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Regularity of the solutions for a Robin problem and some applications

Regularidad de las soluciones para un problema de Robin y algunas aplicaciones

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Abstract. In this paper we study the regularity of the solutions for a Robin problem, with a nonlinear term with sub-critical growth respect to a variable. We establish the Sobolev space $H^1(\Omega)$ as the orthogonal sum of two subspaces, and we give the first step to demonstrate the existence of solutions of our problem.

Key words and phrases. Robin problems, trace operators, variational formulation, weak solutions, Sobolev spaces, bootstrapping, Green formula, orthogonal sum of subspace.

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Resumen. En este artículo estudiamos la regularidad de las soluciones de un problema de Robin, con término no lineal con crecimiento subcrítico respecto a una variable. Expresamos el espacio de Sobolev $H^1(\Omega)$ como la suma de dos subespacios dando el primer paso para la demostración de existencia de soluciones de nuestro problema.

Palabras y frases clave. Problemas de Robin, operador trazo, formulación variacional, soluciones débiles, espacios de Sobolev, argumento iterativo, formula de Green, suma ortogonal de subespacios.

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1. Introduction

We examine the regularity of the solutions to the problem

$$(\mathbb{P}) \qquad \begin{cases} u \in H^1(\Omega, -\Delta)\,, & \\ -\Delta u = f(x, u(x))\,, & \text{in} \quad \Omega\,, \\ \gamma_1 u + \alpha\gamma_0 u = 0\,, & \text{on} \quad \partial\Omega\,, \end{cases}$$

here $-\Delta$ is the Laplacian operator, Ω is a bounded domain in $\mathbb{R}^n (n \geq 2)$ simply connected and with smooth boundary $\partial\Omega$. The function $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, satisfies the conditions:

f$_0$) The function f is continuous.

f$_1$) There exists a constant $c > 0$ such that

$$|f(x, s)| \leq c\,(1 + |s|^\sigma)\,, \qquad \forall x \in \overline{\Omega} \quad \text{and} \quad \forall s \in \mathbb{R}\,,$$

where the exponent σ is a constant such that

$$\begin{array}{ll} 1 < \sigma < \frac{n+2}{n-2} & \text{if} \quad n \geq 3\,, \\ 1 < \sigma < \infty & \text{if} \quad n = 2\,. \end{array}$$

The boundary condition $\gamma_1 u + \alpha\gamma_0 u = 0$ involves the trace operators:

$$\gamma_0 : H^1(\Omega) \to H^{1/2}(\partial\Omega)\,, \qquad \text{and} \qquad \gamma_1 : H^1(\Omega, -\Delta) \to H^{-1/2}(\partial\Omega)\,,$$

where $H^1(\Omega, -\Delta) = \{u \in H^1(\Omega) : -\Delta u \in L^2(\Omega)\}$ with the norm

$$\|u\|_{H^1(\Omega,-\Delta)} = \left(\|u\|^2_{H^1(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}\right)^{1/2}\,,$$

for each $u \in H^1(\Omega, -\Delta)$, $\gamma_1 u \in H^{-1/2}(\partial\Omega)$, and $\gamma_0 u \in H^{1/2}(\partial\Omega)$. Identifying the element $\gamma_0 u$ with the functional $\gamma_0^* u \in H^{-1/2}(\partial\Omega)$ defined by

$$\langle \gamma_0^* u, w\rangle = \int_{\partial\Omega} (\gamma_0 u) w\, ds\,, \quad \forall w \in H^{1/2}(\partial\Omega)\,,$$

the boundary condition makes sense in $H^{-1/2}(\partial\Omega)$ (Theorem 2.3), and it appears in the functional of the problem ($\mathbb{P}$), see Theorems 3.1, 3.2 and 3.3 in section 3.

This work was motivated by Arcoya-Villegas [1], who proved the existence of nontrivial solution of the nonlinear Neumann's problem

$$\begin{cases} -\Delta u = f(x, u)\,, & \text{in} \quad \Omega\,, \\ \frac{\partial u}{\partial \eta} = 0\,, & \text{on} \quad \partial\Omega\,. \end{cases}$$

We study the existence of solutions of the problem ($\mathbb{P}$) in [4]. We write this paper to separate the regularity conditions of the additional existence conditions.

2. **Preliminary results**

In this section, we introduce some notations and known theorems to familiarize the reader with the boundary condition $\gamma_1 u+\alpha\gamma_0 u=0$. We also remember that $-\Delta$ is an isomorphism between some spaces, properly enunciated in Theorem 2.5. Using the Trace Theorem, Theorem 2.5 and the bootstrapping we prove that the weak solutions of Problem $(\mathbb{P})$ belong to the space $H^2(\Omega)$. (Theorem 3.2).

The norm

$$\|u\|^2=\int_\Omega u^2+\int_\Omega |\nabla u|^2\,, \tag{2.1}$$

is the usual norm of the Sobolev space $H^1(\Omega)$.

Theorem 2.1. *Suppose Ω bounded with boundary $\partial\Omega$ of class C^1. Then, there exists a continuous and linear function $\gamma_0 : H^1(\Omega)\to L^2(\partial\Omega)$ such that:*

a) $\gamma_0 u = u|_{\partial\Omega}\,,\quad$ *if* $u\in W^{1,2}(\Omega)\cap C\left(\overline{\Omega}\right)$,

b) $\|\gamma_0 u\|_{L^2(\partial\Omega)}\le c_1\|u\|\,,\ \forall u\in H^1(\Omega)\,,$

here the constant c_1 depends on Ω. The value $\gamma_0 u$ is called trace of u on $\partial\Omega$.

Proof. See [7, Theorem 1, p. 258]. ☑

Theorem 2.2 (Trace Theorem). *Let $\Omega\subset\mathbb{R}^n$ be open, bounded, and with smooth boundary $\partial\Omega$. Then the trace operators γ_j can be extended to continuous linear operators, mapping $H^m(\Omega)$* onto *$H^{m-j-1/2}(\partial\Omega)$, for $0\le j\le m-1$. Moreover, the operator $\gamma=(\gamma_0,\gamma_1,\dots,\gamma_j,\dots,\gamma_{m-1})$ maps $H^m(\Omega)$ onto the product space $\prod_{j=0}^{m-1}H^{m-j-1/2}(\partial\Omega)$. Finally, the space $H_0^m(\Omega)$ is the Kernel of the operator γ.*

Proof. See [2, Theorem 3.1, p. 189] or [5, Theorem 5, p. 905]. ☑

Theorem 2.3 (Green Formula). *There exists a* unique *operator γ_1 mapping $H^1(\Omega,-\Delta)$ into $H^{-1/2}(\partial\Omega)$ such that Green Formula holds:*

$$\langle\gamma_1 u,\gamma_0 v\rangle=\int_\Omega v\Delta u+\int_\Omega \nabla u\cdot\nabla v\,,\quad u\in H^1(\Omega,-\Delta)\,,\quad v\in H^1(\Omega). \tag{2.2}$$

Remark. The space $C^\infty\left(\overline{\Omega}\right)$ is dense in $H^1(\Omega,-\Delta)$.

