ and $p \sim \sum_{j\geq 0} p_j$ in $FS^{m,\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$.

First of all we observe that the sum (16) is locally finite so it defines a function $p \in C^{\infty}(\mathbb{R}^{2n})$.

Consider

$$D^{\alpha}_{\xi}D^{\beta}_{x}p(x,\xi) = \sum_{j\geq 0}\sum_{\substack{\gamma\leq\alpha\\ \delta\leq\beta}}\binom{\alpha}{\gamma}\binom{\beta}{\delta}D^{\beta-\delta}_{x}D^{\alpha-\gamma}_{\xi}p_j(x,\xi)\cdot D^{\delta}_{x}D^{\gamma}_{\xi}\varphi_j(x,\xi).$$

Choosing $R \geq B$ where B is the constant in Definition 4, we can apply the estimates (14) and obtain

$$\left|D_\xi^\alpha D_x^\beta p(x,\xi)\right| \leq C^{|\alpha|+|\beta|+1}\alpha!\beta!\langle x\rangle^{-|\beta|}\langle\xi\rangle^{-|\alpha|}\exp\left[\varepsilon(|x|^{\frac{1}{\theta}}+|\xi|^{\frac{1}{\theta}})\right]\sum_{j\geq 0}H_{j\alpha\beta}(x,\xi)$$

where

$$H_{j\alpha\beta}(x,\xi) = \sum_{\substack{\gamma\leq\alpha\\ \delta\leq\beta}}\frac{[(\alpha-\gamma)!]^{\mu-1}[(\beta-\delta)!]^{\nu-1}}{\gamma!\delta!}\cdot$$

$$\cdot C^{2j-|\gamma|-|\delta|}(j!)^{\mu+\nu-1}\langle x\rangle^{|\delta|-j}\langle\xi\rangle^{|\gamma|-j}\left|D_x^\delta D_\xi^\gamma\varphi_j(x,\xi)\right|.$$

Now the condition (15) and the fact that $D_x^\delta D_\xi^\gamma\varphi_j(x,\xi) = 0$ in $Q^e_{2Rj^{\mu+\nu-1}}$ for $(\delta,\gamma)\neq(0,0)$ imply that

$$H_{j\alpha\beta}(x,\xi) \leq C_1^{|\alpha|+|\beta|+1}(\alpha!)^{\mu-1}(\beta!)^{\nu-1}\left(\frac{C_2}{R}\right)^j$$

where C_2 is independent of R. Enlarging R, we obtain that

$$\sum_{j\geq 0}H_{j\alpha\beta}(x,\xi) \leq C_3^{|\alpha|+|\beta|+1}(\alpha!)^{\mu-1}(\beta!)^{\nu-1} \quad \forall(x,\xi)\in\mathbb{R}^{2n}$$

from which we deduce that $p\in\Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})$.
It remains to prove that $p \sim \sum_{j\geq 0} p_j$. Let us fix $N\in\mathbb{N}\setminus\{0\}$. We observe that if $(x,\xi)\in Q^e_{2RN^{\mu+\nu-1}}$, then

$$p(x,\xi) - \sum_{j<N}p_j(x,\xi) = \sum_{j\geq N}\varphi_j(x,\xi)p_j(x,\xi).$$

Thus we have

$$\left|\sum_{j\geq N}D_\xi^\alpha D_x^\beta\left[\varphi_j(x,\xi)p_j(x,\xi)\right]\right| \leq$$

$$C^{|\alpha|+|\beta|+1}\alpha!\beta!\langle x\rangle^{-|\beta|-N}\langle\xi\rangle^{-|\alpha|-N}\exp\left[\varepsilon(|x|^{\frac{1}{\theta}}+|\xi|^{\frac{1}{\theta}})\right]\sum_{j\geq N}H_{jN\alpha\beta}(x,\xi)$$

where

$$H_{jN\alpha\beta}(x,\xi) = \sum_{\substack{\gamma\leq\alpha\\ \delta\leq\beta}}\frac{[(\alpha-\gamma)!]^{\mu-1}[(\beta-\delta)!]^{\nu-1}}{\gamma!\delta!}\cdot$$

$$\cdot C^{2j-|\gamma|-|\delta|}(j!)^{\mu+\nu-1}\langle x\rangle^{|\delta|+N-j}\langle\xi\rangle^{|\gamma|+N-j}|D_x^\delta D_\xi^\gamma\varphi_j(x,\xi)|.$$

Arguing as above we can estimate

$$H_{jN\alpha\beta}(x,\xi) \leq C_4^{2N+|\alpha|+|\beta|+1}(N!)^{\mu+\nu-1}(\alpha!)^{\mu-1}(\beta!)^{\nu-1}$$

and this concludes the proof.

□

PROPOSITION 5. *Let* $p \in \Gamma^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$ *such that* $p \sim 0$. *Then the operator* P *is* $\theta-$*regularizing.*

To prove this assertion we need a preliminary result.

LEMMA 3. *Let* $M, r, \varrho, \overline{B}$ *be positive numbers,* $\varrho \geq 1$. *We define*

$$h(\lambda) = \inf_{0 \leq N \leq \overline{B}\lambda^{\frac{1}{\varrho}}} \frac{M^{rN}(N!)^r}{\lambda^{\frac{rN}{\varrho}}}, \quad \lambda \in \mathbb{R}^+.$$

Then there exist positive constants C, τ *such that*

$$h(\lambda) \leq Ce^{-\tau\lambda^{\frac{1}{\varrho}}}, \quad \lambda \in \mathbb{R}^+.$$

Proof. See Lemma 3.2.4 in [27] for the proof.

□

Proof of Proposition 5. It is sufficient to prove that if $p \sim 0$, then the kernel of P

$$K(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} p(x, \xi) d\xi$$

belongs to $S_\theta(\mathbb{R}^{2n})$. By Definition 5, there exist $B, C > 0$ and for all $\varepsilon > 0$ there exists a positive constant C_ε such that, for every $(x, \xi) \in \mathbb{R}^{2n}$

$$\left|D^\alpha_\xi D^\beta_x p(x, \xi)\right| \leq C_\varepsilon C^{|\alpha|+|\beta|}(\alpha!)^\mu(\beta!)^\nu \langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|} \exp\left[\varepsilon(|x|^{\frac{1}{\theta}} + |\xi|^{\frac{1}{\theta}})\right]\cdot$$

$$\cdot \inf_{0 \leq N \leq (B^{-1}\langle\xi\rangle\langle x\rangle)^{\frac{1}{\mu+\nu-1}}} \frac{C^{2N}(N!)^{\mu+\nu-1}}{\langle\xi\rangle^N\langle x\rangle^N}.$$

Applying Lemma 3 with $\varrho = r = \mu + \nu - 1$, $\lambda = \langle\xi\rangle\langle x\rangle$ and taking into account the condition $\theta \geq \mu+\nu-1$, and the obvious estimate $|x|^{\frac{1}{\theta}} + |\xi|^{\frac{1}{\theta}} \leq c\langle\xi\rangle^{\frac{1}{\theta}}\langle x\rangle^{\frac{1}{\theta}}$, we obtain that for all $\varepsilon > 0$

$$\left|D^\alpha_\xi D^\beta_x p(x, \xi)\right| \leq C'_\varepsilon C^{|\alpha|+|\beta|}(\alpha!)^\mu(\beta!)^\nu \exp\left[-(\tau - \varepsilon)(|x|^{\frac{1}{\theta}} + |\xi|^{\frac{1}{\theta}})\right] \tag{17}$$

for a certain positive τ. For $0 < \varepsilon < \tau$, it follows that $p \in S_\theta(\mathbb{R}^{2n})$. By Theorem 3, it is sufficient to show that there exists $k \in (0, 1)$ such that

$$\sup_{(x,y)\in\mathbb{R}^{2n}\setminus\Omega_k} C^{-|\alpha|-|\gamma|}(\alpha!\gamma!)^{-\theta} \exp\left[a(|x|^{\frac{1}{\theta}} + |y|^{\frac{1}{\theta}})\right] \left|D^\alpha_x D^\gamma_y K(x, y)\right| < +\infty$$

for some positive constants a, C. From the estimates (17) we obtain, for $\tau' < \tau$,

$$\left|D^\alpha_x D^\gamma_y K(x, y)\right| \leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} C^{|\alpha|-|\beta|}\left[(\alpha - \beta)!\right]^\nu e^{-\tau'|x|^{\frac{1}{\theta}}} \int_{\mathbb{R}^n} |\xi|^{|\beta|+|\gamma|}e^{-\tau'|\xi|^{\frac{1}{\theta}}}d\xi.$$

