

# Nontrivial solutions for a Robin problem with a nonlinear term asymptotically linear at $-\infty$ and superlinear at $+\infty$

Soluciones no triviales para un problema de Robin con un término no lineal asintótico en  $-\infty$  y superlineal en  $+\infty$

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ABSTRACT. In this paper we study the existence of solutions for a Robin problem, with a nonlinear term with subcritical growth respect to a variable.

*Key words and phrases.* Robin problems, weak solutions, Sobolev spaces, functional, Palais-Smale condition, critical points.

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RESUMEN. En este artículo estudiamos la existencia de soluciones de un problema de Robin, con término no lineal con crecimiento subcrítico respecto a una variable.

*Palabras y frases clave.* Problemas de Robin, soluciones débiles, espacios de Sobolev, funcionales, condición de Palais-Smale, puntos críticos.

## 1. Introduction

In this paper, we study the existence of nontrivial solutions of the following problem with the real parameter  $\alpha \neq 0$ :

$$(\mathbb{P}) \quad \begin{cases} u \in H^1(\Omega, -\Delta), \\ -\Delta u = f(x, u(x)), & \text{in } \Omega, \\ \gamma_1 u + \alpha \gamma_0 u = 0, & \text{on } \partial\Omega. \end{cases}$$

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Here  $\Delta$  is the Laplace operator,  $\Omega$  is a bounded domain in  $\mathbb{R}^n (n \geq 2)$  simply connected and with smooth boundary  $\partial\Omega$ . The case  $\alpha = 0$  was studied by Arcoya and Villegas in [1].

The function  $f : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ , satisfies the following conditions:

$f_0$ ) The function  $f$  is continuous.

$f_1$ )  $|f(x, s)| \leq c(1 + |s|^\sigma)$ ,  $\forall x \in \overline{\Omega}$  and  $\forall s \in \mathbb{R}$ , where the exponent  $\sigma$  is a constant such that

$$\begin{aligned} 1 < \sigma < \frac{n+2}{n-2} & \text{ if } n \geq 3, \\ 1 < \sigma < \infty & \text{ if } n = 2. \end{aligned}$$

$f_2$ ) There exists  $\lambda > 0$  such that

$$\lim_{s \rightarrow -\infty} [f(x, s) - \lambda s] = 0, \quad \text{uniformly in } x \in \overline{\Omega}.$$

$f_3$ ) There exist  $s_0 > 0$  and  $\theta \in (0, \frac{1}{2})$  such that

$$0 < F(x, s) \leq \theta s f(x, s), \quad \forall x \in \overline{\Omega}, \quad \forall s \geq s_0,$$

where  $F(x, s) = \int_0^s f(x, t) dt$  is a primitive of  $f$ .

The boundary condition  $\gamma_1 u + \alpha \gamma_0 u = 0$  involves the trace operators:  $\gamma_0 : H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$  and  $\gamma_1 : H^1(\Omega, -\Delta) \rightarrow H^{-1/2}(\partial\Omega)$ , where  $H^1(\Omega, -\Delta) = \{u \in H^1(\Omega) : -\Delta u \in L^2(\Omega)\}$  with the norm

$$\|u\|_{H^1(\Omega, -\Delta)} = \left( \|u\|_{H^1(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2 \right)^{1/2},$$

for each  $u \in H^1(\Omega, -\Delta)$ ,  $\gamma_1 u \in H^{-1/2}(\partial\Omega)$  and  $\gamma_0 u \in H^{1/2}(\partial\Omega)$ . Identifying the element  $\gamma_0 u$  with the functional  $\gamma_0^* u \in H^{-1/2}(\partial\Omega)$  defined by

$$\langle \gamma_0^* u, w \rangle = \int_{\partial\Omega} (\gamma_0 u) w ds, \quad \forall w \in H^{1/2}(\partial\Omega),$$

the boundary condition makes sense in  $H^{-1/2}(\partial\Omega)$ . The mathematical difficulties that arise by involving this type of boundary conditions are in the Condition of Palais -Smale.

## 2. Preliminary results

To get the results of existence Theorems 4.2 and 4.1 we will use the following Theorem.

**Theorem 2.1** (Theorem of Silva, E. A.). *Let  $X = X_1 \oplus X_2$  be a real Banach space, with  $\dim(X_1) < +\infty$ . If  $\Phi \in C^1(X, \mathbb{R})$  satisfies the Palais-Smale condition and the following conditions:*

I)  $\Phi(u) \leq 0, \quad \forall u \in X_1$ .

II) *There exists  $\rho_0 > 0$  such that  $\Phi(u) \geq 0, \forall u \in \partial B_{\rho_0}(0) \cap X_2$ .*

III) There exist  $e \in X_2 - \{0\}$  and a constant  $M$  such that

$$\Phi(v + te) \leq M, \quad \forall v \in X_1 \quad \text{and} \quad \forall t > 0.$$

Then  $\Phi$  has at least a critical point different from zero.

*Proof.* See [6, Lemma 1.13, p. 460].  $\square$

We use the decomposition of  $H^1(\Omega)$  as orthogonal sum of two subspaces established in [3]. We denote the sequence of eigenvalues of the problem

$$\begin{cases} -\Delta u = \mu u, & \text{in } \Omega, \\ \gamma_1 u + \alpha \gamma_0 u = 0, & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

in the case  $\alpha < 0$  with  $\{\mu_j\}_{j=1}^\infty$ , where

$$\mu_1 = \inf_{\substack{u \neq 0 \\ u \in H^1(\Omega)}} \frac{\int_{\Omega} |\nabla u|^2 + \alpha \int_{\partial\Omega} (\gamma_0 u)^2 ds}{\int_{\Omega} u^2} < 0. \quad (2.2)$$

With  $X_1$  we denote the space associated to the first eigenvalue  $\mu_1$ , and with  $X_2 = X_1^\perp$  the orthogonal complement of  $X_1$  respect to the inner product defined by

$$(u, v)_k = \int_{\Omega} \nabla u \cdot \nabla v + \alpha \int_{\partial\Omega} (\gamma_0 u)(\gamma_0 v) ds + k \int_{\Omega} uv, \quad u, v \in H^1(\Omega), \quad (2.3)$$

where  $k$  is a positive constant suitably selected in [3]. Then

$$H^1(\Omega) = X_1 \oplus X_2, \quad (2.4)$$

and

$$\int_{\Omega} |\nabla \varphi|^2 + \alpha \int_{\partial\Omega} (\gamma_0 \varphi)^2 ds = \mu_1 \int_{\Omega} \varphi^2, \quad \forall \varphi \in X_1. \quad (2.5)$$

In the case  $\alpha > 0$ , the constant  $k$  in (2.3) is positive and arbitrary. We denote with  $\{\beta_j\}_{j=1}^\infty$  the eigenvalues of Problem (2.1), in particular, we have

$$\beta_1 = \inf_{\substack{u \neq 0 \\ u \in H^1(\Omega)}} \frac{\int_{\Omega} |\nabla u|^2 + \alpha \int_{\partial\Omega} (\gamma_0 u)^2 ds}{\int_{\Omega} u^2} > 0. \quad (2.6)$$

With  $Y_1$  we denote the space associated to  $\beta_1$  and  $Y_2 = Y_1^\perp$  the orthogonal complement of  $Y_1$  with respect to the inner product defined by the formula (2.3). Then

$$H^1(\Omega) = Y_1 \oplus Y_2, \quad (2.7)$$

and

$$\int_{\Omega} |\nabla \varphi|^2 + \alpha \int_{\partial\Omega} (\gamma_0 \varphi)^2 ds = \beta_1 \int_{\Omega} \varphi^2, \quad \forall \varphi \in Y_1. \quad (2.8)$$

### 3. Condition of Palais-Smale

Following Arcoya - Villegas [1], Figueiredo [4], and using theorems 3.1, 3.2 and 3.3 of [3] we establish the conditions under which the functional

$$\Phi(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{\alpha}{2} \int_{\partial\Omega} (\gamma_0 u)^2 ds - \int_{\Omega} F(x, u), \quad \forall u \in H^1(\Omega),$$

satisfies the Palais-Smale condition. We prove the cases  $\alpha > 0$  and  $\alpha < 0$ . The condition of Palais-Smale (*P.S*) affirms: any sequence  $\{u_n\}_{n=1}^{\infty}$  in  $H^1(\Omega)$  such that  $|\Phi(u_n)| \leq c$  and  $\lim_{n \rightarrow \infty} \Phi'(u_n) = 0$  in  $H^{-1}(\Omega)$ , contains a convergent subsequence in the norm of  $H^1(\Omega)$ . In virtue of the density of  $C^\infty(\overline{\Omega})$  in  $H^1(\Omega)$  and by the continuity of the operator  $\gamma_0 : H^1(\Omega) \rightarrow L^2(\partial\Omega)$ , we have the following lemma:

**Lemma 3.1.** *Let us suppose  $\Omega \subset \mathbb{R}^n$  bounded with boundary of class  $C^1$ . If  $u \in H^1(\Omega)$ ,  $u^+(x) = \max\{u(x), 0\}$  and  $u^-(x) = \max\{-u(x), 0\}$  then*

$$\int_{\partial\Omega} \gamma_0(u^+) \gamma_0(u^-) ds = 0. \quad (3.1)$$

*Proof.* Let  $\{u_n\}_{n=1}^{\infty}$  be a sequence in  $C^\infty(\overline{\Omega})$  such that  $u_n \rightarrow u$  in  $H^1(\Omega)$  then  $u_n^+ \rightarrow u^+$  and  $u_n^- \rightarrow u^-$  in  $H^1(\Omega)$ , see [2]. By the continuity of the operator  $\gamma_0 : H^1(\Omega) \rightarrow L^2(\partial\Omega)$  we have  $\gamma_0(u_n^+) \rightarrow \gamma_0(u^+)$  and  $\gamma_0(u_n^-) \rightarrow \gamma_0(u^-)$  in  $L^2(\partial\Omega)$  then:

$$\begin{aligned} \int_{\partial\Omega} \gamma_0(u^+) \gamma_0(u^-) ds &= \lim_{n \rightarrow \infty} \int_{\partial\Omega} \gamma_0(u_n^+) \gamma_0(u_n^-) ds \\ &= \lim_{n \rightarrow \infty} \int_{\partial\Omega} u_n^+ u_n^- ds \\ &= 0. \end{aligned}$$