Proof. See [2, Theorem 2.1, p. 174]. ☑

Theorem 2.4 (Compactness of the trace operator). *Let $\Omega\subset\mathbb{R}^n$ be an open set, bounded, and with boundary $C^{0,1}$, $m\ge 1$ an integer and $1\le p<+\infty$. Then the next claims are true :*

a) If $mp<n$ and $1\le q<\frac{p(n-1)}{n-mp}$, then there exists a unique mapping $\gamma_0 : W^{m,p}(\Omega)\to L^q(\partial\Omega)$, linear and continuous, such that if $u\in C\left(\overline{\Omega}\right)$ then $\gamma_0 u=u|_{\partial\Omega}$. If $p>1$ then γ_0 is compact.

b) If $mp=n$, then the affirmation (a) is valid for every $q\ge 1$.

c) *If $mp > n$, the trace $\gamma_0 u$ of $u \in W^{m,p}(\Omega) \subset C\left(\overline{\Omega}\right)$ is the classic restriction $u|_{\partial\Omega}$.*

Proof. See [10, Theorem 6.2 p. 107], [8, Theorem 6.10.5] or [3]. ☑

For the reader's convenience, we recall some notation and definitions of *"normality"* and *"covering"*.

If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ an $n-tuple$ of nonnegative integers α_j, then x^α denotes the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, which has degree $|\alpha| = \sum_{j=1}^n \alpha_j$. Similarly, if $D_j = \frac{\partial}{\partial x_j}$ for $1 \leq j \leq n$, then $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ is the differential operator of order $|\alpha|$.

Let A be a partial differential linear operator of order $2m$, defined by

$$Au = \sum_{|k|,|h| \leq m} (-1)^{|k|} D^k \left(a_{kh} D^h u\right) , \tag{2.3}$$

where $a_{kh} \in \mathcal{C}^\infty\left(\overline{\Omega}\right)$, with real values.

Let $B_0, \ldots, B_{m-1}$ be m boundary operators defined by

$$B_j u = \sum_{|h| \leq m_j} b_{jh} D^h u\, , \quad j = 0, 1, \ldots, m-1\, , \tag{2.4}$$

where $0 \leq m_j \leq 2m-1$ and the functions b_{jh} are infinitely differential functions over $\partial\Omega$.

Definition 2.1. We say that the set $\{B_j\}_{j=0}^{m-1}$ is *normal* according to Aronszajn and Milgram, if for $j \neq i$, $m_j \neq m_i$, $x \in \partial\Omega$, and normal vector $\eta \neq 0$ to $\partial\Omega$ in x, then

$$Q_j(x, \eta) = \sum_{|h| = m_j} b_{jh} \eta^h \neq 0\, , \quad j = 0\, , 1, \ldots, m-1.$$

Definition 2.2. The set $\{B_j\}_{j=0}^{m-1}$ *"covers"* the operator A defined in (2.3), if for each $x \in \partial\Omega$ and every pair of real non-null vectors ξ, ξ' tangent and normal to $\partial\Omega$ in x respectively, then the polynomial in the variable τ,

$$P\left(x, \xi + \tau\xi'\right) = \sum_{|k|=|h|=m_j} a_{kh} \left(\xi + \tau\xi'\right)^{k+h} ,$$

has m roots: $\tau_1(\xi, \xi'), \cdots, \tau_m(\xi, \xi')$, with positive imaginary parts, and the m polynomials in τ

$$Q_j(\tau) := Q_j(x, \xi + \tau\xi') = \sum_{|h| = m_j} b_{j,h} (\xi + \tau\xi')^h\, ,$$

are linearly independent module the polynomial $\prod_{i=1}^m (\tau - \tau_i(\xi, \xi'))$.

Definition 2.3. We say that the set of operators

$$\{A, B_0, B_1, \cdots, B_{m-1}\}\, ,$$

satisfies condition (NR) if:

a) The set of boundary operators $B = \{B_j : j = 0, 1, \cdots, m-1\}$ is *n*ormal according to Aronszajn and Milgram, and
b) The set B "*c*overs" operator A.

Proposition 2.1. *The operators*

$$-\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\,, \quad \textit{and the boundary operator} \quad B_0 u = \sum_{j=1}^{n} \eta_j(x) \frac{\partial u}{\partial x_j}\,,$$

where $\eta(x) = (\eta_1, \cdots, \eta_n)$ *is a unit normal vector field over* $\partial\Omega$ *and external to* Ω*, satisfies Condition* (NR)*.*

Proof. a) Let $x \in \partial\Omega$ and $v(x) = (v_1, \cdots, v_N) \neq 0$ a normal vector field to $\partial\Omega$,

$$Q_0(x, \eta) = \sum_{j=1}^{N} \eta_j(x) \cdot v_j(x) = \eta \cdot v \neq 0\,,$$

therefore, the operator $\{B_0\}$ is normal according to Aronszajn and Milgram.

b) Now, the operator B_0 *"covers"* the operator $-\Delta$. In fact, let $\xi = (\xi_1, \cdots, \xi_n)$ and $\xi' = (\xi_1', \cdots, \xi_n')$ be tangent and normal vectors respectively to $\partial\Omega$ at the point x. By the ellipticity of the operator $-\Delta$ we have

$$P(x, \xi + \tau\xi') = -\|\xi + \tau\xi'\|^2 = -\|\xi\|^2 - \tau^2\|\xi'\|^2\,,$$

then $P(x, \xi + \tau\xi')$ has 2 roots: $\tau_1 = \frac{\|\xi\|}{\|\xi'\|} i$, and $\tau_2 = -\tau_1$. On the other hand, the polynomial in τ

$$Q_0(x, \xi + \tau\xi') = \sum_{j=1}^{N} n_j(x)(\xi_j + \tau\xi_j') = (\eta(x) \cdot \xi')\tau = \|\eta\|\|\xi'\|\tau\,,$$

is not multiple of the polynomial $\tau - \tau_1$. ☑

Let $1 < p < \infty$, $r \geq 0$ and $\eta(x) = (\eta_1, \cdots, \eta_n)$ be a real vector normal field over $\partial\Omega$ and external to Ω. We define :

$$N = \left\{ u \in W^{2,p}(\Omega) : -\Delta u = 0\,, \quad B_0 u = \sum_{j=1}^{n} \eta_j(x) \frac{\partial u}{\partial x_j} = 0 \right\}\,,$$
$$N^* = \left\{ u \in W^{2,p'}(\Omega) : (-\Delta)^* u = 0V, \quad B_0^* u = 0 \right\}\,,$$

where $p' = \frac{p}{p-1}$ and $((-\Delta)^*, B_0^*)$ is the formal adjoint operator of $(-\Delta, B_0)$. It is easy to see that $(-\Delta, B_0)$ is self-adjoint. Spaces N and N^* are subspaces of $C^\infty\left(\overline{\Omega}\right)$, and thus they neither depend on p nor on p'. If Ω is simply connected then $N = N^*$ is the space of the constants. In this work, the set Ω is simply connected.