Now, for every $\varepsilon > 0$ there exists a positive constant $C = C(\varepsilon)$ such that

$$|\xi|^{|\beta|+|\gamma|} \leq C^{|\beta|+|\gamma|+1}(\beta!\gamma!)^{\theta} e^{\varepsilon|\xi|^{\frac{1}{\theta}}}$$

Furthermore, we observe that there exists $C'_k > 0$ such that in $\mathbb{R}^{2n} \setminus \Omega_k$

$$-\frac{\tau'}{2}|x|^{\frac{1}{\theta}} \leq \frac{\tau'}{2}k^{\frac{1}{\theta}} + \frac{\tau'}{2}k^{\frac{1}{\theta}}|x|^{\frac{1}{\theta}} - C'_k|y|^{\frac{1}{\theta}}.$$

So we can conclude that there exist $a_k > 0$ for which

$$\sup_{\mathbb{R}^{2n}\setminus\Omega_k} \exp\left[a_k(|x|^{\frac{1}{\theta}} + |y|^{\frac{1}{\theta}})\right]\left|D_x^{\alpha}D_y^{\gamma}K(x,y)\right| \leq C^{|\alpha|+|\gamma|+1}(\alpha!\gamma!)^{\theta}$$

and this concludes the proof. □

Let us give now the main results of this section.

PROPOSITION 6. *Let $P = p(x, D) \in OPS^{\infty}_{\mu\nu\theta}(\mathbb{R}^n)$ and let ${}^tP$ be its transpose defined by*

$$\langle {}^tPu, v\rangle = \langle u, Pv\rangle, \quad u \in S'_{\theta}(\mathbb{R}^n), v \in S_{\theta}(\mathbb{R}^n). \tag{18}$$

Then, ${}^tP = Q + R$, where R is a $\theta-$regularizing operator and $Q = q(x, D)$ is in $OPS^{\infty}_{\mu\nu\theta}(\mathbb{R}^n)$ with

$$q(x,\xi) \sim \sum_{j\geq 0}\sum_{|\alpha|=j}(\alpha!)^{-1}\partial_{\xi}^{\alpha}D_x^{\alpha}p(x,-\xi)$$

in $FS^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$.

THEOREM 5. *Let $P = p(x, D), Q = q(x, D) \in OPS^{\infty}_{\mu\nu\theta}(\mathbb{R}^n)$. Then $PQ = T + R$ where R is $\theta-$regularizing and $T = t(x,\xi) \in OPS^{\infty}_{\mu\nu\theta}(\mathbb{R}^n)$ with*

$$t(x,\xi) \sim \sum_{j\geq 0}\sum_{|\alpha|=j}(\alpha!)^{-1}\partial_{\xi}^{\alpha}p(x,\xi)D_x^{\alpha}q(x,\xi)$$

in $FS^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$.

To prove these results it is convenient to enlarge the class of our operators by considering more general classes of symbols.

Let μ, ν, θ be real numbers satisfying the condition (13).

DEFINITION 6. *We shall denote by $\Pi^{\infty}_{\mu\nu\theta}(\mathbb{R}^{3n}; C)$ the Fréchet space of all functions $a(x, y, \xi) \in C^{\infty}(\mathbb{R}^{3n})$ such that for every $\varepsilon > 0$*

$$\sup_{\alpha,\beta\in\mathbb{N}^n}\ \sup_{(x,y,\xi)\in\mathbb{R}^{3n}} C^{-|\alpha|-|\beta|-|\gamma|}(\alpha!)^{-\mu}(\beta!\gamma!)^{-\nu}\langle\xi\rangle^{|\alpha|}\left(|x|^2+|y|^2\right)^{\frac{1}{2}|\beta+\gamma|}.$$

$$\cdot\langle x-y\rangle^{-|\beta+\gamma|}\exp\left[-\varepsilon(|x|^{\frac{1}{\theta}}+|y|^{\frac{1}{\theta}}+|\xi|^{\frac{1}{\theta}})\right]\left|D_\xi^\alpha D_x^\beta D_y^\gamma a(x,y,\xi)\right|<+\infty.$$

We set

$$\Pi^\infty_{\mu\nu\theta}(\mathbb{R}^{3n})=\varinjlim_{C\to+\infty}\Pi^\infty_{\mu\nu\theta}(\mathbb{R}^{3n},C).$$

It is immediate to verify the following relations:
i) if $a(x,y,\xi)\in\Pi^\infty_{\mu\nu\theta}(\mathbb{R}^{3n})$, then the function $(x,\xi)\to a(x,x,\xi)$ belongs to $\Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})$.
ii) if $p(x,\xi)\in\Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})$, then $p((1-\tau)x+\tau y,\xi)\in\Pi^\infty_{\mu\nu\theta}(\mathbb{R}^{3n})$ for every $\tau\in[0,1]$.

Given $a\in\Pi^\infty_{\mu\nu\theta}(\mathbb{R}^{3n})$, we can associate to a a pseudodifferential operator defined by

$$Au(x)=(2\pi)^{-n}\int_{\mathbb{R}^{2n}}e^{i\langle x-y,\xi\rangle}a(x,y,\xi)u(y)dyd\xi,\quad u\in S_\theta(\mathbb{R}^n). \tag{19}$$

We remark that the integral written above is not absolutely convergent in general. Let us give a more precise meaning to (19).

LEMMA 4. *Let $\chi\in S_\theta^\theta(\mathbb{R}^n)$, $\chi(0)=1$. Then, for every $x\in\mathbb{R}^n$ and $u\in S_\theta(\mathbb{R}^n)$, the function*

$$I_{\chi,\delta}(x)=(2\pi)^{-n}\int_{\mathbb{R}^{2n}}e^{i\langle x-y,\xi\rangle}a(x,y,\xi)\chi(\delta\xi)u(y)dyd\xi \tag{20}$$

has a limit when $\delta\to 0^+$ and this limit is independent of χ.

Proof. We remark that for every positive ζ, η the following relations hold:

$$\frac{1}{m_{2\theta,\zeta}(x)}\sum_{p=0}^\infty\frac{\zeta^p}{(p!)^{2\theta}}(1-\Delta_\xi)^p e^{i\langle x,\xi\rangle}=e^{i\langle x,\xi\rangle} \tag{21}$$

$$\frac{1}{m_{2\theta,\eta}(\xi)}\sum_{q=0}^\infty\frac{\eta^q}{(q!)^{2\theta}}(1-\Delta_y)^q e^{i\langle x-y,\xi\rangle}=e^{i\langle x-y,\xi\rangle}. \tag{22}$$

Substituting (21) in (20) and integrating by parts, we obtain

$$I_{\chi,\delta}(x)=\frac{(2\pi)^{-n}}{m_{2\theta,\zeta}(x)}\sum_{p=0}^\infty\frac{\zeta^p}{(p!)^{2\theta}}\cdot$$

$$\cdot\int_{\mathbb{R}^{2n}}e^{i\langle x,\xi\rangle}(1-\Delta_\xi)^p\left[e^{-i\langle y,\xi\rangle}a(x,y,\xi)\chi(\delta\xi)\right]u(y)dyd\xi=$$

$$\frac{(2\pi)^{-n}}{m_{2\theta,\zeta}(x)}\sum_{p=0}^\infty\frac{\zeta^p}{(p!)^{2\theta}}\int_{\mathbb{R}^{2n}}e^{i\langle x-y,\xi\rangle}\lambda_{p,\delta}(x,y,\xi)dyd\xi$$

where

$$\lambda_{p,\delta}(x,y,\xi) = \sum_{r=0}^{p} \binom{p}{r} (-1)^r \sum_{|\alpha|=r} \frac{r!}{\alpha_1!\ldots\alpha_n!} \cdot$$

$$\cdot \sum_{\beta \leq 2\alpha} \binom{2\alpha}{\beta} (-iy)^{\beta} \partial_{\xi}^{2\alpha-\beta} \left[a(x,y,\xi)\chi(\delta\xi)\right] u(y)$$

