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On groups and normal polymorphic functions

Sobre grupos y funciones polimorfas normales

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Abstract. Let Γ be a Fuchsian group acting on the unit disk $\mathbb{D}$. A function f meromorphic in $\mathbb{D}$ is polymorphic if there exists a homomorphism f_* of Γ onto a group Σ of Möbius transformations such that $f \circ \gamma = f_*(\gamma) \circ f$ for $\gamma \in \Gamma$. A function is normal if $\sup \left(1-|z|^2\right)|f'(z)|/\left(1+|f(z)|^2\right) < \infty$. First we study the behaviour of a normal polymorphic function at the fixed points of Γ and then the existence of such functions for a given type of group Σ.

Key words and phrases. Kleinian group, polymorphic function, normal function, projective structure.

Resumen. Sea Γ un grupo fuchsiano que actúa en el disco unitario $\mathbb{D}$. Una función f meromorfa en $\mathbb{D}$ es polimorfa si existe un homomorfismo f_* de Γ sobre un grupo Σ de transformaciones de Möbius tal que $f \circ \gamma = f_*(\gamma) \circ f$ para $\gamma \in \Gamma$. Una función es normal si $\sup \left(1-|z|^2\right)|f'(z)|/\left(1+|f(z)|^2\right) < \infty$. Primero estudiamos el comportamiento de una función polimorfa normal en los puntos fijos de Γ y después la existencia de tales funciones para un tipo de grupo Σ dado.

Palabras y frases clave. Grupo kleiniano, función polimorfa, función normal, estructura proyectiva.

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1. Introduction

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$ and $\mathbb{T} = \partial\mathbb{D}$ and let $\mathbb{H}$ denote the upper halfplane. We consider the group Möb $= \mathrm{PSL}(2, \mathbb{C})$ of all Möbius transformations

$$\tau(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1. \tag{1.1}$$

If X is any subset of $\widehat{\mathbb{C}}$ then we write

$$\text{Möb}(X) := \{\tau \in \text{ Möb: } \tau(X) = X\},$$

so that Möb (X) is the stabilizer of X in Möb.

Let Γ be a Fuchsian group acting on $\mathbb{D}$, that is, any discrete subgroup of Möb $(\mathbb{D})$. A Γ-*polymorphic function* is a non-constant function f meromorphic in $\mathbb{D}$ such that, for every $\gamma \in \Gamma$,

$$f \circ \gamma = \sigma \circ f \quad \text{for some} \quad \sigma \in \text{ Möb}. \tag{1.2}$$

Defining $f_*(\gamma) = \sigma$ we obtain a homomorphism

$$f_* : \Gamma \to \text{ Möb}, \quad f_*(\Gamma) = \Sigma.$$

The image group Σ need not be discrete. Note that the function f need not be locally univalent and that the groups may be infinitely generated.

The name "polymorphic" is not standard; we follow the usage of Heyhal [8], [9], [22]. Other names for similar concepts are "deformation" [15] and "projective structure", in particular in connection with Riemann surfaces, see [7], [4] and for instance [14]. Polymorphic functions that are not locally univalent correspond to "branched projective structures", see e.g. [19].

A meromorphic function f in $\mathbb{D}$ is called *normal* if

$$\sup_{z\in\mathbb{D}} \left(1 - |z|^2\right) f^{\#}(z) < \infty, \tag{1.3}$$

where $f^{\#} = |f'| / \left(1 + |f|^2\right)$ is the *spherical derivative*. Every meromorphic function omitting three values in $\widehat{\mathbb{C}}$ is normal. See [18] and for instance [16]. A function f analytic in $\mathbb{D}$ is called a *Bloch function* if

$$\|f\|_{\mathcal{B}} := |f(0)| + \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right) |f'(z)| < \infty. \tag{1.4}$$

The Banach space with this norm is denoted by $\mathcal{B}$; see for instance [2] and [23, Section 4.2]. If $f \in \mathcal{B}$ then f and $\exp f$ are normal.