We will consider the sets:

$$\begin{aligned} W_{B_0}^{2+r,p}(\Omega) &= \left\{ u \in W^{2+r,p}(\Omega) : B_0 u = 0 \right\}, \qquad \text{and} \\ \{W^{r,p}(\Omega); N^*\} &= \left\{ u \in W^{r,p}(\Omega) : \textstyle\int_\Omega u(x)dx = 0 \right\}. \end{aligned}$$

Theorem 2.5. *The operator* $-\Delta$ *is an isomorphism on* $\{W^{r,p}(\Omega); N^*\}$ *of* $W^{2+r,p}_{B_0}(\Omega)/N$ *for all real* $r \geq 0$ *and* $1 < p < +\infty$.

Proof. See [9, Theorem 3.1]. ☑

3. Regularity results

In Theorem 3.1 we establish the derivative of the functional $\Phi : H^1(\Omega) \to \mathbb{R}$, corresponding to our problem $(\mathbb{P})$. In Theorem 3.2 we show that the critical points of Φ belong to the space $H^2(\Omega)$. Finally in Theorem 3.3 we obtain the variational formulation of the problem $(\mathbb{P})$. Thus Theorem 3.2 is a regularity result for weak solutions of the problem $(\mathbb{P})$.

Theorem 3.1. *The functional* $\Phi : H^1(\Omega) \to \mathbb{R}$, *defined by*

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\alpha}{2}\int_{\partial\Omega}(\gamma_0 u)^2 ds - \int_\Omega F(x,u)\,,$$

where $\alpha \in \mathbb{R}$, $F(x,s) = \int_0^s f(x,t)\,dt$ *belongs to class* $C^1(H^1(\Omega),\mathbb{R})$ *and its derivative is*

$$\langle \Phi'(u), v\rangle = \int_\Omega \nabla u \cdot \nabla v + \alpha\int_{\partial\Omega}(\gamma_0 u)(\gamma_0 v) ds - \int_\Omega f(x,u)v\,, \quad \forall u, v \in H^1(\Omega)\,. \tag{3.1}$$

Proof. If

$$\Phi_1(u) = \frac{1}{2}\int_{\partial\Omega}(\gamma_0 u)^2 ds\,,$$

$$\frac{\Phi_1(u+tv) - \Phi_1(u)}{t} = \int_{\partial\Omega}\left\{\gamma_0 u . \gamma_0 v + \frac{t}{2}(\gamma_0 v)^2\right\} ds\,,$$

then,

$$\langle \Phi_1'(u), v\rangle = \lim_{t\to 0}\frac{\Phi_1(u+tv) - \Phi_1(u)}{t} = \int_{\partial\Omega}(\gamma_0 u)(\gamma_0 v) ds\,.$$

Using Theorem 2.1, the Gateaux derivative of Φ_1 is continuous; indeed,

$$|\,\langle \Phi_1'(u), v\rangle\,| \leq \|\gamma_0 u\|_{L^2(\partial\Omega)}\|\gamma_0 v\|_{L^2(\partial\Omega)} \leq C\|u\|\,\|v\|\,.$$

Hence, $\Phi_1 \in C^1((\Omega),\mathbb{R})$. It is well known that the functional

$$\Phi_2(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x,u)\,,$$

belongs to the class $C^1(H^1(\Omega),\mathbb{R})$ whose derivative is

$$\langle \Phi_2'(u), v\rangle = \int_\Omega \nabla u . \nabla v - \int_\Omega f(x,u)v.$$

Then we obtain (3.1). ☑

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded and simply connected domain with smooth boundary $\partial\Omega$, $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ a function satisfying (f_0) and (f_1), and α a real number.*

If $u \in H^1(\Omega)$ satisfies

$$\langle \Phi'(u), v\rangle = \int_\Omega \nabla u \cdot \nabla v + \alpha \int_{\partial\Omega} (\gamma_0 u)(\gamma_0 v) ds - \int_\Omega f(x,u)v = 0\,, \tag{3.2}$$

for all $v \in H^1(\Omega)$, then $u \in H^2(\Omega)$.

Proof. Since $\gamma_1 : H^2(\Omega) \to H^{1/2}(\partial\Omega)$ is onto (Trace Theorem), and $-\alpha\gamma_0 u \in H^{1/2}(\partial\Omega)$, there exists $w \in H^2(\Omega)$ such that

$$\frac{\partial w}{\partial \eta} = -\alpha\gamma_0 u\,. \tag{3.3}$$

From (3.2), (3.3) and the Green formula we obtain

$$\int_\Omega \nabla h \cdot \nabla v = \int_\Omega gv\,, \quad \forall v \in H^1(\Omega)\,, \tag{3.4}$$

where

$$h = u - w\,, \quad \textit{and} \quad g(x) = f(x,u) + \Delta w(x)\,. \tag{3.5}$$

(i) First we prove that

$$g \in \left\{W^{0,p_1}(\Omega); N^*\right\}\,, \quad \text{where} \quad 1 < p_1 \leq 2\,. \tag{3.6}$$

Using Condition (f_1) in the cases:

a) If $n = 2$, then

$$f \in L^2(\Omega)\,, \quad \text{and so} \quad g \in L^2(\Omega)\,. \tag{3.7}$$

b) If $n > 2$, taking into account that

$$1 < \sigma < \frac{n+2}{n-2}\,, \quad \text{and} \quad 2^* = \frac{2n}{n-2}\,.$$

If $p_1 = \frac{2^*}{\sigma}$ then

$$1 < \frac{2n}{n+2} < p_1 < \frac{2n}{n-2}\,. \tag{3.8}$$

Consequently

$$\int_\Omega |f(x,u)|^{p_1} \leq C \int_\Omega \left(1 + |u|^{2^*}\right) < +\infty\,, \tag{3.9}$$

thus, $f \in L^{p_1}(\Omega)$, then $g \in L^{p_1}(\Omega)$ for $1 < p_1 \leq 2$. From (3.4) with $v = 1$ over $\overline{\Omega}$, we have

$$\int_\Omega g(x)dx = 0\,.$$

Thus, $g \in \left\{W^{0,p_1}(\Omega); N^*\right\}$.

(ii) Now, we will prove that

$$u \in W^{2,p_1}(\Omega)\,. \tag{3.10}$$

By virtue of Theorem 2.5 , there exists a unique $\widetilde{h} \in W^{2,p_1}_{B_0}(\Omega)/N$ such that

$$\begin{cases} -\Delta\widetilde{h} &= g\,, \quad \text{in} \quad \Omega\,, \\ \frac{\partial \widetilde{h}}{\partial \eta} &= 0\,, \quad \text{on} \quad \partial\Omega\,. \end{cases} \tag{3.11}$$

In the case $n \geq 3$, we have $\frac{1}{p_1'} = 1 - \frac{1}{p_1} > 1 - \frac{n+2}{2n} = \frac{1}{2^*}$, then $p_1' < 2^*$, and hence $H^1(\Omega) \subset L^{p_1'}(\Omega)$. If $n = 2$, $H^1(\Omega) \subset L^{p_1'}(\Omega)$.