Applying (22) we obtain that

$$I_{\chi,\delta}(x) = \frac{(2\pi)^{-n}}{m_{2\theta,\zeta}(x)} \sum_{p=0}^{\infty} \frac{\zeta^p}{(p!)^{2\theta}} \sum_{q=0}^{\infty} \frac{\eta^q}{(q!)^{2\theta}} \cdot$$

$$\cdot \int_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} \frac{1}{m_{2\theta,\eta}(\xi)} (1-\Delta_y)^q \lambda_{p,\delta}(x,y,\xi)dyd\xi.$$

The hypotheses on a, u, χ imply that there exist $C_1, C_2, C_3 > 0$ and for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left|(1-\Delta_y)^q \lambda_{p,\delta}(x,y,\xi)\right| \leq C_\varepsilon C_1^p C_2^q (p!q!)^{2\theta} e^{\varepsilon|x|^{\frac{1}{\theta}}} e^{-(C_3-\varepsilon)|y|^{\frac{1}{\theta}}} e^{\varepsilon|\xi|^{\frac{1}{\theta}}}.$$

Hence, choosing $\zeta < \frac{1}{C_1}, \eta < \frac{1}{C_2}$ and ε sufficiently small, we can re-arrange the sums under the integral sign and obtain an estimate independent of δ. By Lebesgue's dominated convergence theorem, it turns out that

$$\lim_{\delta\to 0^+} I_{\chi,\delta}(x) = \frac{(2\pi)^{-n}}{m_{2\theta,\zeta}(x)} \int_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{\zeta^p\eta^q}{(p!q!)^{2\theta}} \sum_{r=0}^{p} \binom{p}{r} (-1)^r \sum_{|\alpha|=r} \frac{r!}{\alpha_1!\ldots\alpha_n!} \cdot$$

$$\cdot \sum_{\beta\leq 2\alpha} \binom{2\alpha}{\beta} (1-\Delta_y)^q \partial_{\xi}^{2\alpha-\beta} \left[(-iy)^{\beta} a(x,y,\xi)u(y)\right] dyd\xi.$$

□

From Lemma 4 we deduce the following natural definition.

DEFINITION 7. *Given* $a \in \Pi^{\infty}_{\mu\nu\theta}(\mathbb{R}^{3n})$, *we define, for every* $u \in S_\theta(\mathbb{R}^n)$

$$Au(x) = (2\pi)^{-n} \lim_{\delta\to 0^+} \int_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x,y,\xi)\chi(\delta\xi)u(y)dyd\xi \tag{23}$$

with $\chi \in S_\theta(\mathbb{R}^n)$, $\chi(0) = 1$.

We denote by $\overline{OPS}^{\infty}_{\mu\nu\theta}(\mathbb{R}^n)$ the space of all operators of the form (19) defined by an amplitude of $\Pi^{\infty}_{\mu\nu\theta}(\mathbb{R}^{3n})$. Theorems 2 and 3 extend without relevant changes in the proofs to these operators; details are left to the reader.
The next theorem states a relation between operators (19) and the elements of $OPS^{\infty}_{\mu\nu\theta}(\mathbb{R}^n)$.

THEOREM 6. *Let A be an operator defined by an amplitude $a \in \Pi^{\infty}_{\mu\nu\theta}(\mathbb{R}^{3n})$. Then we may write $A = P + R$, where R is a $\theta-$regularizing operator and $P = p(x, D) \in OPS^{\infty}_{\mu\nu\theta}(\mathbb{R}^n)$, with $p \sim \sum_{j\geq 0} p_j$, where*

$$p_j(x,\xi) = \sum_{|\alpha|=j} (\alpha!)^{-1} \partial^{\alpha}_{\xi} D^{\alpha}_{y} a(x,y,\xi)_{|y=x}\,. \tag{24}$$

Proof. Let $\chi \in C^{\infty}(\mathbb{R}^{2n})$ such that

$$\chi(x,y) = \begin{cases} 1 & \text{if} \quad |x-y| \leq \frac{1}{4}\langle x\rangle \\ 0 & \text{if} \quad |x-y| \geq \frac{1}{2}\langle x\rangle \end{cases} \tag{25}$$

and

$$\left|D^{\beta}_{x} D^{\gamma}_{y} \chi(x,y)\right| \leq C^{|\beta|+|\gamma|+1}(\beta!\gamma!)^{\nu}$$

for all $\beta, \gamma \in \mathbb{N}^n$ and $(x,y) \in \mathbb{R}^{2n}$. We may decompose a as the sum of two elements of $\Pi^{\infty}_{\mu\nu\theta}(\mathbb{R}^{3n})$ writing

$$a(x,y,\xi) = \chi(x,y)a(x,y,\xi) + (1-\chi(x,y))a(x,y,\xi).$$

Furthermore, it follows from Theorem 3 that $(1-\chi(x,y))a(x,y,\xi)$ defines a $\theta-$regularizing operator. Hence, eventually perturbing A with a $\theta-$regularizing operator, we can assume that $a(x,y,\xi)$ is supported on $\left(\mathbb{R}^{2n} \setminus \Omega_{\frac{1}{2}}\right) \times \mathbb{R}^n$, where $\Omega_{\frac{1}{2}}$ is defined as in Theorem 3.

It is trivial to verify that $\sum_{j\geq 0} p_j$ defined by (24) belongs to $FS^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$. By Theorem 4 we can find a sequence $\varphi_j \in C^{\infty}(\mathbb{R}^{2n})$ depending on a parameter R such that

$$p(x,\xi) = \sum_{j\geq 0} \varphi_j(x,\xi) p_j(x,\xi)$$

defines an element of $\Gamma^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$ for R large and $p \sim \sum_{j\geq 0} p_j$ in $FS^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$. Let $P = p(x,D)$. To prove the Theorem it is sufficient to show that the kernel $K(x,y)$ of $A-P$ is in $S_{\theta}(\mathbb{R}^{2n})$.

We can write

$$a(x,y,\xi) - p(x,\xi) = (1-\varphi_0(x,\xi))a(x,y,\xi)$$
$$+ \sum_{N=0}^{\infty} (\varphi_N - \varphi_{N+1})(x,\xi) \left(a(x,y,\xi) - \sum_{j\leq N} p_j(x,\xi) \right).$$

Consequently,

$$K(x,y) = \overline{K}(x,y) + \sum_{N=0}^{\infty} K_N(x,y) \tag{26}$$

where

$$\overline{K}(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle}(1-\varphi_0(x,\xi))a(x,y,\xi)d\xi,$$

$$K_N(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle}(\varphi_N - \varphi_{N+1})(x,\xi) \left(a(x,y,\xi) - \sum_{j\le N} p_j(x,\xi) \right) d\xi.$$

A power expansion in the second argument gives for $N = 1, 2, ...$

$$a(x,y,\xi) = \sum_{|\alpha|\le N} (\alpha!)^{-1}(y-x)^\alpha \partial_y^\alpha a(x,x,\xi) + \sum_{|\alpha|=N+1} (\alpha!)^{-1}(y-x)^\alpha w_\alpha(x,y,\xi)$$

with

$$w_\alpha(x,y,\xi) = (N+1)\int_0^1 \partial_y^\alpha a(x, x+t(y-x),\xi)(1-t)^N dt.$$

In view of our definition of the $p_j(x,\xi)$, integrating by parts, we obtain that

$$K_N(x,y) = W_N(x,y) + (2\pi)^{-n} \sum_{1\le|\alpha|\le N} \sum_{0\ne\beta\le\alpha} \frac{1}{\beta!(\alpha-\beta)!}\cdot$$

$$\cdot \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} D_\xi^\beta(\varphi_N - \varphi_{N+1})(x,\xi)(D_\xi^{\alpha-\beta}\partial_y^\alpha a)(x,x,\xi)d\xi,$$

where

$$W_N(x,y) = (2\pi)^{-n} \sum_{|\alpha|=N+1} \sum_{\beta\le\alpha} \frac{1}{\beta!(\alpha-\beta)!}\cdot$$

$$\cdot \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} D_\xi^\beta(\varphi_N - \varphi_{N+1})(x,\xi) D_\xi^{\alpha-\beta} w_\alpha(x,y,\xi)d\xi$$

for all $N = 1, 2, ...$
Using an absolute convergence argument, we may re-arrange the sums under the integral sign. We also observe that

$$\sum_{N\ge|\alpha|} D_\xi^\beta(\varphi_N - \varphi_{N+1})(x,\xi) = D_\xi^\beta \varphi_{|\alpha|}(x,\xi).$$