A Stolz angle S at $\zeta \in \mathbb{T}$ is a sector in $\mathbb{D}$ with vertex ζ. We say that f has the *angular limit* $f(\zeta) := \omega$ at ζ if $f(z) \to \omega \in \widehat{\mathbb{C}}$ as $z \to \zeta$, $z \in S$ for every Stolz angle S. The *Lehto-Virtanen theorem* [18] states:

Let f be normal and $\zeta \in \mathbb{T}$. If there exists an arc $C \subset \mathbb{D}$ ending at ζ such that

$$f(z) \to \omega \in \widehat{\mathbb{C}} \quad \text{as} \quad z \to \zeta, \quad z \in C,$$

then $f(z) \to \omega$ as $z \to \zeta$ holds in every Stolz angle S at ζ and in every domain between C and S. In particular f has an angular limit.

Now let f be Γ-polymorphic. Then, by (1.2),

$$\gamma'(z)\, f'(\gamma(z)) = \sigma'(f(z)) f'(z)\ , \tag{1.5}$$

for $\gamma \in \Gamma$ and $\sigma = f_*(\gamma)$. It follows that

$$\left(1 - |\gamma(z)|^2\right) f^{\#}(\gamma(z)) = \left(1 - |z|^2\right) |f'(z)|\, \sigma^{\#}(f(z))\,. \tag{1.6}$$

If f is replaced by $\tau \circ f$ with $\tau \in$ Möb, then σ is replaced by $\tau \circ \sigma \circ \tau^{-1}$ and the supremum in (1.3) is changed by a bounded factor. Hence normality is not affected so that we may assume that one given element of Σ has standard form.

In Section 2, we consider Γ-polymorphic functions f that are assumed to be normal and we study their behaviour at the fixed points of Γ. An important property is that the limit set $L(f_*(\Gamma))$ lies on the boundary of $f(\mathbb{D})$. The *limit set* $L(\Sigma)$ of a subgroup Σ of Möb is defined as the closure of the set of all loxodromic fixed points of Σ, see [3, p. 97]. It is Σ-invariant.

In Section 3, we start from a given subgroup Σ of Möb and investigate whether there is a Fuchsian group Γ and a normal Γ-polymorphic function f with $f_*(\Gamma) = \Sigma$. We also study what further properties the function f must have in order to be normal.

Section 4 is devoted to examples to illustrate the results of the previous sections and to show that certain phenomena can occur. The groups and functions are constructed at the same time. The first seven constructions follow the pattern described at the beginning of that section.

We apologize for the clash with the usual notation. Papers on Kleinian groups tend to use roman letters for the groups and Greek letters for functions. We use the conventions of function theory where the role of roman and Greek letters tend to be reversed.

2. The behaviour at the fixed points

For a Möbius transformation $\tau \neq \mathrm{id}$, we denote the set of fixed points by $\mathrm{Fix}(\tau)$. The classical distinction was between parabolic, elliptic, hyperbolic and loxodromic transformations. We follow the current usage and include the hyperbolic among the loxodromic transformations. Thus τ is called *loxodromic* if it has two fixed points with multipliers q and q^{-1} where $|q| \neq 1$. This holds if and only if $\mathrm{tr}\,\tau \notin [-2, 2]$ where $\mathrm{tr}\,\tau = a + d$ is the trace in the notation (1.1). If $\mathrm{tr}\ \tau \notin \mathbb{R}$ then τ is strictly loxodromic. The elliptic elements may be of infinite order.

Now let Γ be a Fuchsian group in $\mathbb{D}$. Then all elements of Γ are hyperbolic, parabolic or elliptic of finite order. First we consider the hyperbolic case, which is the most important case.

Theorem 1. *Let f be a normal Γ-polymorphic function. Let $\gamma \in \Gamma$ be hyperbolic with fixed points $\zeta^{\pm}$ and suppose that $\sigma = f_*(\gamma)$ is loxodromic or parabolic.*

Then the angular limits $f(\zeta^{\pm})$ exist and

$$\mathit{Fix}(\sigma)=\left\{f(\zeta^+),\ f(\zeta^-)\right\}\subset\partial f(\mathbb{D})\,. \tag{2.1}$$

This leaves out the case that σ is elliptic. If σ is elliptic then the angular limits $f(\zeta^{\pm})$ do not exist, as is easy to see. Moreover there are normal Γ-polymorphic functions f such that $\mathrm{Fix}(\sigma)\cap\overline{f(\mathbb{D})}=\emptyset$, see Example 8.