From (3.11) we get

$$-\int_\Omega v\Delta\widetilde{h} = \int_\Omega vg\,, \quad \forall v \in H^1(\Omega)\,. \tag{3.12}$$

Integrating by parts, the first integral of (3.12) gives

$$\int_\Omega \nabla\widetilde{h}\cdot\nabla v = \int_\Omega gv\,, \quad \forall v \in H^1(\Omega)\,. \tag{3.13}$$

Subtracting (3.13) from (3.4) we obtain

$$\int_\Omega \nabla(h - \widetilde{h})\cdot\nabla v = 0\,, \quad \forall v \in H^1(\Omega)\,. \tag{3.14}$$

Let us see that $\widetilde{h} \in W^{1,2}(\Omega)$. Since $\widetilde{h} \in W^{2,p_1(\Omega)}$, if $p_1 = 2$ then $\widetilde{h} \in W^{1,2}(\Omega)$; if $p_1 < 2$, then we have the cases: $n = 2$ and $n > 2$. If $n = 2$, given that $p_1 > 1$ then $2 < \frac{2p_1}{2-p_1}$; if $n > 2$, as $p_1 > \frac{2n}{n+2}$, then $2 < \frac{np_1}{n-p_1}$. Therefore, in both cases $W^{2,p_1}(\Omega) \subset W^{1,2}(\Omega)$, then $\widetilde{h} \in W^{1,2}(\Omega)$.

Making $v = h - \widetilde{h}$ in (3.14) we obtain $\int_\Omega |\nabla v|^2 dx = 0$. By the Poincaré inequality we have that $v(x) = v_\Omega = \frac{1}{|\Omega|}\int_\Omega v\,dx$, $x \in \Omega$. Then $h(x) = \widetilde{h}(x) + v_\Omega$, thus $D^\alpha h = D^\alpha \widetilde{h}$ for any index α, with $|\alpha| \leq 2$. Since $\widetilde{h} \in W^{2,p_1}(\Omega)$, then $h \in W^{2,p_1}(\Omega)$, therefore $u \in W^{2,p_1}(\Omega)$.

(iii) Finally we will prove that

$$u \in W^{2,2}(\Omega). \tag{3.15}$$

If $n = 2$, by (3.7) and case (i), $g \in \left\{W^{0,2}(\Omega); N^*\right\}$, and by case (ii), $u \in W^{2,2}(\Omega)$. If $n > 2$, we have the following cases:

a) $2p_1 > n$,
b) $2p_1 = n$,
c) $2p_1 < n$.

In case a), $W^{2,p_1}(\Omega) \subset C\left(\overline{\Omega}\right) \subset L^\infty(\Omega)$, thus $u \in L^q(\Omega), \forall q \in [1, +\infty)$, and on the other hand, we have

$$\int_\Omega |f(x,u)|^2 \leq c\int_\Omega \left(1 + |u|^{2\sigma}\right) < +\infty\,, \tag{3.16}$$

then $f \in L^2(\Omega)$, $g \in L^2(\Omega)$, thus by cases (i) and (ii), $u \in W^{2,2}(\Omega)$.

In case b), $W^{2,p_1}(\Omega) \subset L^q(\Omega)$, $\forall q \in [p_1, \infty)$, $2\sigma > 2 > p_1$ and by (3.16), then $f \in L^2(\Omega)$. Similarly to the last case $u \in W^{2,2}(\Omega)$.

In case c), we use bootstrapping. As $W^{2,p_1}(\Omega) \subset L^{p_1^*}(\Omega)$, where $p_1^* = \frac{np_1}{n-2p_1}$, thus $u \in L^{p_1^*}(\Omega)$. Let $p_2 = \frac{p_1^*}{\sigma}$ be, and

$$\int_\Omega |f(x,u)|^{p_2} \leq \widetilde{c} \int_\Omega \left(1 + |u|^{p_1^*}\right) < +\infty\,.$$

Then, $f \in L^{p_2}(\Omega)$, $g \in L^{p_2}(\Omega)$, thus $u \in W^{2,p_2}(\Omega)$. If $p_2 < 2$ and $2p_2 < n$ then $u \in L^{p_2^*}(\Omega)$, $p_2^* = \frac{np_2}{n-2p_2}$. Considering $p_3 = \frac{p_2^*}{\sigma}$ then $f \in L^{p_3}$, $g \in L^{p_3}(\Omega)$, thus $u \in W^{2,p_3}(\Omega)$. Therefore, we obtain the sequence $\{p_j\}$, such that

$$1 < p_1 < p_2 < \cdots < p_s\,,$$

where $p_s \geq 2$. In fact, since $p_1 > \frac{2n}{n+2}$, there exists $\varepsilon > 0$ such that $p_1 = \frac{2n}{n+2}(1+\varepsilon)$ and we obtain

$$\frac{p_2}{p_1} = \frac{p_1^*/\sigma}{2^*/\sigma} = \frac{(n-2)(1+\varepsilon)}{n-2-4\varepsilon} > 1 + \varepsilon\,,$$

then $p_2 - p_1 > \varepsilon$. If $1 < p_1 < p_2 < \cdots < p_{i-1} < p_i$, where $\frac{p_j}{p_{j-1}} > 1+\varepsilon$ for $j = 2, 3, \cdots, i$, then

$$\frac{p_{i+1}}{p_i} = \frac{p_i^*}{p_{i-1}^*} = \frac{p_i(n-2p_{i-1})}{p_{i-1}(n-2p_i)} > \frac{p_i}{p_{i-1}} > 1 + \varepsilon\,.$$

Hence $\{p_j\}$ is an strictly increasing sequence, therefore, there exists $p_s \geq 2$ and $u \in L^{p_{s-1}^*}(\Omega)$, then $f \in L^{p_s}(\Omega)$, $g \in L^{p_s}(\Omega)$ and by case (ii) $u \in W^{2,p_s}(\Omega)$. Therefore $u \in W^{2,2}(\Omega)$. ☑

Theorem 3.3. *Based on the conditions in Theorem 3.2 on Ω, f and α, the problems* ($\mathbb{P}$) *and* (Q) *are equivalent:*

$$(\mathbb{P}) \quad \begin{cases} \text{i)} & u \in H^1(\Omega, -\Delta)\,, & \\ \text{ii)} & -\Delta u = f(x, u(x))\,, & x \in \Omega\,, \\ \text{iii)} & \gamma_1 u + \alpha\gamma_0 u = 0\,, & \text{on } \partial\Omega\,, \end{cases}$$

and

$$(Q) \quad \begin{cases} \text{i)} & u \in H^1(\Omega)\,, \\ \text{ii)} & \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f(x,u)v - \alpha \int_{\partial\Omega} (\gamma_0 u)(\gamma_0 v) ds\,, \quad \forall v \in H^1(\Omega)\,. \end{cases}$$

Proof. Let u be a solution to problem ($\mathbb{P}$), then

$$\int_\Omega (-\Delta u) v = \int_\Omega f(x, u(x)) v\,, \quad \forall v \in H^1(\Omega)\,.$$

Using the Green Formula (2.2) we obtain

$$\int_\Omega \nabla u \cdot \nabla v - \langle \gamma_1 u, \gamma_0 v\rangle = \int_\Omega f(x,u) v dx\,, \quad \forall v \in H^1(\Omega)\,.$$

Now, taking the boundary condition, u is a solution to problem (Q).