□

From (3.1) we have:

$$\int_{\partial\Omega} (\gamma_0 u) (\gamma_0 u^+) ds = \int_{\partial\Omega} (\gamma_0 u^+)^2 ds, \quad (3.2)$$

$$\int_{\partial\Omega} (\gamma_0 u) (\gamma_0 u^-) ds = - \int_{\partial\Omega} (\gamma_0 u^-)^2 ds. \quad (3.3)$$

**Lemma 3.2** (Condition of Palais-Smale). *If  $\alpha < 0$  we suppose  $(f_0), (f_1), (f_2)$  and  $(f_3)$ . In the case  $\alpha > 0$ , moreover, we also suppose the following conditions*

*S<sub>1</sub>) The number  $\lambda$  of condition  $(f_2)$  is not an eigenvalue of the operator  $-\Delta$  with boundary condition  $\gamma_1 u + \alpha \gamma_0 u = 0$ .*

*S<sub>2</sub>) The numbers  $\sigma$  and  $\theta$  of the conditions  $(f_1)$  and  $(f_3)$  are such that*

$$\begin{aligned} \sigma\theta &\leq \frac{1}{2} + \frac{1}{n} \quad \text{if } n \geq 3 \quad \text{and} \\ \sigma\theta &< 1 \quad \text{if } n = 2. \end{aligned}$$

Then  $\forall u \in H^1(\Omega)$  the functional

$$\Phi(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{\alpha}{2} \int_{\partial\Omega} (\gamma_0 u)^2 ds - \int_{\Omega} F(x, u),$$

satisfies the condition of Palais-Smale (P.S.).

*Proof.* Let  $\{u_n\}_{n=1}^{\infty}$  be a sequence in  $H^1(\Omega)$  such that

$$|\Phi(u_n)| = \left| \frac{1}{2} \int_{\Omega} |\nabla u_n|^2 + \frac{\alpha}{2} \int_{\partial\Omega} (\gamma_0 u_n)^2 ds - \int_{\Omega} F(x, u_n) \right| \leq C, \quad (3.4)$$

and  $\forall v \in H^1(\Omega)$

$$|\langle \Phi'(u_n), v \rangle| = \left| \int_{\Omega} \nabla u_n \cdot \nabla v + \alpha \int_{\partial\Omega} (\gamma_0 u_n)(\gamma_0 v) ds - \int_{\Omega} f(x, u_n)v \right| \leq \varepsilon_n \|v\|, \quad (3.5)$$

for some constant  $C > 0$  and  $\varepsilon_n \rightarrow 0^+$ .

To show that  $\{u_n\}_{n=1}^{\infty}$  has a convergent subsequence it is enough to prove that  $\{u_n\}_{n=1}^{\infty}$  is bounded.

**Case  $\alpha < 0$ .** We argue by contradiction. Let us consider a subsequence of  $\{u_n\}_{n=1}^{\infty}$ , which we denote in the same way, such that

$$\lim_{n \rightarrow \infty} \|u_n\| = +\infty.$$

Let  $z_n = \frac{u_n}{\|u_n\|}$ . Then there exists a subsequence of  $\{z_n\}$  which we denote in the same way, such that

$$\begin{aligned} z_n &\rightharpoonup z_0 && \text{weakly in } H^1(\Omega), && z_0 \in H^1(\Omega), \\ z_n &\rightarrow z_0 && \text{in } L^2(\Omega), \\ \gamma_0 z_n &\rightarrow \gamma_0 z_0 && \text{in } L^2(\partial\Omega), \\ z_n(x) &\rightarrow z_0(x) && \text{a.e. } x \in \Omega, \\ |z_n(x)| &\leq q(x) && \text{a.e. } x \in \Omega, \quad q \in L^2(\Omega). \end{aligned} \quad (3.6)$$

Dividing the terms of (3.5) by  $\|u_n\|$  and taking the limit  $\forall v \in H^1(\Omega)$  we obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|} v = \int_{\Omega} \nabla z_0 \cdot \nabla v + \alpha \int_{\partial\Omega} (\gamma_0 z_0)(\gamma_0 v) ds. \quad (3.7)$$

From (3.7) with  $v = 1$  in  $\overline{\Omega}$ , we get

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|} = \alpha \int_{\partial\Omega} \gamma_0 z_0 ds < +\infty. \quad (3.8)$$

We obtain the desired contradiction in three steps.

*First step.* We shall prove

$$z_0(x) = 0 \quad \text{a.e. } x \in \Omega, \quad \text{and} \quad \gamma_0 z_0(x) = 0 \quad \text{a.e. } x \in \partial\Omega. \quad (3.9)$$

First we prove

$$z_0(x) \leq 0 \quad \text{a.e. } x \in \Omega, \quad \text{and} \quad \gamma_0 z_0(x) \leq 0 \quad \text{a.e. } x \in \partial\Omega. \quad (3.10)$$

Let  $\Omega^+ = \{x \in \Omega : z_0(x) > 0\}$  and  $|\Omega^+|$  be the measure of Lebesgue of  $\Omega^+$ . Choosing  $v = z_0^+$  in (3.7) we obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega^+} \frac{f(x, u_n)}{\|u_n\|} z_0 = \int_{\Omega^+} |\nabla z_0|^2 + \alpha \int_{\partial\Omega} (\gamma_0 z_0^+)^2 ds < \infty. \quad (3.11)$$

Using conditions  $(f_3)$  and  $(f_2)$ , for  $x \in \Omega^+$  we obtain

$$\frac{f(x, u_n(x))z_0(x)}{\|u_n\|} \geq -(\lambda q(x) + K_1)z_0(x). \quad (3.12)$$

Indeed, condition  $(f_3)$  implies the existence of a constant  $c > 0$  such that

$$f(x, s) \geq cs^{\frac{1}{p}-1}, \quad \forall s \geq s_0. \quad (3.13)$$

Then we can choose  $s^* > s_0$  such that

$$f(x, s) \geq \lambda s, \quad \forall s \geq s^*. \quad (3.14)$$

On the other hand, by  $(f_2)$ , for  $\varepsilon > 0$  there is  $s' < 0$  such that

$$|f(x, s) - \lambda s| \leq \varepsilon, \quad \forall s \leq s' \quad \text{and} \quad \forall x \in \overline{\Omega}, \quad (3.15)$$

by the continuity of the function  $f$  there exists a constant  $K_1$  such that

$$|f(x, s) - \lambda s| \leq K_1, \quad \forall s \in (-\infty, s^*] \quad \text{and} \quad \forall x \in \overline{\Omega}. \quad (3.16)$$

From (3.14) and (3.16) we get

$$f(x, s) \geq \lambda s - K_1 \quad \forall s \in \mathbb{R}, \quad \forall x \in \overline{\Omega}. \quad (3.17)$$

Now, using (3.17) with  $x \in \Omega^+$  we have

$$\begin{aligned} \frac{f(x, u_n(x))z_0(x)}{\|u_n\|} &\geq \frac{(\lambda u_n(x) - K_1)}{\|u_n\|} z_0(x) \\ &\geq (\lambda z_n(x) - K_1)z_0(x) \\ &\geq -(\lambda q(x) + K_1)z_0(x). \end{aligned}$$

From (3.6) we have  $\lim_{n \rightarrow \infty} u_n(x) = +\infty$  for a.e.  $x \in \Omega^+$  and using (3.13) the superlinearity of  $f$  in  $+\infty$  we have for a. e.  $x \in \Omega^+$

$$\lim_{n \rightarrow +\infty} \frac{f(x, u_n)z_0(x)}{\|u_n\|} = \lim_{n \rightarrow \infty} \frac{f(x, u_n)}{u_n(x)} z_n(x)z_0(x) = +\infty.$$

If  $|\Omega^+| > 0$ , by the Fatou's Lemma, we get

$$\begin{aligned} +\infty &= \int_{\Omega^+} \underline{\lim}_{n \rightarrow \infty} \frac{f(x, u_n(x))}{\|u_n\|} z_0(x) \\ &\leq \underline{\lim}_{n \rightarrow \infty} \int_{\Omega^+} \frac{f(x, u_n)}{\|u_n\|} z_0(x), \end{aligned}$$

then

$$\lim_{n \rightarrow +\infty} \int_{\Omega^+} \frac{f(x, u_n(x))}{\|u_n\|} z_0(x) = +\infty,$$

in contradiction with (3.11). Hence  $|\Omega^+| = 0$  and  $z_0(x) \leq 0$  a.e.  $x \in \Omega$ . If  $y \in \partial\Omega$ , then

$$\gamma_0 z_0(y) = \lim_{r \rightarrow 0} \frac{1}{|B(y, r) \cap \Omega|} \int_{B(y, r) \cap \Omega} z_0(x) dx \leq 0.$$

See [5, p. 143]. Below we prove that

$$\int_{\Omega} z_0(x) dx = 0 = \int_{\partial\Omega} \gamma_0 z_0(s) ds. \tag{3.18}$$