Proof. (a) Let σ be loxodromic. We may assume that $\sigma(w)=aw$ with $|a|>1$ so that $\mathrm{Fix}(\sigma)=\{0,\infty\}$. Let $z\in\mathbb{D}$ be such that $f(z)\neq 0,\infty$. We can choose $n\in\mathbb{Z}$ such that $1\leq|a^nf(z)|<|a|$. Then it follows from (1.6) and (1.3) that

$$\begin{aligned}|a|\,\frac{1-|z|^2}{1+|a|^2}\left|\frac{f'(z)}{f(z)}\right|&\leq\frac{|a|^n\left(1-|z|^2\right)|f'(z)|}{1+|a^nf(z)|^2}\\&=\left(1-|\gamma^n(z)|^2\right)f^{\#}(\gamma^n(z))\;,\end{aligned} \tag{2.2}$$

is bounded in $\mathbb{D}$. We conclude that f cannot assume the fixed points 0 and ∞. Hence $\mathrm{Fix}(\sigma)\cap f(\mathbb{D})=\emptyset$. See [22, Th. 8] for this statement.

Now let C be a circular arc in $\mathbb{D}$ from ζ^- to ζ^+. Then $\gamma(C)=C$. Since $f(z)\neq 0,\infty$ we see that

$$f(\gamma^n(z))=\sigma^n(f(z))=a^nf(z)\to\infty\text{ or }0\text{ as }n\to\pm\infty\,,$$

and it follows that $f(z)\to\infty$ or 0 as $z\to\zeta^{\pm}$, $z\in C$. Since f is normal we conclude from the Lehto-Virtanen theorem (Section 1) that the angular limits exist and $f(\zeta^+)=\infty$, $f(\zeta^-)=0$. In particular it follows that $\mathrm{Fix}(\sigma)\in\overline{f(\mathbb{D})}$. This proves (2.1).

(b) Let σ be parabolic. We may assume that $\sigma(w)=w+b$ with $b\neq 0$ so that $\mathrm{Fix}(\sigma)=\{\infty\}$. Then $f(\gamma^n(z))=f(z)+nb$ by (1.2) and it follows from (1.6) and (1.3) that, for $z\in\mathbb{D}$,

$$\frac{\left(1-|z|^2\right)|f'(z)|}{1+|f(z)+nb|^2}=\left(1-|\gamma^n(z)|^2\right)f^{\#}(\gamma^n(z))\leq M<\infty. \tag{2.3}$$

Now suppose that f has a pole z^* in $\mathbb{D}$. Then there exist $z_n\to z^*$ such that $f(z_n)+nb=0$ and it follows from (2.3) that $\left(1-|z_n|^2\right)|f'(z_n)|\leq M$, which contradicts $f'(z^*)=\infty$. Since $f(\gamma^n(z_n))=f(z)+nb\to\infty$ as $n\to\pm\infty$ we obtain that

$$f(z)\to\infty\text{ as }z\to\zeta^{\pm},\ z\in C\,;$$

The Lehto-Virtanen theorem shows that the angular limits exist and $f(\zeta^{\pm})=\infty\in\overline{f(\mathbb{D})}$. This proves (2.1) since $f(z)\neq\infty$ for $z\in\mathbb{D}$. ☑

Much more can be said if $\gamma\in\Gamma$ is parabolic. This is perhaps not surprising because a parabolic fixed point corresponds to a puncture of the Riemann surface $\mathbb{D}/\Gamma$. A horodisk at $\zeta\in\mathbb{T}=\partial\mathbb{D}$ is a disk in $\mathbb{D}$ that touches $\mathbb{T}$ at ζ.