Then we have

$$K = \overline{K} + \sum_{\alpha\ne 0} I_\alpha + \sum_{N=0}^{\infty} W_N$$

where

$$I_\alpha(x,y) = (2\pi)^{-n} \sum_{0\ne\beta\le\alpha} \frac{1}{\beta!(\alpha-\beta)!} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} D_\xi^\beta \varphi_{|\alpha|}(x,\xi) D_\xi^{\alpha-\beta}\partial_y^\alpha a(x,x,\xi)d\xi$$

and we may write $W_0(x, y)$ for $K_0(x, y)$. To conclude the proof, we want to show that $\overline{K}, \sum\limits_{\alpha\neq 0} I_\alpha, \sum\limits_{N=0}^{\infty} W_N \in S_\theta(\mathbb{R}^{2n})$. First of all, we have to estimate the derivatives of $\overline{K}$ for $(x, \xi) \in supp(1 - \varphi_0(x, \xi))$, i.e. for $\langle x\rangle \leq R$, $\langle \xi\rangle \leq R$. We have

$$\left|x^k y^h D_x^\delta D_y^\gamma \overline{K}(x, y)\right| = (2\pi)^{-n} \left| x^k y^h \sum_{\substack{\gamma_1+\gamma_2=\gamma \\ \delta_1+\delta_2+\delta_3=\delta}} \frac{\gamma!\delta!}{\gamma_1!\gamma_2!\delta_1!\delta_2!\delta_3!} \cdot \right.$$

$$\left. \cdot (-1)^{|\gamma_1|} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \xi^{\gamma_1+\delta_1} D_x^{\delta_2} D_y^{\gamma_2} a(x, y, \xi) D_x^{\delta_3}(1 - \varphi_0(x, \xi)) d\xi \right| \leq$$

$$|x|^{|k|}|y|^{|h|} \sum_{\substack{\gamma_1+\gamma_2=\gamma \\ \delta_1+\delta_2+\delta_3=\delta}} \frac{\gamma!\delta!}{\gamma_1!\gamma_2!\delta_1!\delta_2!\delta_3!} C^{|\gamma_2|+|\delta_2|+|\delta_3|} (\gamma_2!\delta_2!\delta_3!)^\nu \langle x - y\rangle^{|\gamma_2+\delta_2|}.$$

$$\cdot \exp\left[\varepsilon(|x|^{\frac{1}{\theta}} + |y|^{\frac{1}{\theta}})\right] \int_{\langle\xi\rangle\leq R} \langle\xi\rangle^{|\gamma_1+\delta_1|} e^{\varepsilon\langle\xi\rangle^{\frac{1}{\theta}}} d\xi.$$

Now, $a(x, y, \xi)$ is supported on $\left(\mathbb{R}^{2n} \setminus \Omega_{\frac{1}{2}}\right) \times \mathbb{R}^n$ and in this region $|y| \leq \frac{3}{2}\langle x\rangle$ so, there exist constants $C_1, C_2 > 0$ depending on R such that

$$\sup_{(x,y)\in\mathbb{R}^{2n}} \left|x^k y^h D_x^\delta D_y^\gamma \overline{K}(x, y)\right| \leq C_1 R^{|k|+|h|} C_2^{|\gamma|+|\delta|} (\gamma!\delta!)^\theta,$$

so $\overline{K} \in S_\theta(\mathbb{R}^{2n})$. Consider now

$$x^k y^h D_x^\delta D_y^\gamma I_\alpha(x, y) = (2\pi)^{-n} \sum_{0\neq\beta\leq\alpha} \frac{1}{\beta!(\alpha-\beta)!} \sum_{\delta_1+\delta_2+\delta_3=\delta} \frac{\delta!}{\delta_1!\delta_2!\delta_3!} (-1)^{|\gamma|} x^k y^h \cdot$$

$$\cdot \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \xi^{\gamma+\delta_1} D_x^{\delta_2} D_\xi^\beta \varphi_{|\alpha|}(x, \xi) D_\xi^{\alpha-\beta} D_x^{\delta_3} \partial_y^\alpha a(x, x, \xi) d\xi =$$

$$(2\pi)^{-n} \sum_{0\neq\beta\leq\alpha} \frac{1}{\beta!(\alpha-\beta)!} \sum_{\delta_1+\delta_2+\delta_3=\delta} \frac{\delta!}{\delta_1!\delta_2!\delta_3!} (-1)^{|\gamma|} (-i)^h x^k \cdot$$

$$\cdot \int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} \partial_\xi^h \left[e^{i\langle x,\xi\rangle} \xi^{\gamma+\delta_1} D_x^{\delta_2} D_\xi^\beta \varphi_{|\alpha|}(x, \xi) D_\xi^{\alpha-\beta} D_x^{\delta_3} \partial_y^\alpha a(x, x, \xi)\right] d\xi.$$

We need the estimates for $(x, \xi) \in suppD_\xi^\beta \varphi_{|\alpha|}(x, \xi) \subset \overline{Q}_{2R|\alpha|^{\mu+\nu-1}} \setminus Q_{R|\alpha|^{\mu+\nu-1}}$. Then, there exist $C_1, C_2, C_3 > 0$ such that

$$\left|x^k y^h D_x^\delta D_y^\gamma I_\alpha(x, y)\right| \leq C_1^{|h|+|k|+1} C_2^{|\alpha|} C_3^{|\gamma|+|\delta|} (k!h!\gamma!\delta!)^\theta (\alpha!)^\nu \langle x\rangle^{-|\alpha|}.$$

$$\cdot \sum_{0\neq\beta\leq\alpha} (\beta!)^{\mu-1} [(\alpha-\beta)!]^{\mu-1} \left(\frac{1}{R|\alpha|^{\mu+\nu-1}}\right)^{|\beta|} \int_{\langle\xi\rangle\leq 2R|\alpha|^{\mu+\nu-1}} \langle\xi\rangle^{-|\alpha-\beta|} d\xi$$

with C_2 independent of R. Now, if $(x,\xi) \in \overline{Q}_{2R|\alpha|^{\mu+\nu-1}} \setminus Q_{R|\alpha|^{\mu+\nu-1}}$, we have that

$$C_2^{|\alpha|}(\alpha!)^{\nu}\langle x\rangle^{-|\alpha|}\sum_{0\neq\beta\leq\alpha}(\beta!)^{\mu-1}[(\alpha-\beta)!]^{\mu-1}\left(\frac{1}{R|\alpha|^{\mu+\nu-1}}\right)^{|\beta|}\cdot$$

$$\cdot\int_{\langle\xi\rangle\leq 2R|\alpha|^{\mu+\nu-1}}\langle\xi\rangle^{-|\alpha-\beta|}d\xi \leq \left(\frac{C_4}{R}\right)^{|\alpha|}$$

with C_4 independent of R. Finally, we conclude that

$$\sup_{(x,y)\in\mathbb{R}^{2n}}\left|x^k y^h D_x^{\delta} D_y^{\gamma} I_{\alpha}(x,y)\right| \leq C^{|h|+|k|+1}C_2^{|\gamma|+|\delta|}(k!h!\gamma!\delta!)^{\theta}\left(\frac{C_4}{R}\right)^{|\alpha|}.$$

Choosing $R > C_4$, we obtain that $\sum\limits_{\alpha\neq 0} I_{\alpha} \in S_{\theta}(\mathbb{R}^{2n})$.