If u is a solution to the problem (Q) then

$$\int_\Omega \nabla u.\nabla v = \int_\Omega f(x,u)v - \alpha\int_{\partial\Omega}(\gamma_0 u)(\gamma_0 v)ds\,, \quad \forall v \in H^1(\Omega)\,, \tag{3.17}$$

thus

$$\int_\Omega \nabla u.\nabla v = \int_\Omega f(x,u)v\,, \quad \forall v \in C_c^\infty(\Omega)\,. \tag{3.18}$$

By Theorem 3.2 $u \in H^2(\Omega)$, and using the Green formula, we obtain

$$-\int_\Omega v\Delta u = \int_\Omega f(x,u)v\,, \quad \forall v \in C_c^\infty(\Omega)\,,$$

then $-\Delta u = f(x,u(x))$.

Changing f for $-\Delta u$ in (3.17) and using the Green formula, we obtain

$$\gamma_1 u + \alpha\gamma_0 u = 0\,.$$

☑

4. Applications

The purpose of this section is to give the first step for the proof of the existence of solutions of our the problem $(\mathbb{P})$ (see [4]), which consists in expressing the Sobolev space $H^1(\Omega)$ as the orthogonal sum of two subspaces. To achieve this goal we consider the problem $(\mathbb{P})$ with $f(x,u(x)) = \mu u(x)$ where μ is a parameter. Making use of spectral analysis of compact symmetric operator, we obtain the existence of eigenvalues and by virtue of theorems 3.2 and 3.3, we establish some properties of the first eigenvalue and its associated eigenfunctions.

4.1. Existence of eigenvalues.

Theorem 4.1. *Let α be a real parameter, $\alpha \neq 0$. The eigenvalue problem*

$$\left\{\begin{array}{rcll} -\Delta u & = & \mu u\,, & \text{in} \quad \Omega\,, \\ \gamma_1 u + \alpha\gamma_0 u & = & 0\,, & \text{on} \quad \partial\Omega\,, \end{array}\right. \tag{4.1.1}$$

has an increasing sequence of eigenvalues. In the case $\alpha < 0$ the first eigenvalue is negative, and in the case $\alpha > 0$ the first eigenvalue is positive.

Proof. 1) Case $\alpha < 0$. In this case, to establish the existence of the eigenvalues of the problem (4.1.1) first we introduce an scalar product $(.,.)_k$ in $H^1(\Omega)$. To make this, first we have that given $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that

$$\alpha\int_{\partial\Omega}(\gamma_0 u)^2 ds \geq \alpha\varepsilon\int_\Omega |\nabla u|^2 + \alpha C(\varepsilon)\int_\Omega u^2\,, \quad \forall u \in H^1(\Omega)\,. \tag{4.1.2}$$

Choosing k and ε such that

$$0 < \varepsilon < -\frac{1}{\alpha} \quad \text{and} \quad k > -\alpha C(\varepsilon)\,, \tag{4.1.3}$$

we define the symmetric bilinear form in $H^1(\Omega)$ in this way for $u, v \in H^1(\Omega)$:

$$a_k[u,v] = \int_\Omega \nabla u \cdot \nabla v + \alpha \int_{\partial\Omega} (\gamma_0 u)(\gamma_0 v) ds + k \int_\Omega uv \,. \tag{4.1.4}$$

This defines an scalar product and we denote it by $(.,.)_k$. In fact, the form $a_k[.,.]$ is coercive: using (4.1.2) we have $a_k[u,u] \geq \delta \|u\|^2$, $\forall u \in H^1(\Omega)$, with $\delta = min\{1+\alpha\varepsilon,\ k+\alpha C(\varepsilon)\}$. On the other hand, using the inequality $\|\gamma_0 u\|_{L^2(\partial\Omega)} \leq c_1\|u\|$, $\forall u \in H^1(\Omega)$, and the Schwarz inequality we have $|a_k[u,v]| \leq c_3\|u\|\ \|v\|$, where $c_3 = 1 + |\alpha| c_1^2 + k$, i.e. $a_k[.,.]$ is continuous.

We define for each $u \in H^1(\Omega)$ a norm $\|u\|_k = \sqrt{(u,u)_k}$. Then

$$\delta \|u\|^2 \leq \|u\|_k^2 \leq c_3 \|u\|^2 \,, \quad \forall u \in H^1(\Omega)\,, \tag{4.1.5}$$

i.e., the norm $\|\cdot\|_k$ is equivalent to the usual norm of $H^1(\Omega)$.

With our k chosen in (4.1.3), the problem (4.1.1) is equivalent to

$$\begin{cases} -\Delta u + ku &= (\mu + k)u\,, \quad \text{in} \quad \Omega\,, \\ \gamma_1 u + \alpha\gamma_0 u &= 0\,, \qquad\qquad \text{on} \quad \partial\Omega\,. \end{cases} \tag{4.1.6}$$

We prove that the eigenvalues $\mu + k$ are positive. Let u be a weak solution of (4.1.6), then

$$(u,v)_k = (\mu + k)\int_\Omega uv\,, \quad \forall v \in H^1(\Omega)\,. \tag{4.1.7}$$

If moreover, $u \neq 0$ and $v = u$ in (4.1.7) we get that $\mu + k > 0$.