Theorem 2. *Let f be a normal Γ-polymorphic function. Let $\gamma \in \Gamma$ be parabolic with fixed point ζ and let $\sigma = f_*(\gamma) \neq id$.*

(i) *It is not possible that σ is loxodromic.*

(ii) *If σ is parabolic then the angular limit $f(\zeta)$ exists and*

$$Fix(\sigma) = \{f(\zeta)\} \subset \partial f(\mathbb{D}),$$

moreover $f(z) \to f(\zeta)$ as $z \to \zeta$ in each horodisk at ζ.

(iii) *If σ is elliptic then the angular limit $f(\zeta)$ exists and*

$$f(\zeta) \in \; Fix(\sigma) \cap \overline{f(\mathbb{D})}.$$

If $\sigma = f_*(\gamma)$ is elliptic then the situation is therefore rather different in the two cases that γ is hyperbolic or parabolic. Both possibilities in (iii), namely that $f(\zeta) \in \partial f(\mathbb{D})$ or $f(\zeta) \in f(\mathbb{D})$, can occur as Example 5 with $\vartheta = 2\pi/n$ or $\vartheta/\pi \notin \mathbb{Q}$ shows. In Example 6 with $\vartheta = 2\pi/n$ we also have $f(\zeta) \in f(\mathbb{D})$.

Proof. a) Let σ be loxodromic or parabolic with fixed points $\omega^{\pm}$. As in part (b) of the proof of Theorem 1, we see that $f(z) \neq \omega^{\pm}$. Let H be a horodisk at ζ. Since $f(\gamma^n(z)) = \sigma^n(f(z)) \to \omega^{\pm}$ as $n \to \pm\infty$ for every $z \in \mathbb{D}$ and, since γ is parabolic, we see that

$$f(z) \to \omega^{\pm} \text{ as } z \to \zeta, \; z \in \partial H.$$

Hence it follows from the Lehto-Virtanen theorem that $f(z) \to \omega^{\pm}$ as $z \to \zeta$ in the two components of $H \setminus [-\zeta, \zeta]$ and therefore in H. In particular we have $\omega^+ = \omega^-$ so that σ is not loxodromic, which proves (i). This proves (ii).

b) Now let σ be elliptic. Let $r_k \to 1^-$ as $k \to \infty$ and

$$h_k(z) = \zeta \frac{z + r_k}{1 + r_k z} \quad (z \in \mathbb{D}), \quad g_k = f \circ h_k. \tag{2.4}$$

Since the supremum in (1.3) is not changed if we replace f by g_k, the sequence (g_k) is normal. Hence (g_k) has a subsequence that converges locally uniformly to a function g meromorphic in $\mathbb{D}$. It follows that

$$g_k \circ \left(h_k^{-1} \circ \gamma \circ h_k\right) = \sigma \circ g_k \to \sigma \circ g \text{ as } k \to \infty. \tag{2.5}$$

Let $z \in \mathbb{D}$. Since γ is parabolic and h_k converges to the fixed point ζ of γ, the non-euclidean distance between $h_k(z)$ and $\gamma \circ h_k(z)$ tends to 0 as $k \to \infty$. Hence it follows from (2.4) that $h_k^{-1} \circ \gamma \circ h_k(z) \to z$. Thus we obtain from (2.5) that $g = \sigma \circ g$. Since $\sigma \neq \text{id}$ it follows that g is constant and equal to one of the fixed points $\omega^{\pm}$ of σ. In view of (2.4) we conclude that the limit set E of $f(\zeta x)$ as $x \to 1^-$ satisfies $E \subset \{\omega^+, \omega^-\}$. Since E is connected it follows that f has an angular limit $f(\zeta) = \omega^+$ or ω^-. This proves (iii). ☑

Corollary 3. *If f is normal and Γ-polymorphic then*

$$L(f_*(\Gamma)) \subset \partial f(\mathbb{D}).$$

This is an immediate consequence of Theorem 1 and Theorem 2(i). Example 8 shows that strict inclusion can occur.