Arguing as for I_{α}, we can prove that also

$$\sup_{(x,y)\in\mathbb{R}^{2n}}\left|x^k y^h D_x^{\delta} D_y^{\gamma} W_N(x,y)\right| \leq C_1^{|h|+|k|+1}C_2^{|\gamma|+|\delta|}(h!k!\gamma!\delta!)^{\theta}\left(\frac{C}{R}\right)^{N}$$

with C independent of R, which gives, for R sufficiently large, that $\sum\limits_{N=0}^{\infty} W_N$ is in $S_{\theta}(\mathbb{R}^{2n})$. This concludes the proof.

□

Proof of Proposition 6. By the formula (18), ${}^tP$ is defined by

$${}^tPu(x) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle}p(y,-\xi)u(y)dyd\xi, \quad u \in S_{\theta}(\mathbb{R}^n).$$

Thus, ${}^tP \in \overline{OPS}^{\infty}_{\mu\nu\theta}(\mathbb{R}^n)$ with amplitude $p(y,-\xi)$. By Theorem 6, ${}^tP = Q + R$ where R is $\theta-$regularizing and $Q = q(x,D) \in OPS^{\infty}_{\mu\nu\theta}(\mathbb{R}^n)$, with

$$q(x,\xi) \sim \sum_{j\geq 0}\sum_{|\alpha|=j}(\alpha!)^{-1}\partial_{\xi}^{\alpha}D_x^{\alpha}p(x,-\xi).$$

□

Proof of Theorem 5. We can write $Q = {}^t({}^tQ)$. Then, by Theorem 6 and Proposition 6, $Q = Q_1 + R_1$, where R_1 is $\theta-$regularizing and

$$Q_1u(x) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle}q_1(y,\xi)u(y)dyd\xi \tag{27}$$

with $q_1(y,\xi) \in \Gamma^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$, $q_1(y,\xi) \sim \sum\limits_{\alpha}(\alpha!)^{-1}\partial_{\xi}^{\alpha}D_y^{\alpha}q(y,-\xi)$. From (27) it follows that

$$\widehat{Q_1u}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle}q_1(y,\xi)u(y)dy, \quad u \in S_{\theta}(\mathbb{R}^n)$$

from which we deduce that

$$PQu(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} p(x,\xi) q_1(y,\xi) u(y) dy d\xi + PR_1 u(x).$$

We observe that $p(x,\xi)q_1(y,\xi) \in \Pi^{\infty}_{\mu\nu\theta}(\mathbb{R}^{3n})$, then we may apply Theorem 6 and obtain that

$$PQu(x) = Tu(x) + Ru(x)$$

wher R is $\theta-$regularizing and $T = t(x, D) \in OPS^{\infty}_{\mu\nu\theta}(\mathbb{R}^n)$ with

$$t(x,\xi) \sim \sum_{\alpha} (\alpha!)^{-1} \partial^{\alpha}_{\xi} p(x,\xi) D^{\alpha}_{x} q(x,\xi).$$

$\square$

REMARK 1. Definitions analogous to 4 and 5 can be given for formal sums of elements of $\Gamma^{m_1,m_2}_{\mu\nu}(\mathbb{R}^{2n})$. Furthermore, under the condition (13), all the results of this section can be extended to the corresponding operators.

4. Construction of a parametrix for the problem (2)

Let μ, ν be real numbers such that $\mu > 1$, $\nu > 1$ and consider the operator in (3) where we assume that $a_j(t, x, D_x)$, $j = 1, ..., m$ are pseudodifferential operators of the form (8) with symbols $a_j(t,x,\xi) \in C\left([0,T], \Gamma^{pj,qj}_{\mu\nu}(\mathbb{R}^{2n})\right)$, for some nonnegative p, q such that $p + q \in [\,0, 1\,[$.

We want to construct a parametrix for the problem (2).We start by considering the homogeneous equation. Namely, let θ be a real number such that $\theta \geq \mu + \nu - 1$ and $p + q \in \left[0, \frac{1}{\theta}\right[$. We want to find an operator $E(t,s) \in OPS^{\infty}_{\mu\nu\theta}(\mathbb{R}^n)$, t, s in $[0, T]$ such that

$$\begin{cases} P(t,x,D_t,D_x)E(t,s) = R(t,s) & (t,s) \in [0,T]^2, x \in \mathbb{R}^n \\ D^j_t E(s,s) = 0 & j = 0, ..., m-2 \\ D^{m-1}_t E(s,s) = iI & \end{cases} \tag{28}$$

where I is the identity operator and $R(t,s)$ has its kernel in $C\left([0,T], S_\theta(\mathbb{R}^{2n})\right)$.

In order to construct the parametrix above, we want to apply the results obtained in Sections 2, 3. To be more precise, we need to reformulate these results for operators with symbols depending with a certain regularity on some parameters. The proofs follow the same arguments of the previous sections.

Denote by I a compact pluri-interval of $\mathbb{R}^d$.

THEOREM 7. *Let* $a \in C(I, \Pi^{\infty}_{\mu\nu\theta}(\mathbb{R}^{3n}))$. *Then, the operator*

$$A(t)u(s,\cdot)(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(t,x,y,\xi) u(s,y) dy d\xi$$

defines a linear continuous map from $C\left(I, S_\theta(\mathbb{R}^n)\right)$ *to* $C\left(I^2, S_\theta(\mathbb{R}^n)\right)$ *which extends to a linear continuous map from* $C\left(I, S_\theta'(\mathbb{R}^n)\right)$ *to* $C\left(I^2, S_\theta'(\mathbb{R}^n)\right)$. *Furthermore, if* $a \in C^k(I, \Pi^\infty_{\mu\nu\theta}(\mathbb{R}^{3n}))$, $k \in \mathbb{N}$, *then*

$$D_t^k A(t)u(s,\cdot)(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} D_t^k a(t,x,y,\xi)u(s,y)dyd\xi$$

for all $x \in \mathbb{R}^n$, $(t,s) \in I^2$.

PROPOSITION 7. *i) Let* $p_j \in C\left(I, \Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})\right)$, $j \geq 0$ *such that* $\sum\limits_{j\geq 0} p_j$ *belongs to* $\mathcal{B}\left(I, FS^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})\right)$, *set of the bounded functions from* I *to* $FS^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})$. *Then, there exists* p *in* $C\left(I, \Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})\right)$ *such that* $p \sim \sum\limits_{j\geq 0} p_j$ *in* $FS^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})$ *uniformly with respect to* $t \in I$.
ii) Let $p(t) \in C\left(I, \Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})\right)$, $p(t) \sim 0$ *uniformly with respect to* $t \in I$ *in* $FS^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})$. *Then the operator* $P(t)$ *has its kernel in* $C\left(I, S_\theta(\mathbb{R}^{2n})\right)$.

PROPOSITION 8. *Let* $p(t) \in C\left(I, \Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})\right)$. *Then there exists* $Q(t) = q(t,x,D_x)$ *in* $OPS^\infty_{\mu\nu\theta}(\mathbb{R}^n), t \in I$, *with symbol* $q(t,x,\xi) \sim \sum\limits_{j\geq 0}\sum\limits_{|\alpha|=j} (\alpha!)^{-1}\partial_\xi^\alpha D_x^\alpha p(t,x,-\xi)$ *in* $FS^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})$ *uniformly with respect to* $t \in I$, *such that* ${}^tP = Q + R$, *where* R *has its kernel in* $C\left(I, S_\theta(\mathbb{R}^{2n})\right)$.

THEOREM 8. *Let* $P(t) = p(t,x,D)$, $Q(t,s) = q(t,s;x,D) \in OPS^\infty_{\mu\nu\theta}(\mathbb{R}^n)$ *for* $t,s \in I$, *such that* $p(t,x,\xi) \sim \sum\limits_{j\geq 0} p_j(t;x,\xi)$, $q(t,s;x,\xi) \sim \sum\limits_{j\geq 0} q_j(t,s;x,\xi)$ *in* $FS^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})$ *uniformly with respect to* $t,s \in I$. *Assume that* $p_j \in C\left(I, \Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})\right)$, $q_j \in C\left(I^2, \Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})\right)$. *Then* $PQ(t,s) = B(t,s) + R(t,s)$, *where* R *has its kernel in* $C\left(I^2, S_\theta^\theta(\mathbb{R}^{2n}\right)$ *and* $B(t,s) = b(t,s;x,D)$ *is in* $OPS^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})$ *with*

$$b(t,s;x,\xi) \sim \sum_{j\geq 0}\sum_{h+k+|\alpha|=j} (\alpha!)^{-1}\partial_\xi^\alpha p_h(t,x,\xi) D_x^\alpha q_k(t,s;x,\xi)$$

in $FS^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})$ *uniformly with respect to* $(t,s) \in I^2$.