Let  $v = \frac{1}{2}u_n$  in (3.5) and subtracting this identity from (3.4), we obtain

$$\left| \int_{\Omega} \left\{ \frac{f(x, u_n)}{2} u_n - F(x, u_n) \right\} \right| \leq \frac{\varepsilon_n}{2} \|u_n\| + C. \tag{3.19}$$

Dividing this inequality by  $\|u_n\|$  and passing to the limit, we get

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{\frac{f(x, u_n)}{2} u_n - F(x, u_n)}{\|u_n\|} dx = 0. \tag{3.20}$$

On the other hand, given  $\varepsilon > 0$ , conditions  $(f_0)$  and  $(f_2)$  imply the existence of a constant  $k_\varepsilon > 0$  such that

$$\left| \frac{1}{2} f(x, s) s - F(x, s) \right| \leq \varepsilon |s| + k_\varepsilon, \quad \forall s \leq s^*. \tag{3.21}$$

Using (3.21) we have

$$\begin{aligned} \left| \int_{u_n(x) \leq s^*} \frac{\frac{f(x, u_n)}{2} u_n - F(x, u_n)}{\|u_n\|} \right| &\leq \varepsilon \int_{\Omega} \frac{|u_n|}{\|u_n\|} + \frac{K_\varepsilon}{\|u_n\|} |\Omega| \\ &\leq \varepsilon c + \frac{K_\varepsilon}{\|u_n\|} |\Omega| \end{aligned}$$

and, since  $\varepsilon$  is arbitrary,

$$\lim_{n \rightarrow \infty} \int_{u_n(x) \leq s^*} \frac{\frac{f(x, u_n)}{2} u_n - F(x, u_n)}{\|u_n\|} = 0. \tag{3.22}$$

The identities (3.20) and (3.22) show that

$$\lim_{n \rightarrow \infty} \int_{u_n(x) > s^*} \frac{\frac{f(x, u_n)}{2} u_n - F(x, u_n)}{\|u_n\|} = 0. \quad (3.23)$$

Using (3.16) and condition  $(f_3)$ , we obtain

$$\begin{aligned} \int_{u_n(x) > s^*} \frac{\frac{f(x, u_n)}{2} u_n - F(x, u_n)}{\|u_n\|} &\geq \left(\frac{1}{2} - \theta\right) s^* \int_{u_n(x) > s^*} \frac{f(x, u_n(x))}{\|u_n\|} \\ &\geq \left(\frac{1}{2} - \theta\right) s^* \left\{ \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|} - \int_{u_n(x) \leq s^*} \frac{f(x, u_n)}{\|u_n\|} \right\} \\ &\geq \left(\frac{1}{2} - \theta\right) s^* \left\{ \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|} - \lambda \int_{\Omega} \chi_n z_n - \frac{K_1}{\|u_n\|} |\Omega| \right\}, \end{aligned}$$

where

$$\chi_n(x) = \begin{cases} 1 & \text{if } u_n(x) \leq s^*, \\ 0 & \text{otherwise.} \end{cases}$$

Using (3.8), (3.20) and getting the limit we have

$$\begin{aligned} 0 &\geq \lim_{n \rightarrow \infty} \left(\frac{1}{2} - \theta\right) s^* \left\{ \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|} - \lambda \int_{\Omega} \chi_n z_n dx - \frac{K_1 |\Omega|}{\|u_n\|} \right\} \\ &= \left(\frac{1}{2} - \theta\right) s^* \left\{ \alpha \int_{\partial\Omega} \gamma_0 z_0 ds - \lambda \int_{\Omega} z_0 \right\} \\ &= \left(\frac{1}{2} - \theta\right) s^* \left\{ -\alpha \int_{\partial\Omega} |\gamma_0 z_0| ds + \lambda \int_{\Omega} |z_0| \right\} \geq 0. \end{aligned}$$

Hence

$$\left(\frac{1}{2} - \theta\right) s^* \left\{ -\alpha \int_{\partial\Omega} |\gamma_0 z_0| ds + \lambda \int_{\Omega} |z_0| \right\} = 0. \quad (3.24)$$

Then

$$\int_{\partial\Omega} |\gamma_0 z_0| ds = 0 = \int_{\Omega} |z_0|.$$

Using (3.10) we have (3.9). Now, the limit (3.7) is

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|} v = 0, \quad \forall v \in H^1(\Omega). \quad (3.25)$$

*Second step.* We shall prove now that

$$\lim_{n \rightarrow \infty} \sup \int_{\Omega} \frac{f(x, u_n(x))}{\|u_n\|} z_n \leq 0. \quad (3.26)$$



We denote:

$$\begin{aligned} I_1 &= \int_{u_n(x) < 0} \frac{f(x, u_n(x))}{\|u_n\|} z_n, \\ I_2 &= \int_{0 \leq u_n(x) \leq s_0} \frac{f(x, u_n(x))}{\|u_n\|} z_n, \\ I_3 &= \int_{u_n(x) > s_0} \frac{f(x, u_n(x))}{\|u_n\|} z_n. \end{aligned}$$

Let us prove that

$$\lim_{n \rightarrow \infty} I_1 = 0. \quad (3.27)$$

From condition  $(f_2)$ , we have  $\lim_{s \rightarrow -\infty} \frac{sf(x,s) - \lambda s^2}{s} = 0$ , so, given  $\varepsilon > 0$  by the continuity of  $f$  there exists a constant  $c_\varepsilon > 0$  such that

$$|f(x, s)s - \lambda s^2| \leq c_\varepsilon + \varepsilon|s|, \quad \forall s \leq 0, \quad (3.28)$$

then

$$\begin{aligned} \left| \int_{u_n < 0} f(x, u_n) u_n \right| &\leq c_\varepsilon \int_{u_n < 0} dx + \varepsilon \int_{u_n < 0} |u_n| + \lambda \int_{u_n < 0} u_n^2 \\ &\leq c + (c + \lambda) \int_{\Omega} u_n^2. \end{aligned}$$

Dividing the last inequality by  $\|u_n\|^2$  and getting the limit yields (3.27), because  $z_n \rightarrow z_0$  in  $L^2(\Omega)$  and  $z_0(x) = 0$  a.e.  $x \in \Omega$ . Let us see that

$$\lim_{n \rightarrow \infty} I_2 = 0. \quad (3.29)$$

If  $L = \max \{|f(x, s)| : (x, s) \in \bar{\Omega} \times [0, s_0]\}$  then

$$\begin{aligned} \left| \int_{0 \leq u_n(x) \leq s_0} \frac{f(x, u_n)}{\|u_n\|} z_n \right| &\leq \int_{0 \leq u_n \leq s_0} \frac{|f(x, u_n)|}{\|u_n\|^2} |u_n(x)| \\ &\leq \frac{L s_0}{\|u_n\|^2} |\Omega|. \end{aligned}$$

Therefore,  $\lim_{n \rightarrow \infty} I_2 = 0$ . To prove that  $\lim_{n \rightarrow \infty} I_3 = 0$ , first we see that

$$\lim_{n \rightarrow \infty} \sup I_3 \leq 0. \quad (3.30)$$

From (3.19) and (3.21) we have

$$\begin{aligned}
 & \left| \int_{u_n > s_0} \left\{ F(x, u_n) - \frac{1}{2} f(x, u_n) u_n \right\} \right| \\
 & \leq \int_{u_n \leq s_0} \left| \frac{1}{2} f(x, u_n) u_n - F(x, u_n) \right| + c + \frac{\varepsilon_n}{2} \|u_n\| \\
 & \leq \int_{u_n \leq s_0} (\varepsilon |u_n| + k_\varepsilon) + c + \frac{\varepsilon_n}{2} \|u_n\| \\
 & \leq \int_{\Omega} (\varepsilon |u_n| + k_\varepsilon) + c + \frac{\varepsilon_n}{2} \|u_n\| \\
 & \leq c\varepsilon \|u_n\| + c + \frac{\varepsilon_n}{2} \|u_n\|.
 \end{aligned}$$

On the other hand, condition  $(f_3)$  implies

$$\left( \frac{1}{2} - \theta \right) \int_{u_n > s_0} f(x, u_n) u_n \leq \int_{u_n > s_0} \left\{ \frac{1}{2} f(x, u_n) u_n - F(x, u_n) \right\}.$$

So,

$$\left( \frac{1}{2} - \theta \right) \int_{u_n > s_0} f(x, u_n) u_n \leq c + \left( c\varepsilon + \frac{\varepsilon_n}{2} \right) \|u_n\|.$$

Dividing by  $\|u_n\|^2$ , we obtain

$$\int_{u_n > s_0} \frac{f(x, u_n)}{\|u_n\|} z_n \leq \frac{c}{\|u_n\|^2} + \left( c\varepsilon + \frac{\varepsilon_n}{2} \right) \frac{1}{\|u_n\|},$$

then  $\lim_{n \rightarrow \infty} \sup I_3 \leq 0$ . Hence

$$\lim_{n \rightarrow \infty} \sup \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|} z_n = \lim_{n \rightarrow \infty} \sup \{I_1 + I_2 + I_3\} \leq 0.$$

*Third step.* Finally we prove

$$\lim_{n \rightarrow \infty} \sup \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|} z_n = 1, \tag{3.31}$$

which contradicts (3.26). From (3.5) with  $v = z_n$  and dividing by  $\|u_n\|$ , we get

$$\frac{\varepsilon_n}{\|u_n\|} \leq \int_{\Omega} z_n^2 - 1 - \alpha \int_{\partial\Omega} (\gamma_0 z_n)^2 ds + \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|} z_n \leq \frac{\varepsilon_n}{\|u_n\|}.$$

By taking superior limit we obtain (3.31).