For fixed $u \in H^1(\Omega)$, the map $v \to \int_\Omega uv$ is a bounded linear functional in $H^1(\Omega)$,

$$\left|\int_\Omega uv\right| \leq \int_\Omega |u|\ |v| \leq \|u\|\ \|v\| \leq \frac{1}{\delta}\|u\|_k \|v\|_k\,, \quad \forall v \in H^1(\Omega)\,. \tag{4.1.8}$$

So, by the Riesz-Frechet Representation Theorem, there is an element in $H^1(\Omega)$ that we denote by Tu, such that

$$\int_\Omega uv = (Tu, v)_k\,, \quad \forall v \in H^1(\Omega)\,. \tag{4.1.9}$$

Thus (4.1.9) defines the operator $T : H^1(\Omega) \to H^1(\Omega)$ that is linear and symmetric: $(Tu,v)_k = (u,Tv)_k$, $\forall u,v \in H^1(\Omega)$. By virtue of (4.1.8) and (4.1.9) we have $\|Tu\|_k \leq \frac{1}{\delta}\|u\|_k$, $\forall u \in H^1(\Omega)$, i.e. T is bounded. Now, we are going to prove that the operator T is compact. Let $\{u_n\}$ be a bounded sequence in $H^1(\Omega)$. By the theorem on compact imbedding of $H^1(\Omega)$ in $L^2(\Omega)$ there exists a subsequence $\{u_{n_j}\}$ and a element u in $H^1(\Omega)$ such that

$$u_{n_j} \rightharpoonup u \quad \text{in} \quad H^1(\Omega)\,, \quad \text{and} \quad u_{n_j} \to u \quad \text{in} \quad L^2(\Omega)\,, \quad \text{when} \quad j \to \infty\,.$$

By (4.1.9) and the definition of the operator T, we have

$$\begin{aligned}\left\|Tu_{n_j} - Tu\right\|_k^2 &= \int_\Omega (u_{n_j} - u)T(u_{n_j} - u)\\ &\le \|u_{n_j} - u\|_{L^2(\Omega)}\|T(u_{n_j} - u)\|_{L^2(\Omega)}\\ &\le \|u_{n_j} - u\|_{L^2(\Omega)}\|T(u_{n_j} - u)\|\\ &\le \frac{1}{\sqrt{\delta}}\|u_{n_j} - u\|_{L^2(\Omega)}\|T(u_{n_j}) - T(u)\|_k\,.\end{aligned}$$

Then

$$\lim_{j\to\infty}\left\|Tu_{n_j} - Tu\right\|_k = 0\,.$$

Let us see that the eigenvalues of problem (4.1.6) are the inverse eigenvalues of the operator T. In fact, if u is a weak solution of (4.1.6), then by (4.1.7) and (4.1.9) we obtain

$$Tu = \left(\frac{1}{\mu + k}\right)u. \tag{4.1.10}$$

Reciprocally, if u satisfies (4.1.10) then u verifies (4.1.7) by (4.1.9).

Our next step is to establish the existence of eigenvalues of operator T. Since,

$$\frac{1}{\mu_1 + k} := \sup\left\{(Tu, u)_k : \|u\|_k = 1\right\} > 0\,, \tag{4.1.11}$$

then by [6, Lemma 1.1, p. 36], there exists $\varphi_1 \in H^1(\Omega)$ with $\|\varphi_1\|_k = 1$ such that

$$(T\varphi_1, \varphi_1)_k = \frac{1}{\mu_1 + k}\,, \quad T\varphi_1 = \left(\frac{1}{\mu_1 + k}\right)\varphi_1\,. \tag{4.1.12}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{\mu_1 + k} &= \sup\left\{(Tu, u)_k : \|u\|_k = 1\right\}\\ &= \sup_{u\neq 0}\left(T\frac{u}{\|u\|_k}, \frac{u}{\|u\|_k}\right)_k\\ &= \sup_{u\neq 0}\frac{\int_\Omega u^2}{\|u\|_k^2}\,,\end{aligned} \tag{4.1.13}$$

hence

$$\mu_1 + k = \inf_{\substack{u\neq 0\\ u\in H^1(\Omega)}}\frac{\|u\|_k^2}{\int_\Omega u^2}\,. \tag{4.1.14}$$

Then

$$\mu_1 = \inf_{\substack{u\neq 0\\ u\in H^1(\Omega)}}\frac{\int_\Omega |\nabla u|^2 + \alpha\int_{\partial\Omega}(\gamma_0 u)^2 ds}{\int_\Omega u^2}\,, \tag{4.1.15}$$

in particular, for the constant function $\varphi(x) = b$, $\forall x \in \overline{\Omega}$, $b \neq 0$, we have

$$\mu_1 \leq \frac{\int_\Omega |\nabla\varphi|^2 + \alpha \int_{\partial\Omega} (\gamma_0\varphi)^2 ds}{\int_\Omega \varphi^2} = \frac{\alpha b^2 |\partial\Omega|_{n-1}}{b^2|\Omega|} < 0\,,$$

where $|\partial\Omega|_{n-1}$ denotes the $(n-1)$-dimensional Lebesgue's measure $\partial\Omega$ and $|\Omega|$ the Lebesgue's measure n-dimensional of Ω.

Let $\varphi_1, \cdots, \varphi_{j-1}$ be the $j-1$ eigenfunctions that correspond to the $j-1$ eigenvalues of T. We define

$$\frac{1}{\mu_j + k} := \sup\{(Tu, u)_k : \|u\|_k = 1,\ u \perp \varphi_1, \cdots, \varphi_{j-1}\} > 0\,, \tag{4.1.16}$$

then by [6, Proposition 1.3, p. 37], there exists $\varphi_j \in H^1(\Omega)$ with $\|\varphi_j\|_k = 1$ such that

$$(T\varphi_j, \varphi_j)_k = \frac{1}{\mu_j + k}\,, \quad T\varphi_j = \frac{1}{\mu_j + k}\varphi_j\,. \tag{4.1.17}$$

Thus, the eigenvalues of Problem (4.1.6) are positive and form the sequence $0 < \mu_1 + k \leq \mu_2 + k \leq \cdots$, then the eigenvalue problem (4.1.1) has a sequence

$$-k < \mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots\,. \tag{4.1.18}$$

2) Case $\alpha > 0$. In this case, the bilinear form

$$a_k[u, v] = \int_\Omega \nabla u \cdot \nabla v + k \int_\Omega uv + \alpha \int_{\partial\Omega} (\gamma_0 u)(\gamma_0 v) ds\,, \quad u, v \in H^1(\Omega)\,,$$

is coercive for every value of the constant $k > 0$. In fact

$$a_k[u, u] \geq \int_\Omega |\nabla u|^2 + k \int_\Omega u^2 \geq m\|u\|^2\,,$$

where $m = \min\{1, k\}$.

On the other hand, $a_k[., .]$ is continuous, hence $|a_k[u, v]| \leq m_2\|u\|\,\|v\|$, where $m_2 = 1 + k + \alpha c_1^2$. This implies

$$m\|u\|^2 \leq \|u\|_k^2 \leq m_2\|u\|^2\,, \quad \forall u \in H^1(\Omega)\,, \tag{4.1.19}$$

i.e., the norm $\|.\|_k$ is equivalent to the usual norm of $H^1(\Omega)$.

The procedure to determine the existence of the eigenvalues of Problem (4.1.1) with $\alpha > 0$ continues in analogous form to the case $\alpha < 0$. Considering the eigenvalue problem (4.1.6) with an arbitrary positive constant k, we find that it has a sequence of eigenvalues $\beta_1 + k \leq \beta_2 + k \leq \beta_3 + k \leq \cdots$. Hence

$$\beta_1 \leq \beta_2 \leq \beta_3 \leq \cdots \tag{4.1.20}$$

are the eigenvalues of problem (4.1.1).