Following a standard argument based on Theorem 8, we can now construct the symbol $e(t,s;x,\xi)$ of $E(t,s)$ starting from its asymptotic expansion. Then we will prove the regularity of e, namely $D_t^k e \in C\left([0,T], \Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})\right)$ for all $s \in [0,T]$, $k = 0, \ldots, m$ with the aid of Proposition 7.
For every $(x,\xi) \in \mathbb{R}^{2n}$, let $e_h(t,s;x,\xi), h \geq 0$ be the solutions of the following

Cauchy problems for ordinary differential equations

$$(29)\qquad \begin{cases} \left(D_t^m + \sum\limits_{j=1}^m a_j(t,x,\xi)D_t^{m-j}\right)e_0 = 0 & (t,s)\in[0,T]^2 \\ D_t^j e_0(s,s;x,\xi) = 0 & j=0,\dots,m-2 \\ D_t^{m-1}e_0(s,s;x,\xi) = i & \end{cases}$$

and for $h \geq 1$,

$$(30)\qquad \begin{cases} \left(D_t^m + \sum\limits_{j=1}^m a_j(t,x,\xi)D_t^{m-j}\right)e_h = d_h(t,s;x,\xi) & (t,s)\in[0,T]^2 \\ D_t^j e_h(s,s;x,\xi) = 0 & j=0,\dots,m-1 \end{cases}$$

where

$$d_h(t,s;x,\xi) = -\sum_{j=1}^m\sum_{l=1}^h\sum_{|\alpha|=l}(\alpha!)^{-1}\partial_\xi^\alpha a_j(t,x,\xi)D_x^\alpha D_t^{m-j}e_{h-l}(t,s;x,\xi).$$

We want to prove that

$$(31)\qquad D_t^k e_h \in C\left([0,T]^2, \Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})\right) \qquad h\geq 0,\quad k=0,\dots,m$$

and

$$(32)\qquad \sum_{h\geq 0} D_t^k e_h \in \mathcal{B}\left([0,T]^2, FS^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})\right) \qquad k=0,\dots,m.$$

LEMMA 5. *Let the functions a_j belong to $C\left([0,T], \Gamma^{pj,qj}_{\mu\nu}(\mathbb{R}^{2n})\right)$ and let e_0 be defined by* (29)*. Then, there exist positive constants C, c such that*

$$(33)\qquad \left|D_\xi^\alpha D_x^\beta D_t^k e_0(t,s;x,\xi)\right| \leq C^{|\alpha|+|\beta|}(\alpha!)^\mu(\beta!)^\nu\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|}\cdot$$

$$\cdot\exp\left[c\langle\xi\rangle^p\langle x\rangle^q|t-s|\right]\sum_{i=\min(|\alpha+\beta|,1)}^{|\alpha+\beta|m}\langle\xi\rangle^{pi}\langle x\rangle^{qi}\frac{|t-s|^{i+m-1-k}}{(i+m-1-k)!} \qquad k=0,\dots,m-1,$$

$$(34)\qquad \left|D_\xi^\alpha D_x^\beta D_t^m e_0(t,s;x,\xi)\right| \leq C^{|\alpha|+|\beta|}(\alpha!)^\mu(\beta!)^\nu\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|}\cdot$$

$$\cdot\exp\left[c\langle\xi\rangle^p\langle x\rangle^q|t-s|\right]\sum_{i=1}^{(|\alpha+\beta|+1)m}\langle\xi\rangle^{pi}\langle x\rangle^{qi}\frac{|t-s|^{i-1}}{(i-1)!}$$

for every $(t,s)\in[0,T]^2$, $(x,\xi)\in\mathbb{R}^{2n}$.

Proof. Let $k = 0, ..., m-1$. For $\alpha = \beta = 0$, (33) follows directly from the initial data of (29) and from well known estimates for the solution of the Cauchy problem for ordinary differential equations. See also [4] and [14]. Let us now assume that (33) holds for $|\alpha + \beta| = N$ and let $l \in \{1, ..., n\}$. By (29), it follows that $D_{\xi_l} e_0$ is a solution of the problem

$$\begin{cases} \left(D_t^m + \sum\limits_{j=1}^m a_j(t,x,\xi) D_t^{m-j} \right) D_{\xi_l} e_0 = - \sum\limits_{j=1}^m D_{\xi_l} a_j(t,x,\xi) D_t^{m-j} e_0 \\ D_t^j D_{\xi_l} e_0(s,s;x,\xi) = 0 \qquad\qquad j = 0, ..., m-1 \end{cases}$$

so we have that

$$D_{\xi_l} e_0(t,s;x,\xi) = - \int_s^t e_0(t,\tau;x,\xi) \sum_{j=1}^m D_{\xi_l} a_j(\tau,x,\xi) D_\tau^{m-j} e_0(\tau,s;x,\xi) d\tau.$$

This remark allows to estimate the left-hand side of (33) inductively for every $\alpha, \beta \in \mathbb{N}^n$. The estimate (34) easily follows from (33) and (29).

□

LEMMA 6. *Let the functions a_j belong to $C\left([0,T], \Gamma_{\mu\nu}^{pj,qj}(\mathbb{R}^{2n})\right)$ and let e_h, $h \geq 1$ be the solutions of* (30)*. Then, there exist positive constants C, c such that, for every $\alpha, \beta \in \mathbb{N}^n$, $(t,s) \in [0,T]^2$, $k = 0, ..., m-1$, $(x,\xi) \in \mathbb{R}^{2n}$, we have*

$$\left| D_\xi^\alpha D_x^\beta D_t^k e_h(t,s;x,\xi) \right| \leq C^{|\alpha|+|\beta|+2h} [(|\alpha|+h)!]^\mu [(|\beta|+h)!]^\nu (h!)^{-1} \cdot \tag{35}$$

$$\cdot \langle\xi\rangle^{-|\alpha|-h} \langle x\rangle^{-|\beta|-h} \exp\left[c\langle\xi\rangle^p \langle x\rangle^q |t-s|\right] \sum_{i=1}^{(|\alpha+\beta|+2h)m} \langle\xi\rangle^{pi} \langle x\rangle^{qi} \frac{|t-s|^{i+m-1-k}}{(i+m-1-k)!}$$

and

$$\left| D_\xi^\alpha D_x^\beta D_t^m e_h(t,s;x,\xi) \right| \leq C^{|\alpha|+|\beta|+2h+1} [(|\alpha|+h)!]^\mu [(|\beta|+h)!]^\nu (h!)^{-1} \cdot \tag{36}$$

$$\cdot \langle\xi\rangle^{-|\alpha|-h} \langle x\rangle^{-|\beta|-h} \exp\left[c\langle\xi\rangle^p \langle x\rangle^q |t-s|\right] \sum_{i=1}^{(|\alpha+\beta|+2h+1)m} \langle\xi\rangle^{pi} \langle x\rangle^{qi} \frac{|t-s|^{i-1}}{(i-1)!}$$

for every $h \geq 1$, $(t,s) \in [0,T]^2$, $(x,\xi) \in \mathbb{R}^{2n}$.

Proof. First of all, we observe that

$$e_h(t,s;x,\xi) = \int_s^t e_0(t,\tau;x,\xi) d_h(\tau,s;x,\xi) d\tau, \quad h \geq 1.$$

From the initial data of (29), it turns out that, for all $\alpha, \beta \in \mathbb{N}^n$, $k = 0, ..., m-1$,

$$D_\xi^\alpha D_x^\beta D_t^k e_h(t,s;x,\xi) = D_\xi^\alpha D_x^\beta \int_s^t D_t^k e_0(t,\tau;x,\xi) d_h(\tau,s;x,\xi) d\tau, \quad h \geq 1$$

which we can easily estimate by induction on $h \geq 1$, obtaining (35). The estimate (36) immediately follows from (35) and (30).