**Case  $\alpha > 0$ .** In this case we have that

$$(u, v)_* = \int_{\Omega} \nabla u \cdot \nabla v + \alpha \int_{\partial\Omega} (\gamma_0 u)(\gamma_0 v) ds, \quad \forall u, v \in H^1(\Omega),$$

defines an inner product in  $H^1(\Omega)$  and the norm  $\|u\|_* = \sqrt{(u, u)_*}$  is equivalent to the usual norm  $\|\cdot\|$  of  $H^1(\Omega)$ . As a matter of fact, from (2.6) we get  $\int_{\Omega} u^2 \leq \beta_1^{-1} \|u\|_*^2$ , then

$$\|u\|^2 \leq (1 + \beta_1^{-1}) \|u\|_*^2 = d_1 \|u\|_*^2, \quad \forall u \in H^1(\Omega).$$

On the other hand, the inequality  $\|\gamma_0 u\|_{L^2(\partial\Omega)} \leq c_1 \|u\|$  implies,

$$\|u\|_*^2 \leq (1 + \alpha c_1^2) \|u\|^2 = d_2 \|u\|^2, \quad \forall u \in H^1(\Omega).$$

Then

$$(d_1)^{-1/2} \|u\|^2 \leq \|u\|_*^2 \leq d_2 \|u\|^2, \quad \forall u \in H^1(\Omega).$$

Henceforth, we denote the constants with the same letter  $c$  and expressions of the form  $c\varepsilon_n$  with  $\varepsilon_n$ . Using the inner product previously defined and its associated norm, the inequalities (3.4) and (3.5) take the form

$$|\Phi(u_n)| = \left| \frac{1}{2} \|u_n\|_*^2 - \int_{\Omega} F(x, u_n) \right| \leq C, \tag{3.32}$$

$$\begin{aligned} |\langle \Phi'(u_n), v \rangle| &= \left| (u_n, v)_* - \int_{\Omega} f(x, u_n)v \right| \\ &\leq \varepsilon'_n \|v\| \leq \sqrt{d_1} \varepsilon'_n \|v\|_* = \varepsilon_n \|v\|_*, \end{aligned} \tag{3.33}$$

where,  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$  and  $v \in H^1(\Omega)$ .

Next we shall prove that the sequence  $\{u_n\}_{n=1}^{\infty}$  is bounded. With this purpose first we establish the inequality  $\|u_n^+\|_*^2 \leq c + c\|u_n^-\|_*$  and second, we prove that  $\|u_n^-\|_*$  is bounded. The desired result will follow from the equality  $\|u\|_*^2 = \|u^+\|_*^2 + \|u^-\|_*^2, \forall u \in H^1(\Omega)$ .

*First step.* We shall prove

$$\int_{u_n \geq s_0} F(x, u_n) dx \leq c + \varepsilon_n \|u_n\|_* + c \|u_n^-\|_{L^1}. \tag{3.34}$$

From (f<sub>3</sub>) we have

$$\int_{u_n \geq s_0} F(x, u_n) \leq \left(\frac{1}{\theta} - 2\right)^{-1} \int_{u_n \geq s_0} \{f(x, u_n)u_n - 2F(x, u_n)\} dx. \tag{3.35}$$

From (3.32) and (3.33) we get

$$\left| \int_{\Omega} \{f(x, u_n)u_n - 2F(x, u_n)\} dx \right| \leq c + \varepsilon_n \|u_n\|_*. \tag{3.36}$$

Hence

$$\int_{u_n \geq s_0} \{f(x, u_n)u_n - 2F(x, u_n)\} dx \leq c + \varepsilon_n \|u_n\|_* + \int_{u_n < s_0} |2F(x, u_n) - f(x, u_n)u_n|. \quad (3.37)$$

Conditions  $(f_0)$  and  $(f_2)$  imply

$$|2F(x, s) - f(x, s)s| \leq c + c|s|, \quad s < 0, \quad \forall x \in \overline{\Omega}, \quad (3.38)$$

from (3.37) and (3.38) we get

$$\left| \int_{u_n \geq s_0} \{f(x, u_n)u_n - 2F(x, u_n)\} dx \right| \leq c + \varepsilon_n \|u_n\|_* + c \|u_n^-\|_{L^1}. \quad (3.39)$$

Now, from (3.35) and (3.39), we obtain (3.34).

*Second step.* We shall prove now that

$$\left| \|u_n^-\|_*^2 - \int_{u_n < 0} 2F(x, u_n) \right| \leq c + \varepsilon_n \|u_n^-\|_* + c \|u_n^-\|_{L^1}. \quad (3.40)$$

Making  $v(x) = u_n^-(x)$  in (3.33) we have

$$\left| \|u_n^-\|_*^2 - \int_{u_n < 0} f(x, u_n)u_n \right| \leq \varepsilon_n \|u_n^-\|_*. \quad (3.41)$$

From (3.38) and (3.41) we obtain

$$\begin{aligned} \left| \|u_n^-\|_*^2 - \int_{u_n < 0} 2F(x, u_n) \right| &= \left| \|u_n^-\|_*^2 - \int_{u_n < 0} f(x, u_n)u_n \right. \\ &\quad \left. + \int_{u_n < 0} f(x, u_n)u_n - \int_{u_n < 0} 2F(x, u_n) \right| \\ &\leq \left| \|u_n^-\|_*^2 - \int_{u_n < 0} f(x, u_n)u_n \right| \\ &\quad + \left| \int_{u_n < 0} \{f(x, u_n)u_n - 2F(x, u_n)\} dx \right| \\ &\leq \varepsilon_n \|u_n^-\|_* + \int_{u_n < 0} |f(x, u_n)u_n - 2F(x, u_n)| dx \\ &\leq c + \varepsilon_n \|u_n^-\|_* + c \|u_n^-\|_{L^1}. \end{aligned}$$

*Third step.* Next we shall verify the inequality

$$\|u_n^+\|_*^2 \leq c + c \|u_n^-\|_* . \tag{3.42}$$

From (3.32) we have

$$\begin{aligned} & \left| \|u_n^+\|_*^2 - \int_{u_n \geq 0} 2F(x, u_n) \right| - \left| \int_{u_n < 0} 2F(x, u_n) - \|u_n^-\|_*^2 \right| \\ & \leq \left| \|u_n^+\|_*^2 + \|u_n^-\|_*^2 - \int_{\Omega} 2F(x, u_n) \right| \\ & = \left| \|u_n\|_*^2 - \int_{\Omega} 2F(x, u_n) \right| \leq c, \end{aligned}$$

and with (3.40) we get

$$\begin{aligned} \left| \|u_n^+\|_*^2 - \int_{u_n \geq 0} 2F(x, u_n) \right| & \leq c + \left| \int_{u_n < 0} 2F(x, u_n) - \|u_n^-\|_*^2 \right| \\ & \leq c + c \|u_n^-\|_* . \end{aligned}$$

Then the above inequality and (3.34) give

$$\begin{aligned} \|u_n^+\|_*^2 & \leq c + c \|u_n^-\|_* + \left| \int_{u_n \geq 0} 2F(x, u_n) \right| \\ & \leq c + c \|u_n^-\|_* + \left| \int_{0 \leq u_n(x) \leq s_0} 2F(x, u_n) \right| \\ & \quad + \left| \int_{u_n > s_0} 2F(x, u_n) \right| \\ & \leq c + c \|u_n^-\|_* + \varepsilon_n \|u_n^+\|_* . \end{aligned}$$

Therefore

$$\|u_n^+\|_*^2 \leq c + c \|u_n^-\|_* + \varepsilon_n \|u_n^+\|_* ,$$

since  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$  this inequality yields (3.42).

*Fourth step.* We consider the following exhaustive cases:

- i) There exists a constant  $c$  such that  $\|u_n^-\|_* \leq c$ , or
- ii)  $\lim_{n \rightarrow \infty} \|u_n^-\|_* = \infty$ , passing to a subsequence if it would be necessary.

In case i), using (3.42) we have  $\|u_n^+\|_* \leq c, \forall n \in \mathbb{N}$  and, from the equality  $\|u\|_*^2 = \|u^+\|_*^2 + \|u^-\|_*^2 \forall u \in H^1(\Omega)$ , we conclude that  $(u_n)_{n=1}^\infty$  is bounded. Next let us prove that case ii) can not occur. First, from (3.28) and (3.41) we get

$$\left| \|u_n^-\|_*^2 - \lambda \int_{\Omega} (u_n^-)^2 \right| \leq c + c \|u_n^-\|_* . \tag{3.43}$$

If  $w_n = \frac{u_n^-}{\|u_n^-\|_*}$  then there exists  $w_0 \in H^1(\Omega)$  and a subsequence from  $\{w_n\}_{n=1}^\infty$  that we denote in the same way, such that it converges to  $w_0$  weakly in  $H^1(\Omega)$

and strongly in  $L^2(\Omega)$ . Let us see that  $w_0 \neq 0$ . Dividing (3.43) by  $\|u_n^-\|_*^2$  we obtain

$$\left| 1 - \lambda \int_{\Omega} w_n^2 \right| \leq \frac{c}{\|u_n^-\|_*^2} + \frac{c}{\|u_n^-\|_*}.$$

Taking limit when  $n \rightarrow \infty$  we get

$$\int_{\Omega} w_0^2 = \frac{1}{\lambda},$$

therefore  $w_0 \neq 0$ . Let us see that  $\lambda$  is an eigenvalue and  $w_0$  its eigenfunction. First we prove

$$\left| (u_n^-, v)_* - \lambda \int_{\Omega} u_n^- v \right| \leq (c + \varepsilon_n + \|u_n^+\|_* + c \|u_n^+\|_{L^{p\sigma}}^\sigma) \|v\|_*. \tag{3.44}$$