3. Normality and types of groups

Through this section, we assume that Γ is a Fuchsian group and that f is a Γ-polymorphic function. We study the problem of the normality of f given the type of the group $\Sigma = f_*(\Gamma)$. We divide the subgroups of Möb into three classes.

1. An *elliptic* group contains only elliptic elements and the identity. These groups are sometimes included among the elementary groups ([3, p. 84]).

2. A group is called *elementary* if any two elements of infinite order have a common fixed point in $\widehat{\mathbb{C}}$.

3. The richest and most interesting class is formed by the groups that are neither elliptic nor elementary. Such a group has infinitely many loxodromic elements no two of which have a common fixed point ([3, Th. 5.1.3]). An important subclass is formed by the groups that are discontinuous in some open subset of $\widehat{\mathbb{C}}$, the Kleinian groups in the classical terminology ([20, p. 16]).

For our investigation of normality, the limit set $L(\Sigma)$ is more important than the discreteness of the group Σ. We need the following known result, see [6, Th. 2], [25, p. 246] and [24].

Proposition 4. *Let Σ be a non-discrete subgroup of Möb that contains a loxodromic element and is not in Möb $(\widehat{\mathbb{C}} \setminus \{a, b\})$ with different a and b. If G is a Σ-invariant domain in $\widehat{\mathbb{C}}$ then there are only three possibilities:*

(a) $G = \widehat{\mathbb{C}}$,

(b) G is a disk in $\widehat{\mathbb{C}}$ and Σ contains no strictly loxodromic elements,

(c) $G = \widehat{\mathbb{C}} \setminus \{a\}$.

3.1. First we consider the elliptic groups Σ. By Corollary 3 this is the only type of group for which a polymorphic function with $f(\mathbb{D}) = \widehat{\mathbb{C}}$ can be normal.

Making a conjugation in Möb we may assume ([3, p. 84]) that Σ is a subgroup of the group $\mathrm{Rot}\left(\widehat{\mathbb{C}}\right)$ of the rotations of the sphere whose elements have the form

$$\sigma(w) = \frac{aw + b}{-\overline{b}w + \overline{a}}, \quad a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1. \tag{3.1}$$

This conjugation does not affect normality, see Section 1.

Theorem 5. *Let $\Sigma = f_*(\Gamma) \subset Rot\left(\widehat{\mathbb{C}}\right)$. If*

$$\sup_{z \in F} \left(1 - |z|^2\right) f^{\#}(z) < \infty, \tag{3.2}$$

for some fundamental domain F of Γ then f is normal.

Example 1 shows that there are normal functions both with $f(\mathbb{D}) = \widehat{\mathbb{C}}$ and with $f(\mathbb{D}) \neq \widehat{\mathbb{C}}$.

Proof. We obtain from (3.1) that $\sigma^{\#}(w) = 1/\left(1+|w|^2\right)$. Hence it follows from (1.6) that

$$\left(1-|\gamma(z)|^2\right) f^{\#}(\gamma(z)) = \left(1-|z|^2\right) f^{\#}(z) \quad \text{for } \gamma \in \Gamma\,,$$

so that (3.2) implies the normality of f because F is a fundamental domain of Γ. ☑

3.2. Next we turn to the non-elliptic elementary groups Σ. Then, up to conjugation, there are two cases ([3, p. 84]): Either Σ is a subgroup of Möb$(\mathbb{C}\setminus\{0\})$ whose elements are

$$\sigma(w) = aw^{\pm 1}\,, \quad a \in \mathbb{C}\,, \quad a \neq 0\,, \tag{3.3}$$

or Σ is a subgroup of Möb$(\mathbb{C})$ whose elements are

$$\sigma(w) = aw + b\,, \quad a, b \in \mathbb{C}\,, \quad a \neq 0\,. \tag{3.4}$$

In the following discussions we always exclude the groups already dealt with.

A. Let the elements $\neq$ id of Σ have the form (3.3) with $|a| \neq 1$; we thus exclude rotations of the sphere. In Example 3 we will construct a normal function with $f(\mathbb{D}) = \mathbb{C}\setminus\{0\}$ and another normal function that omits infinitely many values.