$\square$

LEMMA 7. *Let the functions* $a_j(t,x,\xi)$ *belong to* $C\left([0,T], \Gamma^{pj,qj}_{\mu\nu}(\mathbb{R}^{2n})\right)$, $j = 1,\dots,m$. *Then, the solutions* e_h *of* (29), (30) *satisfy the conditions* (31) *and* (32).

Proof. We observe that for all $k = 0,\dots,m-1$, $h \geq 0$,

$$\sum_{i=0}^{(|\alpha+\beta|+2h)m} \langle\xi\rangle^{pi}\langle x\rangle^{qi}\frac{|t-s|^{i+m-1-k}}{(i+m-1-k)!} \leq \frac{|t-s|^{m-1-k}}{(m-1-k)!}\exp\left[\langle\xi\rangle^{p}\langle x\rangle^{q}|t-s|\right].$$

Then, by the condition $p+q \in \left[0, \frac{1}{\theta}\right[$ and the obvious estimate

$$\langle\xi\rangle^{p}\langle x\rangle^{q} \leq C_1(|x|^{p+q} + |\xi|^{p+q} + C_2),$$

it follows immediately that there exists $C_1 > 0$ and for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\text{(37)}\qquad \sup_{\alpha,\beta\in\mathbb{N}^n}\sup_{(x,\xi)\in\mathbb{R}^{2n}} C_1^{-|\alpha|-|\beta|-2h}(\alpha!)^{-\mu}(\beta!)^{-\nu}(h!)^{-\mu-\nu+1}\langle\xi\rangle^{|\alpha|+h}\langle x\rangle^{|\beta|+h}\cdot$$
$$\cdot\exp\left[-\varepsilon(|x|^{\frac{1}{\theta}} + |\xi|^{\frac{1}{\theta}})\right]\left|D^\alpha_\xi D^\beta_x D^k_t e_h(t,s;x,\xi)\right| \leq C_\varepsilon\frac{|t-s|^{m-1-k}}{(m-1-k)!}$$

for every $(t,s) \in [0,T]^2$, $k = 0,\dots,m-1$. Analogously, we obtain that there exists $C_2 > 0$ and for all $\varepsilon > 0$ there exists $C'_\varepsilon > 0$ such that

$$\text{(38)}\qquad \sup_{\alpha,\beta\in\mathbb{N}^n}\sup_{(x,\xi)\in\mathbb{R}^{2n}} C_2^{-|\alpha|-|\beta|-2h}(\alpha!)^{-\mu}(\beta!)^{-\nu}(h!)^{-\mu-\nu+1}\langle\xi\rangle^{|\alpha|+h}\langle x\rangle^{|\beta|+h}\cdot$$
$$\cdot\exp\left[-\varepsilon(|x|^{\frac{1}{\theta}} + |\xi|^{\frac{1}{\theta}})\right]\left|D^\alpha_\xi D^\beta_x D^m_t e_h(t,s;x,\xi)\right| \leq C'_\varepsilon.$$

The estimates (37), (38) imply that $D^k_t e_h \in C\left([0,T]^2, \Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})\right)$ for all $k = 0,\dots,m-1$. The continuity of $D^m_t e_h$ follows from the relations (29), (30). Furthermore, (37) and (38) give directly (32).

$\square$

THEOREM 9. *Let* $P(t,x,D_t,D_x)$ *be defined by* (3), *where* $a_j(t,x,\xi)$ *belong to* $C\left([0,T], \Gamma^{pj,qj}_{\mu\nu}(\mathbb{R}^{2n})\right)$, $j = 1,\dots,m$. *Then, for every* $(t,s) \in [0,T]^2$, *there exists* $E(t,s) \in OPS^\infty_{\mu\nu\theta}(\mathbb{R}^n)$ *satisfying* (28) *with symbol* $e(t,s;x,\xi)$ *such that*

$$D^j_t e \in C\left([0,T]^2, \Gamma^\infty_{\mu\nu\theta}(\mathbb{R}^{2n})\right) \qquad j = 0,\dots,m.$$

Proof. Starting from $\sum_{h\geq 0} e_h$ and applying i) of Proposition 7, we can construct a symbol $e \in C\left([0,T]^2, \Gamma^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})\right)$. The same argument can be repeated for the derivatives of e. By construction, the corresponding operator E satisfies (28).

$\square$

As an immediate consequence of Theorem 9, we obtain a parametrix for the inhomogeneous equation.

COROLLARY 1. *Let $f \in C([0,T], S_\theta(\mathbb{R}^n))$ and $s \in [0,T]$. Under the same hypotheses of Theorem 9 the function*

$$u(t,x) = \int_s^t E(t,\tau) f(\tau,\cdot)(x)d\tau$$

is in $C^m([0,T], S_\theta(\mathbb{R}^n))$ and

$$\begin{cases} P(t,x,D_t,D_x)u = f(t,x) + \int_s^t R(t,\tau) f(\tau,\cdot)(x)d\tau & (t,x) \in [0,T]\times\mathbb{R}^n \\ D_t^k u(s,x) = 0 & k = 0,\dots,m-1, x\in\mathbb{R}^n \end{cases}$$

where $E(t,\tau)$, $R(t,\tau)$ are the same of Theorem 9. The same result holds when we replace $S_\theta(\mathbb{R}^n)$ with $S'_\theta(\mathbb{R}^n)$.

5. Existence and uniqueness

With the help of the parametrix constructed in the previous section, we are able to prove existence and uniqueness of the solution for the problem (2). For sake of simplicity we prove the existence only for regular data, but we remark that the result holds when we replace $S_\theta(\mathbb{R}^n)$ with $S'_\theta(\mathbb{R}^n)$.

THEOREM 10. *Let $f \in C([0,T], S_\theta(\mathbb{R}^n))$ and $g_k \in S_\theta(\mathbb{R}^n)$, $k = 0,\dots,m-1$. Under the same hypotheses of Theorem 9, for any given $s \in [0,T]$, there exists a unique function $u \in C^m([0,T], S_\theta(\mathbb{R}^n))$ such that*

$$\begin{cases} P(t,x,D_t,D_x)u = f(t,x) & (t,x)\in[0,T]\times\mathbb{R}^n \\ D_t^k u(s,x) = g_k(x) & x\in\mathbb{R}^n, k=0,\dots,m-1. \end{cases}$$

Proof: Let us start by considering the case in which $g_k(x) = 0$, $k = 0,\dots,m-1$. We shall find $h \in C([0,T], S_\theta(\mathbb{R}^n))$ such that, for every given $s\in[0,T]$, the function

$$u(t,x) = \int_s^t E(t,\tau)\left[f(\tau,\cdot) + h(\tau,\cdot)\right](x)d\tau$$

belonging to $C^m([0,T], S_\theta(\mathbb{R}^n))$, is a solution of the Cauchy problem

$$\begin{cases} P(t,x,D_t,D_x)u = f(t,x) & (t,x)\in[0,T]\times\mathbb{R}^n \\ D_t^k u(s,x) = 0 & k=0,\dots,m-1, x\in\mathbb{R}^n. \end{cases} \tag{39}$$

Hypotheses and notations are the same as in Corollary 1, in particular $E(t,\tau)$ is the parametrix in Theorem 9. To this end, for $g \in C([0,T], S_\theta(\mathbb{R}^n))$, define

$$\mathcal{R}g(t,x) = \int_s^t R(t,\tau)g(\tau,\cdot)(x)d\tau \qquad (t,x) \in [0,T] \times \mathbb{R}^n$$

where $R(t,\tau)$ is the operator with kernel K_R in $C\left([0,T]^2, S_\theta(\mathbb{R}^{2n})\right)$ appearing in (28). By Corollary 1,we have to find a function $h \in C([0,T], S_\theta(\mathbb{R}^n))$ such that

$$h(t,x) + \mathcal{R}h(t,x) + \mathcal{R}f(t,x) = 0$$

for every $(t,x) \in [0,T] \times \mathbb{R}^n$. To conclude, it is then sufficient to show that the series $\sum_{\nu=1}^{\infty}(-1)^\nu \mathcal{R}^\nu f(t,\cdot)$ converges in $S_\theta(\mathbb{R}^n)$ to a function $h(t,\cdot)$ in $C([0,T], S_\theta(\mathbb{R}^n))$ uniformly with respect to $t \in [0,T]$. Now we have that

$$R(t,\tau)f(\tau,\cdot)(x) = \int_{\mathbb{R}^n} K_R(t,\tau,x,y)f(\tau,y)dy \qquad (t,\tau) \in [0,T]^2, x \in \mathbb{R}^n.$$