From (3.33) we get

$$\begin{aligned} & \left| (u_n^-, v)_* - \lambda \int_{\Omega} u_n^- v \right| - \left| (u_n^+, v)_* - \lambda \int_{\Omega} u_n^- v - \int_{\Omega} f(x, u_n) v \right| \leq \\ & \leq \left| (u_n^-, v)_* - \lambda \int_{\Omega} u_n^- v - (u_n^+, v)_* + \lambda \int_{\Omega} u_n^- v + \int_{\Omega} f(x, u_n) v \right| \\ & = \left| (u_n, v)_* - \int_{\Omega} f(x, u_n) v \right| \leq \varepsilon_n \|v\|_*. \end{aligned}$$

Then

$$\begin{aligned} & \left| (u_n^-, v)_* - \lambda \int_{\Omega} u_n^- v \right| \leq \varepsilon_n \|v\|_* + \left| (u_n^+, v)_* - \lambda \int_{\Omega} u_n^- v - \int_{\Omega} f(x, u_n) v \right| \\ & \leq \varepsilon_n \|v\|_* + \|u_n^+\|_* \|v\|_* + \left| \lambda \int_{\Omega} u_n^- v + \int_{\Omega} f(x, u_n) v \right|. \end{aligned} \tag{3.45}$$

Next we estimate  $\left| \lambda \int_{\Omega} u_n^- v + \int_{\Omega} f(x, u_n) v \right|$ . Conditions  $(f_0)$  and  $(f_2)$  imply

$$|f(x, s) - \lambda s| \leq c, \quad \forall s \leq 0, \quad \forall x \in \overline{\Omega}. \tag{3.46}$$

using (3.46), we obtain

$$\begin{aligned} \left| \lambda \int_{\Omega} u_n^- v + \int_{\Omega} f(x, u_n) v \right| &\leq \left| \int_{u_n < 0} \{f(x, u_n) - \lambda u_n\} v \right| \\ &\quad + \left| \int_{u_n \geq 0} f(x, u_n) v \right| \\ &\leq \int_{\Omega} \chi_{u_n} |f(x, u_n) - \lambda u_n| |v| + \int_{\Omega} |f(x, u_n^+)| |v| \\ &\leq c \int_{\Omega} \chi_{u_n} |v| + \int_{\Omega} |f(x, u_n^+)| |v| \\ &\leq c \int_{\Omega} |v| + \int_{\Omega} |f(x, u_n^+)| |v| \\ &\leq c \int_{\Omega} |v| + c \int_{\Omega} |v| + c \int_{\Omega} |u_n^+|^{\sigma} |v| \\ &\leq c \|v\|_* + c \|u_n^+\|_{L^{p\sigma}}^{\sigma} \|v\|_{L^q}, \end{aligned}$$

where the function  $\chi_{u_n}$  is defined by

$$\chi_{u_n}(x) = \begin{cases} 1 & \text{if } u_n(x) < 0, \\ 0 & \text{if } u_n(x) \geq 0, \end{cases}$$

and  $p = \frac{2n}{n+2}$ ,  $q = \frac{2n}{n-2}$  for  $n \geq 3$ , and we take  $1 < p < 1/\sigma\theta$  as long as  $\sigma\theta < 1$  for  $n = 2$ . Then we obtain (3.44). Now, dividing (3.44) by  $\|u_n^-\|_*$ , we get

$$\left| (w_n, v)_* - \lambda \int_{\Omega} w_n v \right| \leq \left( \frac{c + \varepsilon_n}{\|u_n^-\|_*} + \frac{\|u_n^+\|_*}{\|u_n^-\|_*} + c \frac{\|u_n^+\|_{L^{p\sigma}}^{\sigma}}{\|u_n^-\|_*} \right) \|v\|_*. \quad (3.47)$$

It is evident that  $\lim_{n \rightarrow \infty} \frac{c + \varepsilon_n}{\|u_n^-\|_*} = 0$ . From (3.42) we have  $\lim_{n \rightarrow \infty} \frac{\|u_n^+\|_*}{\|u_n^-\|_*} = 0$ .

Let us prove that

$$\lim_{n \rightarrow \infty} \frac{\|u_n^+\|_{L^{p\sigma}}^{\sigma}}{\|u_n^-\|_*} = 0. \quad (3.48)$$

Conditions  $(f_0)$  and  $(f_3)$  imply the existence of positive constants  $K$  and  $c_2$  such that

$$F(x, s) \geq \theta K s^{1/\theta} - c_2, \quad \text{for } s > 0. \quad (3.49)$$

Then (3.34) and (3.49) give

$$\int_{\Omega} |u_n^+|^{1/\theta} \leq c + \varepsilon_n \|u_n^+\|_* + c \|u_n^-\|_*. \quad (3.50)$$

Dividing (3.50) by  $\|u_n^-\|_*^{1/\sigma\theta}$  we have

$$\frac{1}{\|u_n^-\|_*^{1/\sigma\theta}} \int_{\Omega} |u_n^+|^{1/\theta} \leq \frac{c}{\|u_n^-\|_*^{1/\sigma\theta}} + \varepsilon_n \frac{\|u_n^+\|_*}{\|u_n^-\|_*^{1/\sigma\theta}} + \frac{c}{\|u_n^-\|_*^{\frac{1}{\sigma\theta}-1}}. \quad (3.51)$$

From  $(S_2)$  we have  $\frac{1}{\sigma\theta} = 1 + \delta$  for some  $\delta > 0$ . Hence

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left( \frac{|u_n^+|}{\|u_n^-\|_*^{1/\sigma}} \right)^{1/\theta} = 0. \tag{3.52}$$

From  $(S_2)$  and the choice of  $p$  in the case  $n = 2$  we have that  $1 < p\sigma \leq \frac{1}{\theta}$ , then

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} \left( \int_{\Omega} \left( \frac{|u_n^+|}{\|u_n^-\|_*^{1/\sigma}} \right)^{p\sigma} \right)^{1/p\sigma} \\ &\leq \lim_{n \rightarrow \infty} \left( \int_{\Omega} \left( \frac{|u_n^+|}{\|u_n^-\|_*^{1/\sigma}} \right)^{1/\theta} \right)^{\theta} = 0. \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} \left( \int_{\Omega} \left( \frac{|u_n^+|}{\|u_n^-\|_*^{1/\sigma}} \right)^{p\sigma} \right)^{1/p} = 0.$$

Then, the limit in (3.47) yields

$$\lim_{n \rightarrow \infty} \left| (w_n, v)_* - \lambda \int_{\Omega} w_n v \right| = \left| (w_0, v)_* - \lambda \int_{\Omega} w_0 v \right| = 0.$$

Hence

$$(w_0, v)_* = \lambda \int_{\Omega} w_0 v, \quad \forall v \in H^1(\Omega),$$

so,  $\lambda$  is an eigenvalue of  $-\Delta$ , with boundary condition  $\gamma_1 u + \alpha \gamma_0 u = 0$ . But this contradicts hypothesis  $(S_1)$ . Hence,  $\|u_n^-\|_*$  cannot tend to  $+\infty$  when  $n \rightarrow \infty$ . □

#### 4. Results of existence

In this section, we establish the existence of solutions of Problem  $(\mathbb{P})$ .

**Theorem 4.1.** *Suppose  $n \geq 2$ ,  $\alpha < 0$ ,  $(f_0), (f_1), (f_2), (f_3)$ , and let  $\mu_1, \mu_2$  be the first and the second eigenvalues of  $-\Delta$  with the boundary condition of the problem*

$$(P_1) \quad \begin{cases} -\Delta u = f(x, u(x)), & \text{in } \Omega, \\ \gamma_1 u + \alpha \gamma_0 u = 0, & \text{on } \partial\Omega, \end{cases}$$

such that

- $f_4) \frac{f(x, s)}{s} \geq \mu_1, \forall s \in \mathbb{R} - \{0\}, \forall x \in \overline{\Omega}.$
- $f_5) \text{ There exist } \varepsilon_0 > 0 \text{ and } p > 0 \text{ such that } \mu_1 < \mu_2 - p < \mu_2, \text{ and}$

$$\frac{f(x, s)}{s} \leq \mu_2 - p, \quad \forall s \in (-\varepsilon_0, \varepsilon_0) - \{0\}, \quad \forall x \in \overline{\Omega}.$$



Then Problem  $(P_1)$  has at least one nontrivial solution.

*Proof.* We prove the conditions of Theorem 2.1. The functional  $\Phi$  associated to the problem  $(P_1)$  is defined by

$$\Phi(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{\alpha}{2} \int_{\partial\Omega} (\gamma_0 u)^2 ds - \int_{\Omega} F(x, u),$$

which satisfies the Palais-Smale condition by Lemma (3.2).