The first eigenvalue β_1 is positive. In fact, by virtue of (4.1.15) we have

$$\beta_1 = \inf_{\substack{u \neq 0 \\ u \in H^1(\Omega)}} \frac{\int_\Omega |\nabla u|^2 + \alpha \int_{\partial\Omega} (\gamma_0 u)^2 ds}{\int_\Omega u^2} \geq 0\,, \tag{4.1.21}$$

if $\beta_1 = 0$, let φ be an eigenfunction associated to β_1. Then

$$0 = \frac{\int_\Omega |\nabla\varphi|^2 + \alpha \int_{\partial\Omega} (\gamma_0\varphi)^2 ds}{\int_\Omega \varphi^2},$$

thus $\gamma_0\varphi = 0$, $\varphi \in H_0^1(\Omega)$ and $|\nabla\varphi| = 0$. Then $\varphi \equiv 0$ which is absurd because $\|\varphi\|_k = 1$.

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4.2. **The first eigenvalue and its associated eigenfunctions.**

Theorem 4.2. *If φ is an eigenfunction of problem* (4.1.1) *associated to the first eigenvalue λ_1, then $\varphi \in C^\infty\left(\overline{\Omega}\right)$.*

Proof. By Theorem 3.3 φ verifies

$$\int_\Omega \nabla\varphi \cdot \nabla v + \alpha \int_{\partial\Omega} \gamma_0\varphi \cdot \gamma_0 v = \mu_1 \int_\Omega \varphi v\,, \quad \forall v \in H^1(\Omega)\,. \tag{4.1.22}$$

Using Theorem 3.2 with $f(x, \varphi(x)) = \lambda_1\varphi(x)$ we have $\varphi \in H^2(\Omega)$, $\gamma_0\varphi \in H^{2-1/2}(\partial\Omega)$. Then there exists $w_1 \in H^3(\Omega)$ (Trace Theorem) such that $\gamma_1 w_1 = -\alpha\gamma_0\varphi$. By the Green formula (2.2) we have

$$\int_\Omega \nabla\varphi.\nabla v = \int_\Omega \{\lambda_1\varphi + \Delta w_1\} v + \int_\Omega \nabla w_1 \cdot \nabla v\,.$$

Making $h = \varphi - w_1$ and $G = \lambda_1\varphi + \Delta w_1$, we obtain

$$\begin{cases} \int_\Omega \nabla h \cdot \nabla v &= \int_\Omega G v\,, \quad \forall v \in H^1(\Omega)\,, \\ \qquad \gamma_1(h) &= 0\,, \end{cases} \tag{4.1.23}$$

thus, $G \in \left\{H^1(\Omega); N^*\right\}$. By Theorem 2.5 there exists $\widetilde{h} \in H^3(\Omega)$ such that

$$\begin{cases} -\Delta\widetilde{h} &= G \quad \text{in} \quad \Omega\,, \\ \frac{\partial\widetilde{h}}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega\,. \end{cases} \tag{4.1.24}$$

Multiplying the first equality in (4.1.24) by v and integrating we obtain

$$\int_\Omega \nabla\widetilde{h}.\nabla v = \int_\Omega G v\,, \quad \forall v \in H^1(\Omega)\,. \tag{4.1.25}$$

Subtracting (4.1.25) from the first equation in (4.1.23) we have

$$\int_\Omega \nabla(h - \widetilde{h}) \cdot \nabla v = 0\,, \quad \forall v \in H^1(\Omega)\,.$$

Making $v = h - \widetilde{h}$ we obtain $\int_\Omega \left|\nabla(h - \widetilde{h})\right|^2 = 0$, then $\nabla\left(h - \widetilde{h}\right) = 0$, $h - \widetilde{h} = M$ for some constant M, then $\varphi = w_1 + \widetilde{h} + M \in H^3(\Omega)$. Since $\varphi \in H^3(\Omega)$ then $\gamma_0\varphi \in H^{3-1/2}(\partial\Omega)$. Using again Trace Theorem there exists $w_2 \in H^4(\Omega)$ such that $\gamma_1 w_2 = -\alpha\gamma_0\varphi$ and proceeding as in the above case, we conclude that $\varphi \in H^4(\Omega)$. Continuing on this way we obtain that $\varphi \in H^m(\Omega)$ for any integer $m \geq 0$. Thus, for $D^\beta\varphi$ there exists an integer m_β such that $2m_\beta > n$ and

$D^\beta \varphi \in H^{m_\beta}(\Omega)$. Then by the inclusion of Sobolev and smoothness of the boundary, $D^\beta \varphi \in C\left(\overline{\Omega}\right)$. ☑

Theorem 4.3. *The first eigenvalue of problem* (4.1.1) *is simple.*

Proof. i) We claim that if φ is an eigenfunction associated to the first eigenvalue λ_1, then φ does not change sign. Looking for by contradiction, suppose that φ changes sign. Then φ verifies the equality

$$(\varphi, \varphi)_k = (\lambda_1 + k) \int_\Omega \varphi^2 \,. \tag{4.1.26}$$

Writing the function φ in the form $\varphi(x) = \varphi^+(x) + \varphi^-(x)$, where $\varphi^+(x) = \max_{x\in\overline{\Omega}}\{\varphi(x), 0\}$ and $\varphi^-(x) = \min_{x\in\overline{\Omega}}\{\varphi(x), 0\}$, the left member of (4.1.26) can be written

$$(\varphi, \varphi)_k = \left(\varphi^+, \varphi^+\right)_k + \left(\varphi^-, \varphi^-\right)_k ,$$

because φ^+ and φ^- are orthogonals, *i.e.*

$$\begin{aligned}\left(\varphi^+, \varphi^-\right)_k &= \int_\Omega \nabla\varphi^+ \cdot \nabla\varphi^- + k\int_\Omega \varphi^+ \cdot \varphi^- + \alpha \int_{\partial\Omega} \gamma_0\varphi^+ \cdot \gamma_0\varphi^- ds \\ &= 0 \,.\end{aligned}$$

Indeed, we have $\int_\Omega \nabla\varphi^+ \cdot \nabla\varphi^- = 0$ and $\int_\Omega \varphi^+\varphi^- = 0$. Furthermore, by Theorem 4.2, $\varphi \in C\left(\overline{\Omega}\right)$. Then $\gamma_0\varphi^+$ and $\gamma_0\varphi^-$ are restrictions of φ^+ and φ^- on $\partial\Omega$, so

$$\int_{\partial\Omega} \gamma_0\varphi^+ \cdot \gamma_0\varphi^- ds = \int_{\partial\Omega} \varphi^+ \cdot \varphi^- ds = 0 \,.$$

Let $b_1 = (\varphi^+, \varphi^+)_k$ and $b_2 = (\varphi^-, \varphi^-)_k$, then $b_1 > 0$ and $b_2 > 0$. On the other hand,

$$\int_\Omega \varphi^2 = \int_\Omega \left(\varphi^+\right)^2 + \int_\Omega \left(\varphi^-\right)^2 = a_1 + a_2.$$