Theorem 6. *Let* $\Sigma = f_*(\Gamma) \subset$ *Möb*$(\mathbb{C}\setminus\{0\})$ *but* $\Sigma \not\subseteq$ *Rot*$\left(\widehat{\mathbb{C}}\right)$. *If* f *is normal then* $f(\mathbb{D}) \subset \mathbb{C}\setminus\{0\}$ *and* $\log f$ *is an unbounded Bloch function. If*

$$\sup_{z\in F}\left(1-|z|^2\right)\log\left|\frac{f'(z)}{f(z)}\right| < \infty \tag{3.5}$$

for some fundamental domain F *of* Γ *then* $\log f \in \mathcal{B}$ *and* f *is normal.*

Proof. Let f be normal. Since $\Sigma \not\subseteq \mathrm{Rot}\left(\widehat{\mathbb{C}}\right)$ there exists a loxodromic $\sigma \in \Sigma$ with fixed points 0 and ∞ and thus $f(\mathbb{D}) \subset \mathbb{C}\setminus\{0\}$ by Theorems 1 and 2. As in (1.6) we see that the function

$$\left(1-|z|^2\right)\frac{d}{dz}\log f(z) = \left(1-|z|^2\right)\frac{f'(z)}{f(z)}\,, \tag{3.6}$$

is bounded in $\mathbb{D}$. It follows that $\log f \in \mathcal{B}$, and $\log f$ is unbounded because $\infty \in \partial f(\mathbb{D})$ by Theorem 1.

Since $\sigma'(w)/\sigma(w) = \pm 1/w$ it follows from (1.5) that

$$\left(1-|\gamma(z)|^2\right)\left|\frac{f'(\gamma(z))}{f(\gamma(z))}\right| = \left(1-|z|^2\right)\frac{f'(z)}{f(z)} \quad \text{for } \gamma \in \Gamma.$$

Hence (3.5) implies $\log f \in \mathcal{B}$ in view of (3.6) and (1.4), and $\log f \in \mathcal{B}$ in turn implies that f is normal. ☑

B. Now we consider the most complicated case, namely that the elements of Σ have the form (3.4) but not (3.3).

Theorem 7. *Let $\Sigma = f_*(\Gamma) \subset Möb(\mathbb{C})$ but $\Sigma \not\subseteq Möb(\mathbb{C} \setminus \{c\})$.*

(i) *Suppose that Σ has no loxodromic elements. If Σ has two parabolic elements with $\mathbb{R}$-independent translations and if f is normal then $f \in \mathcal{B}$. If*

$$\sup_{z \in F} \left(1 - |z|^2\right) |f'(z)| < \infty \tag{3.7}$$

for some fundamental domain F of Γ then $f \in \mathcal{B}$ and f is normal.

(ii) *Suppose that Σ has a loxodromic element. If $f(\mathbb{D})$ is a half plane then f is normal and Σ is conjugate to $Möb(\mathbb{H})$. If $f(\mathbb{D})$ is not a half plane then $f(\mathbb{D}) = \widehat{\mathbb{C}}$ and f is not normal.*

Note that (3.7) holds automatically if f has no poles and if $\overline{F} \subset \mathbb{D}$, in other words if $\mathbb{D}/\Gamma$ is a closed Riemann surface.

In Example 2 we have a normal function that omits a rectangular lattice while the normal function f of Example 7 satisfies $f(\mathbb{D}) = \mathbb{C}$. In Example 9 we have a normal function f with $f \notin \mathcal{B}$, which shows that it is not possible to omit the assumption of Theorem 6 that there are two $\mathbb{R}$-independent translations.

We do not have an example where $\Sigma \subset$ Möb($\mathbb{C}$) has a loxodromic element and $f(\mathbb{D})$ is a halfplane. Thus it is conceivable that f is never normal if Σ contains a loxodromic element.