Using decomposition (2.4),  $H^1(\Omega) = X_1 \oplus X_2$ , we have

I)  $\Phi(u) \leq 0, \forall u \in X_1$ . Indeed, condition  $(f_4)$  implies that  $F(x, s) \geq \mu_1 \frac{s^2}{2}, \forall s \in \mathbb{R}, \forall x \in \overline{\Omega}$ . Then for each  $u \in X_1$

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{\alpha}{2} \int_{\partial\Omega} (\gamma_0 u)^2 ds - \int_{\Omega} F(x, u) \\ &= \frac{1}{2} \mu_1 \int_{\Omega} u^2 - \int_{\Omega} F(x, u) \quad (\text{by (2.5)}) \\ &\leq \frac{1}{2} \mu_1 \int_{\Omega} u^2 - \frac{1}{2} \mu_1 \int_{\Omega} u^2 = 0. \end{aligned}$$

II) There exists  $\rho_0 > 0$  such that  $\Phi(u) \geq 0, \forall u \in \partial B_{\rho_0}(0) \cap X_2$ . Condition  $(f_5)$  implies

$$F(x, s) \leq (\mu_2 - p) \frac{s^2}{2}, \quad \text{for } |s| < \varepsilon_0, \quad \text{and } \forall x \in \overline{\Omega}. \quad (4.1)$$

On the other hand, for  $|s| \geq \varepsilon_0$ , condition  $(f_1)$  implies the existence of a positive constant  $m_0$  such that

$$|f(x, s)| \leq m_0 |s|^\sigma, \quad \text{for } |s| \geq \varepsilon_0, \quad \text{and } \forall x \in \overline{\Omega}. \quad (4.2)$$

Now, (4.1) and (4.2) implies

$$F(x, s) \leq \begin{cases} (\mu_2 - p) \frac{s^2}{2}, & \text{if } |s| < \varepsilon_0, \\ m |s|^{\sigma+1}, & \text{if } |s| \geq \varepsilon_0, \end{cases} \quad (4.3)$$

for any constant  $m$  and  $x \in \overline{\Omega}$ .

Using (4.3), the variational characterization of  $\mu_2$ , the Sobolev Imbedding Theorem, and the norm  $\|u\|_k = \sqrt{(u, u)_k}$ , where the inner product  $(u, v)_k$  is

defined in (2.3), we get for  $u \in X_2$ ,

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \|u\|_k^2 - \frac{k}{2} \int_{\Omega} u^2 - \int_{\Omega} F(x, u) \\ &\geq \frac{1}{2} \|u\|_k^2 - \frac{k}{2} \int_{|u| < \varepsilon_0} u^2 - \frac{k}{2} \int_{|u| \geq \varepsilon_0} u^2 - \frac{1}{2} (\mu_2 - p) \\ &\quad \int_{|u| < \varepsilon_0} u^2 - m \int_{|u| \geq \varepsilon_0} |u|^{\sigma+1} \\ &= \frac{1}{2} \|u\|_k^2 - \frac{1}{2} (\mu_2 + k - p) \int_{|u| < \varepsilon_0} u^2 - \frac{k}{2} \int_{|u| \geq \varepsilon_0} u^2 - m \int_{|u| \geq \varepsilon_0} |u|^{\sigma+1} \\ &\geq \frac{1}{2} \|u\|_k^2 - \frac{1}{2} (\mu_2 + k - p) \int_{\Omega} u^2 - \tilde{c} \int_{|u| \geq \varepsilon_0} |u|^{\sigma+1} - m \int_{|u| \geq \varepsilon_0} |u|^{\sigma+1} \\ &\text{(where } \tilde{c} = k/2\varepsilon_0^{\sigma-1}\text{)} \\ &= \frac{1}{2} \|u\|_k^2 - \frac{1}{2} (\mu_2 + k - p) \int_{\Omega} u^2 - m_1 \int_{|u| \geq \varepsilon_0} |u|^{\sigma+1} \\ &\text{(} m_1 = \tilde{c} + m\text{)} \\ &\geq \frac{1}{2} \|u\|_k^2 - \frac{(\mu_2 + k - p)}{2(\mu_2 + k)} \|u\|_k^2 - m_1 \int_{\Omega} |u|^{\sigma+1} \\ &\geq \frac{1}{2} \left( \frac{p\delta}{\mu_2 + k} \right) \|u\|^2 - m_1 \int_{\Omega} |u|^{\sigma+1} \\ &= m_4 \|u\|^2 - m_3 \|u\|^{\sigma+1}. \end{aligned}$$

So,

$$\Phi(u) \geq \|u\| (m_4 \|u\| - m_3 \|u\|^{\sigma}), \quad u \in X_2. \tag{4.4}$$

Recalling that  $\sigma > 1$  by condition  $f_1$ , the function  $d : [0, +\infty) \rightarrow \mathbb{R}$  defined by  $d(\rho) = m_4 \rho - m_3 \rho^{\sigma}$  achieves its global maximum in  $\rho_0 = \left( \frac{m_4}{m_3 \sigma} \right)^{1/(\sigma-1)}$ . Then

$$\Phi(u) \geq \rho_0 d(\rho_0) = \rho_0^2 \left( 1 - \frac{1}{\sigma} \right) m_4 > 0, \quad \forall u \in \partial B_{\rho_0}(0) \cap X_2.$$

III) There exists  $e \in X_2 - \{0\}$  and a constant  $M$  such that

$$\Phi(v + te) \leq M, \quad \forall v \in X_1 \quad \text{and} \quad \forall t > 0.$$

If  $n \geq 2$  the space  $H^1(\Omega)$  is not contained in  $L^\infty(\Omega)$ . Let  $e \in X_2$  be a function which is unbounded from above, and  $\lambda_*$  the number given by

$$\lambda_* = \frac{\int_{\Omega} |\nabla e|^2 + \alpha \int_{\partial\Omega} (\gamma_0 e)^2 ds}{\int_{\Omega} e^2}. \tag{4.5}$$

Then,  $\mu_2 \leq \lambda_*$  and  $\mu_1 < \lambda_*$ . The value of  $\lambda_*$  does not change by substituting  $e$  by  $te$ , then we suppose that  $e$  satisfies the condition

$$2(\lambda_* - \lambda) \int_{\Omega} e^2 < -\delta^*(\mu_1 - \lambda), \tag{4.6}$$

where  $\lambda$  is the positive constant of condition  $(f_2)$ ,  $\delta^* = \frac{\delta_1}{\mu_1+k}$  and  $\delta_1$  is a positive constant such that  $\delta_1 \|v\|_{L^\infty}^2 \leq \|v\|_k^2, \forall v \in X_1$  where  $\|v\|_{L^\infty} = \sup_{x \in \bar{\Omega}} |v(x)|$ . Moreover  $\delta^*$  satisfies

$$\delta^* \|v\|_{L^\infty}^2 \leq \int_{\Omega} v^2, \quad \forall v \in X_1. \tag{4.7}$$

If  $v \in X_1$  we get

$$\begin{aligned} 0 = (v, e)_k &= \int_{\Omega} \nabla v \cdot \nabla e + k \int_{\Omega} ve + \alpha \int_{\partial\Omega} (\gamma_0 v)(\gamma_0 e) ds \\ &= (\mu_1 + k) \int_{\Omega} ve, \end{aligned}$$

where  $\mu_1 + k > 0$ , then  $\int_{\Omega} ve = 0$  and we obtain

$$\int_{\Omega} \nabla v \cdot \nabla e + \alpha \int_{\partial\Omega} (\gamma_0 v)(\gamma_0 e) ds = 0. \tag{4.8}$$

From  $(f_3)$  there exist  $m_5 > 0$  and  $s_1 \geq s_0$  such that

$$F(x, s) \geq \frac{\lambda}{2} s^2 + m_5 s^{1/\theta}, \quad \forall s \geq s_1 \quad \text{and} \quad \forall x \in \bar{\Omega}. \tag{4.9}$$

Conditions  $(f_0)$  and  $(f_2)$  imply the existence of a positive constant  $m_6 > 0$  such that

$$F(x, s) \geq \frac{\lambda}{2} s^2 - m_6 |s|, \quad \forall s \leq s_1 \quad \text{and} \quad \forall x \in \bar{\Omega}. \tag{4.10}$$

Now, if  $v \in X_1$  and  $t > 0$  then (4.5), (4.8), (4.9) and (4.10) yield

$$\begin{aligned} \Phi(v + te) &= \frac{1}{2} \int_{\Omega} |\nabla(v + te)|^2 + \frac{\alpha}{2} \int_{\partial\Omega} (\gamma_0(v + te))^2 - \int_{\Omega} F(x, v + te) \\ &= \frac{1}{2} \mu_1 \int_{\Omega} v^2 + \frac{t^2}{2} \lambda_* \int_{\Omega} e^2 - \int_{\Omega} F(x, v + te) \\ &= \frac{1}{2} \mu_1 \int_{\Omega} v^2 + \frac{t^2}{2} \lambda_* \int_{\Omega} e^2 - \int_{v(x)+te(x) \leq s_1} F(x, v + te) \\ &\quad - \int_{v(x)+te(x) > s_1} F(x, v + te) \\ &\leq \frac{1}{2} \mu_1 \int_{\Omega} v^2 + \frac{t^2}{2} \lambda_* \int_{\Omega} e^2 - \frac{\lambda}{2} \int_{\Omega} (v + te)^2 \\ &\quad + m_6 \int_{v(x)+te(x) \leq s_1} |v + te| - m_5 \int_{v(x)+te(x) > s_1} (v + te)^{1/\theta} \\ &\leq \frac{1}{2} (\mu_1 - \lambda) \int_{\Omega} v^2 + \frac{t^2}{2} (\lambda_* - \lambda) \int_{\Omega} e^2 + m_6 \int_{\Omega} |v| \\ &\quad + m_6 t \int_{\Omega} |e| - m_5 \int_{v(x)+te(x) > s_1} (v + te)^{1/\theta}. \end{aligned}$$

From (4.7) we have

$$\begin{aligned} \Phi(v + te) &\leq \frac{\delta^*}{2}(\mu_1 - \lambda)\|v\|_{L^\infty}^2 + m_6|\Omega| \|v\|_{L^\infty} + \frac{t^2}{2}(\lambda_* - \lambda) \int_{\Omega} e^2 \\ &\quad + m_6t \int_{\Omega} |e| - m_5 \int_{v(x)+te(x) > s_1} (v + te)^{1/\theta}. \end{aligned} \quad (4.11)$$

Observing (4.11) we have the following cases:

*Case 1.* If  $\lambda > \lambda_*$ , then

$$\begin{aligned} \Phi(v + te) &\leq \frac{\delta^*}{2}(\mu_1 - \lambda)\|v\|_{L^\infty}^2 + m_6|\Omega| \|v\|_{L^\infty} \\ &\quad + \frac{t^2}{2}(\lambda_* - \lambda) \int_{\Omega} e^2 + m_6t \int_{\Omega} |e|, \end{aligned}$$

where the coefficients of  $\|v\|_{L^\infty}^2$  and  $t^2$  are negative, therefore there exists a constant  $M_1 > 0$  such that  $\Phi(v + te) \leq M_1$ ,  $\forall v \in X_1$  and  $\forall t > 0$ .