Using the notations of the Introduction, we deduce that there exist positive integers A, B for which

$$(40) \quad \|\mathcal{R}f(t,\cdot)\|_{A,B,n} \le \sup_{[0,T]^2} \|K_R(t,s,\cdot,\cdot)\|_{A,B,2n} \int_s^t \left(\int_{\mathbb{R}^n} |f(\tau,y)|dy\right) d\tau \le$$

$$\sup_{[0,T]^2} \|K_R(t,s,\cdot,\cdot)\|_{A,B,2n} \int_s^t \|f(\tau,\cdot)\|_{A,B,n} d\tau.$$

In particular, from (40) we deduce that

$$\|\mathcal{R}f(t,\cdot)\|_{A,B,n} \le \sup_{[0,T]^2} \|K_R(t,s,\cdot,\cdot)\|_{A,B,2n} \cdot \sup_{[0,T]} \|f(t,\cdot)\|_{A,B,n} \cdot |t-s|.$$

Arguing by induction, let us suppose that for a fixed $\nu > 1$

$$\|\mathcal{R}^\nu f(t,\cdot)\|_{A,B,n} \le \left(\sup_{[0,T]^2} \|K_R(t,s,\cdot,\cdot)\|_{A,B,2n}\right)^\nu \sup_{[0,T]} \|f(t,\cdot)\|_{A,B,n} \frac{|t-s|^\nu}{\nu!}.$$

Then, we have

$$\|\mathcal{R}^{\nu+1} f(t,\cdot)\|_{A,B,n} \le \sup_{[0,T]^2} \|K_R(t,s,\cdot,\cdot)\|_{A,B,2n} \int_s^t \|\mathcal{R}^\nu f(\tau,\cdot)\|_{A,B,n} d\tau \le$$

$$\left(\sup_{[0,T]^2} \|K_R(t,s,\cdot,\cdot)\|_{A,B,2n}\right)^{\nu+1} \sup_{[0,T]} \|f(t,\cdot)\|_{A,B,n} \int_s^t \frac{|\tau-s|^\nu}{\nu!} d\tau \le$$

$$\left(\sup_{[0,T]^2} \|K_R(t,s,\cdot,\cdot)\|_{A,B,2n}\right)^{\nu+1} \sup_{[0,T]} \|f(t,\cdot)\|_{A,B,n} \frac{|t-s|^{\nu+1}}{(\nu+1)!}.$$

Hence, $\sum\limits_{\nu=1}^{\infty}(-1)^{\nu}\mathcal{R}^{\nu}f(t,\cdot)$ converges in $S_\theta(\mathbb{R}^n)$ uniformly with respect to t in $[0,T]$. This gives solution to the problem with zero initial data. It is now standard to obtain a result of existence of the solution for a homogeneous problem with non-zero initial data. In fact, let $g_k \in S_\theta(\mathbb{R}^n)$, $k = 0, ..., m-1$ and let

$$v(t,x) = \sum_{k=0}^{m-1} i^k (t-s)^k \frac{g_k(x)}{k!}.$$

Then, arguing as before, we may construct a function $h \in C\left([0,T], S_\theta(\mathbb{R}^n)\right)$ such that

$$u(t,x) = v(t,x) - \int_s^t E(t,\tau)\,[Pv(\tau,\cdot) + h(\tau,\cdot)]\,d\tau$$

is a solution of the Cauchy problem

$$\begin{cases} P(t,x,D_t,D_x)u = 0 & (t,x) \in [0,T]\times\mathbb{R}^n \\ D_t^k u(s,x) = g_k(x) & k = 0,...,m-1, x \in \mathbb{R}^n. \end{cases} \tag{41}$$

The existence of a solution for the problem (2) directly follows from the existence for (39) and (41). To conclude the proof of Theorem 10, we want to show that if u in $C^m\left([0,T], S'_\theta(\mathbb{R}^n)\right)$ is such that $D_t^j u(s,\cdot) = 0$ for some $s \in [0,T]$ and $Pu(t,\cdot) = 0$ for all $t \in [0,T]$, then $u(t,\cdot) = 0$ on $[0,T]$. The argument we will follow is the same developed in [4], [6], so we will give only the main lines of the proof and leave the details to the reader.

Let us consider the transpose ${}^tP$ of the operator P given by

$${}^tP =^t a_m - D_t({}^ta_{m-1} - D_t(... - D_t({}^ta_1 - D_t)...))$$

where ${}^ta_j$ is the transpose of the operator a_j, $j = 1,...,m$. By Proposition 8 we can write ${}^ta_j(t,x,D_x) = b_j(t,x,D_x) + r_j(t,x,D_x)$, where $b_j \in OPS^{pj,qj}_{\mu\nu}(\mathbb{R}^n)$ and r_j are $\theta-$regularizing operators with kernel in $C\left([0,T], S_\theta(\mathbb{R}^{2n})\right)$.

Given $f \in C\left([0,T], S_\theta(\mathbb{R}^n)\right)$ and $s_0 \in [0,T]$, we want to prove the existence of a function $v \in C^m\left([0,T], S_\theta(\mathbb{R}^n)\right)$, such that

$$\begin{cases} {}^tP(t,x,D_t,D_x)v = f(t,x) & (t,x) \in [0,T]\times\mathbb{R}^n \\ v(s_0,x) = 0 & \\ \left({}^ta_j - D_t(... - D_t({}^ta_1 - D_t)...)\right)v(s_0,x) = 0 & j = 0,...,m-1. \end{cases} \tag{42}$$

If such a function exists, then, given u as before, we can write

$$\int_s^{s_0}\langle u(t,\cdot), f(t,\cdot)\rangle dt = \int_s^{s_0}\langle u(t,\cdot),^t Pv(t,\cdot)\rangle dt = \int_s^{s_0}\langle Pu(t,\cdot), v(t,\cdot)\rangle dt = 0$$

from which it follows that $u = 0$.

The existence of a solution of the problem (42) can be obtained from the following lemma.

LEMMA 8. *Let* $b_j(t,x,\xi)$, $j = 1,...,m$, *be as before and assume that* $b_j \sim \sum_{r\geq 0} b_{j,r}$ *in* $FS^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})$ *uniformly with respect to* $t \in [0,T]$.*Then for every* (t,s) *in* $[0,T]^2$ *there exists a* $m \times m$ *matrix of operators* $F^{k,j}(t,s) \in OPS^{\infty}_{\mu\nu\theta}(\mathbb{R}^n)$,$k,j = 0,...,m-1$ *such that:*
i) their symbols $f^{kj}(t,s;x,\xi)$ *belong to* $C\left([0,T]^2, \Gamma^{\infty}_{\mu\nu\theta}(\mathbb{R}^{2n})\right)$ *together with their first order derivatives;*
ii) $F^{kj}(s,s) = -i\delta^k_j I \qquad j,k = 0,...,m-1$
iii) $b_j F^{k,0} - D_t F^{k,j-1} = F^{k,j} + R^{k,j} \qquad j = 1,...,m-1, k = 0,...,m-1$
and $b_m F^{k,0} - D_t F^{k,m-1} = R^{k,m}$
where $R^{k,j}$, $R^{k,m}$ *have their kernels in* $C\left([0,T]^2, S_\theta(\mathbb{R}^{2n})\right)$.

Proof. The lemma can be proved following the arguments in [4],[6] combined with the global results obtained in the previous sections. We omit the details for sake of brevity. □

From Lemma 8, using again the same arguments of [4], we can conclude that there exist $h_k \in C\left[0,T], S_\theta(\mathbb{R}^n)\right)$ such that the function

$$v(t,x) = \int_{s_0}^{t} \sum_{k=0}^{m-1} F^{k,0}(t,\tau)\left(h_k(\tau,\cdot) + \delta_k^{m-1} f(\tau,\cdot)\right)(x)d\tau$$

is a solution of the problem (42).This concludes the proof of Theorem 10.

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