Then using the above notation we can write (4.1.26) in the following way:

$$b_1 + b_2 = (\lambda_1 + k)(a_1 + a_2) \,. \tag{4.1.27}$$

From (4.1.27) and (4.1.14) we have

$$\frac{1}{\lambda_1 + k} = \frac{a_1 + a_2}{b_1 + b_2} = \sup_{u\neq 0} \frac{\int_\Omega u^2}{\|u\|_k^2} \,. \tag{4.1.28}$$

For the numbers $\frac{a_1}{b_1}$, $\frac{a_2}{b_2}$, and $\frac{a_1+a_2}{b_1+b_2}$, we have the unique possibility $\frac{a_1+a_2}{b_1+b_2} = \frac{a_1}{b_1} = \frac{a_2}{b_2}$. So

$$\frac{1}{\lambda_1 + k} = \frac{\int_\Omega \left(\varphi^+\right)^2}{\|\varphi^+\|_k^2} = \frac{\int_\Omega \left(\varphi^-\right)^2}{\|\varphi^-\|_k^2} \,. \tag{4.1.29}$$

Using (4.1.13), the previous equalities imply that φ^+ and φ^- are eigenfunctions corresponding to λ_1. Then $\forall v \in H^1(\Omega)$ we have

$$\lambda_1 \int_\Omega \varphi^+ \cdot v = \int_\Omega \nabla\varphi^+ \cdot \nabla v + \alpha \int_{\partial\Omega} \gamma_0\varphi^+ \cdot \gamma_0 v \,, \tag{4.1.30}$$

By Theorem 4.2, we have that $\varphi^+ \in C^\infty\left(\overline{\Omega}\right)$.

On the other side we observe that:

$$-\Delta\varphi^+ + k\varphi^+ = (\lambda_1 + k)\varphi^+ \geq 0\,, \quad \text{in} \quad \Omega\,.$$

Since φ changes sign in Ω, there exists x_0 inside of Ω such that $\varphi^+(x_0) = 0$ which is its minimum value in $\overline{\Omega}$. Then by the Strong Maximum Principle φ^+ is constant in Ω, thus $\varphi^+ = 0$ in Ω, which is absurd. Therefore φ can not change sign in Ω.

ii) Next we prove that the geometric multiplicity of λ_1 is one. Let φ_1 and φ_2 be two eigenfunctions associate to the eigenvalue λ_1. By step i), for each $t \in \mathbb{R}$ the eigenfunction $\varphi_1 + t\varphi_2$ has definite sign in Ω, so the sets $A = \{t \in \mathbb{R} : \varphi_1 + t\varphi_2 \geq 0\,, \text{in}\,\Omega\}$ and $B = \{t \in \mathbb{R} : \varphi_1 + t\varphi_2 \leq 0\,, \text{in}\,\Omega\}$ are non empty, closed and $A \cup B = \mathbb{R}$. Since the set of real number $\mathbb{R}$ is connected, there exists $\overline{t} \in A \cap B$, $\overline{t} \neq 0$ such as $\varphi_1 + \overline{t}\varphi_2 = 0$, *i.e.* φ_1 and φ_2 are linearly dependent. Therefore, the eigenspace associate to the eigenvalue λ_1 is generated by a single eigenfunction, that we denote by φ_1.

iii) The algebraic multiplicity of λ_1 is one, (λ_1 is simple). Let

$$N(\lambda_1 I + \Delta) = \{u : \lambda_1 u + \Delta u = 0 \quad \text{and} \quad \gamma_1 u + \alpha\gamma_0 u = 0\}\,,$$
$$N(\lambda_1 I + \Delta)^2 = \{u : (\lambda_1 I + \Delta)(\lambda_1 u + \Delta u) = 0 \quad \text{and} \quad \gamma_1 u + \alpha\gamma_0 u = 0\}\,.$$

It is clear that $N(\lambda_1 I+\Delta) \subset N(\lambda_1 I+\Delta)^2$. Next we show that $N(\lambda_1 I+\Delta)^2 \subset N(\lambda_1 I + \Delta)$. Indeed, let $\varphi \in N(\lambda_1 I + \Delta)^2$ be, then $\lambda_1\varphi + \Delta\varphi \in N(\lambda_1 I + \Delta)$, $\lambda_1\varphi + \Delta\varphi = t\varphi_1$ for some real t.

By the other side, we have

$$\begin{aligned}
t(\varphi_1, \varphi_1)_k &= (t\varphi_1, \varphi_1)_k = (\lambda_1\varphi + \Delta\varphi, \varphi_1)_k \\
&= \lambda_1(\varphi, \varphi_1)_k + (\Delta\varphi, \varphi_1)_k \\
&= \lambda_1(\varphi, \varphi_1)_k + (\lambda_1 + k)\int_\Omega \varphi_1\Delta\varphi, \quad \text{(by (4.1.7))} \\
&= (\lambda_1 - \lambda_1)(\varphi, \varphi_1)_k = 0\,.
\end{aligned}$$

Hence $t(\varphi_1, \varphi_1)_k = 0$, then $t = 0$. Thus $\lambda_1\varphi+\Delta\varphi = 0$ and $\varphi \in N(\lambda_1 I+\Delta)$. ☑

4.3. **Orthogonal sum.** Finally we express the Sobolev space $H^1(\Omega)$ as the orthogonal sum of two subspaces, where an addend has finite dimension. In the case $\alpha < 0$, from (4.1.18) and Theorem (4.3), the sequence of eigenvalues has the form

$$-k < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \tag{4.2.1}$$

Let X_1 be the space associated to the first eigenvalue μ_1 and $X_2 = X_1^\perp$ the orthogonal complement of X_1 with respect to the inner product of $(\cdot,\cdot)_k$ defined in (4.1.4). Then we have

$$H^1(\Omega) = X_1 \oplus X_2\,, \tag{4.2.2}$$

and

$$\int_\Omega |\nabla\varphi|^2 + \alpha \int_{\partial\Omega} (\gamma_0\varphi)^2 ds = \lambda_1 \int_\Omega \varphi^2\,, \quad \forall\varphi \in X_1\,. \tag{4.2.3}$$

In the case $\alpha > 0$, from (4.1.20) and Theorem (4.3), the sequence of eigenvalues has the form

$$\beta_1 < \beta_2 \leq \beta_3 \cdots \tag{4.2.4}$$

Let Y_1 be the space associate to β_1 and $Y_2 = Y_1^\perp$ the orthogonal complement of Y_1 with respect to the inner product $(\cdot,\cdot)_k$, but here $k > 0$ is an arbitrary constant. Then

$$H^1(\Omega) = Y_1 \oplus Y_2\,, \tag{4.2.5}$$

and

$$\int_\Omega |\nabla\varphi|^2 + \alpha \int_{\partial\Omega} (\gamma_0\varphi)^2 ds = \beta_1 \int_\Omega \varphi^2\,, \quad \forall\varphi \in Y_1\,. \tag{4.2.6}$$

by virtue of Theorem 4.3, the subspaces X_1 and Y_1 have dimension one.

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