*Case 2.* If  $0 < \lambda \leq \lambda_*$ , and  $v_0 = \min\{v(x) : x \in \overline{\Omega}\}$  then  $v_0 + t \leq s_1$  or  $v_0 + t > s_1$ . Let  $t \leq s_1 - v_0$ .

• If  $v_0 = 0$ , from (4.11) we have

$$\begin{aligned} \Phi(v + te) &\leq \frac{\delta^*}{2}(\mu_1 - \lambda)\|v\|_{L^\infty}^2 + m_6|\Omega| \|v\|_{L^\infty} \\ &\quad + \frac{s_1^2}{2}(\lambda_* - \lambda) \int_{\Omega} e^2 + m_6s_1 \int_{\Omega} |e|. \end{aligned}$$

Since the coefficient of  $\|v\|_{L^\infty}^2$  is negative, there is  $M_2 > 0$  such that  $\Phi(v + te) \leq M_2$ .

• If  $v_0 \neq 0$  then  $|v_0| \leq \|v\|_{L^\infty}$  and from (4.11) we have,

$$\begin{aligned} \Phi(v + te) &\leq \frac{\delta^*}{2}(\mu_1 - \lambda)\|v\|_{L^\infty}^2 + m_6|\Omega| \|v\|_{L^\infty} \\ &\quad + \frac{1}{2}(s_1 - v_0)^2(\lambda_* - \lambda) \int_{\Omega} e^2 + (s_1 - v_0)m_6 \int_{\Omega} |e|. \end{aligned}$$

Using the inequality

$$(s_1 - v_0)^2 \leq 2(s_1^2 + |v_0|^2), \quad (4.12)$$

and calling

$$c = m_6 \left( |\Omega| + \int_{\Omega} |e| \right), \quad (4.13)$$

we obtain

$$\begin{aligned} \Phi(v + te) &\leq \left[ \frac{\delta^*}{2}(\mu_1 - \lambda) + (\lambda_* - \lambda) \int_{\Omega} e^2 \right] \|v\|_{L^\infty}^2 \\ &\quad + c\|v\|_{L^\infty} + s_1^2(\lambda_* - \lambda) \int_{\Omega} e^2 + s_1m_6 \int_{\Omega} |e|. \end{aligned}$$

The coefficient of  $\|v\|_{L^\infty}^2$  is negative, therefore there exists  $M_3 > 0$  such that  $\Phi(v + te) \leq M_3$ .

• In the case  $t > s_1 - v_0$ , let  $\Omega_1 = \{x \in \Omega : e(x) > 1\}$  then  $|\Omega_1| > 0$ . Since the function  $e$  is not bounded from above, and  $\Omega_1 \subset \{x \in \Omega : v(x) + te(x) > s_1\}$  then (4.11) yields

$$\begin{aligned} \Phi(v + te) &\leq \frac{\delta^*}{2}(\mu_1 - \lambda)\|v\|_{L^\infty}^2 + m_6|\Omega| \|v\|_{L^\infty} + \frac{t^2}{2}(\lambda_* - \lambda) \int_{\Omega} e^2 \\ &\quad + m_6t \int_{\Omega} |e| - m_5|\Omega_1|(v_0 + t)^{1/\theta}. \end{aligned} \tag{4.14}$$

Setting  $v_0 + t = s$  we have

$$\begin{aligned} \Phi(v + te) &\leq \frac{\delta^*}{2}(\mu_1 - \lambda)\|v\|_{L^\infty}^2 + m_6|\Omega| \|v\|_{L^\infty} + \frac{(s - v_0)^2}{2}(\lambda_* - \lambda) \int_{\Omega} e^2 \\ &\quad + m_6(s - v_0) \int_{\Omega} |e| - m_5|\Omega_1|s^{1/\theta}. \end{aligned}$$

From  $(s - v_0)^2 \leq 2s^2 + 2\|v\|_{L^\infty}^2$  and (4.13) we obtain

$$\begin{aligned} \Phi(v + te) &\leq \left( \frac{\delta^*}{2}(\mu_1 - \lambda) + (\lambda_* - \lambda) \int_{\Omega} e^2 \right) \|v\|_{L^\infty}^2 + c\|v\|_{L^\infty} \\ &\quad + m_6s \int_{\Omega} |e| + s^2(\lambda_* - \lambda) \int_{\Omega} e^2 - m_5|\Omega_1|s^{1/\theta}. \end{aligned}$$

Since the coefficients of  $\|v\|_{L^\infty}^2$  and  $s^{1/\theta}$  are negative, then there exists  $M_4 > 0$  such that  $\Phi(v + te) \leq M_4$ . If  $M = \max\{M_2, M_3, M_4\}$ , then  $\Phi(v + te) \leq M \forall v \in X_1$ , and  $t > 0$ . □

In the following theorem we consider the case  $\alpha > 0$ , and we use the following condition  $(f_2^*)$ : the number  $\lambda$  of condition  $(f_2)$  is such that  $\lambda > \beta_1$ , and  $\lambda \neq \beta_j$ , for  $j = 2, 3, \dots$ , ( $\lambda$  is not an eigenvalue).

**Theorem 4.2.** *Suppose:  $n \geq 2$ ,  $\alpha > 0$ ,  $(f_0)$ ,  $(f_1)$ ,  $(f_3)$ ,  $(f_2^*)$ ,  $(S_2)$ , and the conditions:*

- $(f_4^*) \frac{f(x,s)}{s} \geq \beta_1, \forall s \in \mathbb{R} - \{0\}, \forall x \in \bar{\Omega}$ ,
- $(f_5^*)$  *there exist  $\varepsilon_0 > 0$  and  $\beta \in (\beta_1, \beta_2)$  such that*

$$\frac{f(x,s)}{s} \leq \beta \quad \forall s \in (-\varepsilon_0, \varepsilon_0) - \{0\} \quad \forall x \in \bar{\Omega}.$$

Then the problem

$$(\mathbb{P}_2) \begin{cases} -\Delta u &= f(x, u(x)), & \text{in } \Omega, \\ \gamma_1 u + \alpha \gamma_0 u &= 0, & \text{on } \partial\Omega, \end{cases}$$

has at least a nontrivial solution.

*Proof.* To prove the conditions of Theorem 2.1 we use the decomposition (2.7),  $H^1(\Omega) = Y_1 \oplus Y_2$ . The functional  $\Phi$  associated to problem  $(\mathbb{P}_2)$  is

$$\Phi(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{\alpha}{2} \int_{\partial\Omega} (\gamma_0 u)^2 ds - \int_{\Omega} F(x, u),$$

which satisfies the Palais-Smale condition by Lemma (3.2).

I)  $\Phi(u) \leq 0, \forall u \in Y_1$ . Let  $u \in Y_1$ , condition  $(f_4^*)$  and the equality (2.8), yields

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{\alpha}{2} \int_{\partial\Omega} (\gamma_0 u)^2 ds - \int_{\Omega} F(x, u) \\ &\leq \frac{1}{2} \beta_1 \int_{\Omega} u^2 - \frac{\beta_1}{2} \int_{\Omega} u^2 = 0. \end{aligned}$$

II) There exists  $\rho_0 > 0$  such that  $\Phi(u) \geq 0 \forall u \in \partial B_{\rho_0}(0) \cap Y_2$ . Condition  $(f_5^*)$  implies

$$F(x, s) \leq \frac{\beta}{2} s^2, \quad |s| \leq \varepsilon_0, \quad \forall x \in \overline{\Omega}. \quad (4.15)$$

On the other hand, for  $|s| \geq \varepsilon_0$  and  $x \in \overline{\Omega}$  condition  $(f_1)$  implies the existence of a positive constant  $m_0$  such that  $|f(x, s)| \leq m_0 |s|^\sigma$  and its integrals yield

$$|F(x, s)| \leq m |s|^{\sigma+1}, \quad \forall |s| \geq \varepsilon_0 \quad \text{and} \quad \forall x \in \overline{\Omega}, \quad (4.16)$$

for any constant  $m > 0$ . From (4.15) and (4.16), we have

$$F(x, s) \leq \frac{\beta}{2} s^2 + m |s|^{\sigma+1}, \quad \forall s \in \mathbb{R}, \quad \forall x \in \overline{\Omega}. \quad (4.17)$$

If  $u \in Y_2$  then

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{\alpha}{2} \int_{\partial\Omega} (\gamma_0 u)^2 ds - \int_{\Omega} F(x, u) \\ &= \frac{1}{2} \|u\|_*^2 - \int_{\Omega} F(x, u) \\ &\geq \frac{1}{2} \|u\|_*^2 - \frac{\beta}{2} \int_{\Omega} u^2 - m \int_{\Omega} |u|^{\sigma+1} \\ &\geq \frac{1}{2} \|u\|_*^2 - \frac{\beta}{2\beta_2} \|u\|_*^2 - mc \|u\|^{\sigma+1} \\ &\geq \frac{1}{2} \left(1 - \frac{\beta}{\beta_2}\right) d_1^{-1} \|u\|^2 - mc \|u\|^{\sigma+1}. \end{aligned}$$

Since  $\sigma + 1 > 2$ , there exist  $\rho_0 > 0$  and  $a > 0$  such that

$$\Phi(u) \geq a > 0, \quad \forall u \in \partial B_{\rho_0}(0) \cap Y_2.$$

III) There exist a function  $e \in Y_2 - \{0\}$  and a constant  $M > 0$  such that

$$\Phi(v + te) \leq M, \quad \forall v \in Y_1 \quad \text{and} \quad \forall t > 0.$$

Let  $e \in Y_2$  be a function which is unbounded from above and  $\lambda_*$  the number defined by

$$\lambda_* = \frac{\int_{\Omega} |\nabla e|^2 + \alpha \int_{\partial\Omega} (\gamma_0 e)^2 ds}{\int_{\Omega} e^2}, \quad (4.18)$$

then  $\beta_2 \leq \lambda_*$  and  $\lambda > \lambda_*$  or  $\lambda \leq \lambda_*$ . We suppose that  $e$  satisfies the condition

$$2(\lambda_* - \lambda) \int_{\Omega} e^2 < -\delta^* \left(1 - \frac{\lambda}{\beta_1}\right), \quad (4.19)$$

where  $\delta^* > 0$  is such that

$$\delta^* \|v\|_{L^\infty}^2 \leq \|v\|_*^2, \quad \forall v \in Y_1. \quad (4.20)$$

For  $v \in Y_1$  and  $k > 0$  we have,

$$(v, u)_k = (v, u)_* + k \int_{\Omega} vu = (\beta_1 + k) \int_{\Omega} vu, \quad \forall u \in H^1(\Omega).$$

Making  $u = e$  we get

$$0 = (v, e)_k = (v, e)_* + k \int_{\Omega} ve = (\beta_1 + k) \int_{\Omega} ve,$$

then

$$\int_{\Omega} ve = 0, \quad (4.21)$$

and

$$(v, e)_* = 0. \quad (4.22)$$

We also use

$$F(x, s) \geq \frac{\lambda}{2} s^2 + m_5 s^{1/\theta}, \quad \forall s \geq s_1, \quad \forall x \in \bar{\Omega} \quad \text{and} \quad (4.23)$$

$$F(x, s) \geq \frac{\lambda}{2} s^2 - m_6 |s|, \quad \forall s \leq s_1 \quad \text{and} \quad \forall x \in \bar{\Omega}. \quad (4.24)$$

If  $v \in Y_1$  and  $t > 0$  then using (4.18), (4.21), (4.22), (4.23) and (4.24), we obtain

$$\begin{aligned} \Phi(v + te) &= \frac{1}{2} \int_{\Omega} |\nabla(v + te)|^2 + \frac{\alpha}{2} \int_{\partial\Omega} (\gamma_0(v + te))^2 - \int_{\Omega} F(x, v + te) \\ &= \frac{1}{2} \beta_1 \int_{\Omega} v^2 + \frac{t^2}{2} \lambda_* \int_{\Omega} e^2 - \int_{v(x)+te(x) \leq s_1} F(x, v + te) \\ &\quad - \int_{v(x)+te(x) > s_1} F(x, v + te) \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{2}\beta_1 \int_{\Omega} v^2 + \frac{t^2}{2}\lambda_* \int_{\Omega} e^2 - \frac{\lambda}{2} \int_{v+te \leq s_1} (v+te)^2 \\
 &\quad + m_6 \int_{v+te \leq s_1} |v+te| \\
 &\quad - \frac{\lambda}{2} \int_{v+te > s_1} (v+te)^2 - m_5 \int_{v+te > s_1} (v+te)^{1/\theta} \\
 &\leq \frac{1}{2}(\beta_1 - \lambda) \int_{\Omega} v^2 + \frac{t^2}{2}(\lambda_* - \lambda) \int_{\Omega} e^2 + m_6 \int_{\Omega} |v+te| \\
 &\quad - m_5 \int_{v+te > s_1} (v+te)^{1/\theta} \\
 &\leq \frac{1}{2}(\beta_1 - \lambda) \int_{\Omega} v^2 + m_6 \int_{\Omega} |v| + \frac{t^2}{2}(\lambda_* - \lambda) \int_{\Omega} e^2 \\
 &\quad + tm_6 \int_{\Omega} |e| - m_5 \int_{v+te > s_1} (v+te)^{1/\theta}.
 \end{aligned}$$

Using  $\beta_1 \int_{\Omega} v^2 = \|v\|_*^2, \forall v \in Y_1$  and (4.20) we get

$$\begin{aligned}
 \Phi(v+te) &\leq \frac{\delta^*}{2} \left(1 - \frac{\lambda}{\beta_1}\right) \|v\|_{L^\infty}^2 + m_6|\Omega| \|v\|_{L^\infty} \\
 &\quad + \frac{t^2}{2}(\lambda_* - \lambda) \int_{\Omega} e^2 + tm_6 \int_{\Omega} |e| - m_5 \int_{v+te > s_1} (v+te)^{1/\theta}.
 \end{aligned} \tag{4.25}$$

Observing (4.25) we have the following cases:

**Case  $\lambda > \lambda_*$ .** In this case, the coefficients of  $\|v\|_{L^\infty}^2$  and  $t^2$  in (4.25) are negative, therefore, there exists a constant  $M_1^* > 0$  such that

$$\Phi(v+te) \leq M_1^* \quad \forall v \in Y_1 \quad \text{and} \quad \forall t > 0.$$

**Case  $0 < \lambda \leq \lambda_*$ .** If  $v_0 = \min \{v(x) : x \in \overline{\Omega}\}$  then  $v_0 + t \leq s_1$  or  $v_0 + t > s_1$ . Let  $t \leq s_1 - v_0$ .

If  $v_0 = 0$  then from (4.25) we have,

$$\begin{aligned}
 \Phi(v+te) &\leq \frac{\delta^*}{2} \left(1 - \frac{\lambda}{\beta_1}\right) \|v\|_{L^\infty}^2 + m_6|\Omega| \|v\|_{L^\infty} \\
 &\quad + \frac{s_1^2}{2}(\lambda_* - \lambda) \int_{\Omega} e^2 + s_1 m_6 \int_{\Omega} |e|,
 \end{aligned} \tag{4.26}$$

then there exists  $M_2^* > 0$  such that  $\Phi(v+te) \leq M_2^*$ .

If  $v_0 \neq 0$  then  $|v_0| \leq \|v\|_{L^\infty} \quad \forall v \in Y_1$ . From (4.25) we have

$$\begin{aligned}
 \Phi(v+te) &\leq \frac{\delta^*}{2} \left(1 - \frac{\lambda}{\beta_1}\right) \|v\|_{L^\infty}^2 + m_6|\Omega| \|v\|_{L^\infty} \\
 &\quad + \frac{1}{2}(s_1 - v_0)^2(\lambda_* - \lambda) \int_{\Omega} e^2 + (s_1 - v_0)m_6 \int_{\Omega} |e|.
 \end{aligned}$$



Using (4.12) and (4.13), we obtain

$$\begin{aligned} \Phi(v + te) &\leq \left[ \frac{\delta^*}{2} \left( 1 - \frac{\lambda}{\beta_1} \right) + (\lambda_* - \lambda) \int_{\Omega} e^2 \right] \|v\|_{L^\infty}^2 \\ &\quad + c\|v\|_{L^\infty} + s_1^2(\lambda_* - \lambda) \int_{\Omega} e^2 + s_1 m_6 \int_{\Omega} |e|. \end{aligned}$$

From (4.19) there exists  $M_3^* > 0$  such that  $\Phi(v + te) \leq M_3^*$ .

In the case  $v_0 + t > s_1$ , let  $\Omega_1 = \{x \in \Omega : e(x) > 1\}$ . Clearly,  $|\Omega_1| > 0$ . From (4.25) we have

$$\begin{aligned} \Phi(v + te) &\leq \frac{\delta^*}{2} \left( 1 - \frac{\lambda}{\beta_1} \right) \|v\|_{L^\infty}^2 m_6 |\Omega| \|v\|_{L^\infty} + \frac{t^2}{2} (\lambda_* - \lambda) \int_{\Omega} e^2 \\ &\quad + t m_6 \int_{\Omega} |e| - m_5 |\Omega_1| (v_0 + t)^{1/\theta}. \end{aligned} \quad (4.27)$$

Making  $v_0 + t = s$ , using (4.12) and (4.13), we obtain

$$\begin{aligned} \Phi(v + te) &\leq \left[ \frac{\delta^*}{2} \left( 1 - \frac{\lambda}{\beta_1} \right) + (\lambda_* - \lambda) \int_{\Omega} e^2 \right] \|v\|_{L^\infty}^2 + c\|v\|_{L^\infty} \\ &\quad + s m_6 \int_{\Omega} |e| + s^2 (\lambda_* - \lambda) \int_{\Omega} e^2 - m_5 |\Omega_1| s^{1/\theta}. \end{aligned}$$

and there exists  $M_4^* > 0$  such that  $\Phi(v + te) \leq M_4^*$ . If  $M^* = \max \{M_2^*, M_3^*, M_4^*\}$  then

$$\Phi(v + te) \leq M^*, \quad \forall v \in Y_1 \quad \text{and} \quad \forall t > 0.$$

✓

Recalling that for  $n = 1$  the space  $H^1(\Omega)$  is contained in  $L^\infty(\Omega)$ , and the fact that the above proofs require an unbounded function in  $H^1(\Omega)$ , we conclude that theorems 4.1 and 4.2 can not be applied to the case  $n = 1$ .

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