Proof. (i) First we assume that Σ has no loxodromic elements. Then $|a| = 1$ in (3.4). It follows that $\sigma^{\#}(w) = 1/\left(1 + |\sigma(w)|^2\right)$ so that, by (1.6),

$$\frac{\left(1 - |\gamma(z)|^2\right) |f'(\gamma(z))|}{1 + |f(\gamma(z))|^2} = \frac{\left(1 - |z|^2\right) |f'(z)|}{1 + |\sigma(f(z))|^2} \tag{3.8}$$

for $\sigma = f_*(\gamma)$.

Let there exist $\sigma_j \in \Sigma$ for $j = 1, 2$ such that $\sigma_j(w) = w + b_j$, $b_j \neq 0$ and $b_2/b_1 \notin \mathbb{R}$ and let f be normal. For every $z \in \mathbb{D}$ there exist $n_j \in \mathbb{Z}$ such that

$$\begin{aligned} |\sigma_1^{n_1} \circ \sigma_2^{n_2}(f(z))| &= |f(z) + n_1 b_1 + n_2 b_2| \\ &\leq |b_1| + |b_2| \,. \end{aligned}$$

Since f is normal it follows from (3.8) that $\left(1 - |z|^2\right) |f'(z)|$ is bounded in $\mathbb{D}$ so that $f \in \mathcal{B}$, see (1.3).

Furthermore it follows from (1.5) and $|\sigma'(w)| = 1$ that

$$\left(1 - |\gamma(z)|^2\right) |f'(\gamma(z))| = \left(1 - |z|^2\right) |f'(z)|$$

for $\gamma \in \Gamma$. Since F is a fundamental domain of Γ we conclude that (3.7) implies $f \in \mathcal{B}$.

(ii) The only discrete subgroups of Möb($\mathbb{C}$) with loxodromic elements belong also to Möb($\mathbb{C} \setminus \{c\}$) for some $c \in \mathbb{C}$ ([3, Section 5.1]). Since these were excluded it follows that Σ is not discrete.

Now $G = f(\mathbb{D})$ is Σ-invariant. Hence we can apply Proposition 4. If $G = \widehat{\mathbb{C}}$ then f is not normal by Corollary 3. If $G \neq \widehat{\mathbb{C}}$ then G is a halfplane in our case so that f is normal. ☑

3.3. Finally we consider the case that Σ is neither elementary nor elliptic. Then $L(\Sigma)$ is uncountable ([3, Th. 5.3.7]); this also holds if Σ is not discrete.

Theorem 8. *Let f be Γ-polymorphic and let $\Sigma = f_*(\Gamma)$ be non-elementary with $L(\Sigma) \neq \emptyset$. Then f is normal if and only if $f(\mathbb{D}) \neq \widehat{\mathbb{C}}$. If f is normal then $\widehat{\mathbb{C}} \setminus L(\Sigma)$ has a Σ-invariant component U such that*

$$f(\mathbb{D}) \subset U, \quad L(\Sigma) = \partial U. \tag{3.9}$$

Proof. If f is normal then $f(\mathbb{D}) \neq \widehat{\mathbb{C}}$ by Corollary 3 because $L(\Sigma) \neq \emptyset$. Conversely suppose that $f(\mathbb{D}) \neq \widehat{\mathbb{C}}$. If $f(\mathbb{D}) = \widehat{\mathbb{C}} \setminus \{a\}$ or $f(\mathbb{D}) = \widehat{\mathbb{C}} \setminus \{a, b\}$ then Σ would be elementary because $f(\mathbb{D})$ is Σ-invariant, see Section 3.2. Hence f omits at least three values and is therefore normal.

Now let f be normal and write $L = L(\Sigma)$. Since $f(\mathbb{D}) \cap L = \emptyset$ it follows from Corollary 3 that $f(\mathbb{D})$ lies in a component U of $\widehat{\mathbb{C}} \setminus L$. If $\sigma \in \Sigma$ then $\sigma(U)$ is a component of $\widehat{\mathbb{C}} \setminus L$ containing $\sigma(f(\mathbb{D})) = f(\mathbb{D})$ so that $\sigma(U) = U$. Hence U is Σ-invariant. It also follows from Corollary 3 that $L \subset \overline{f(\mathbb{D})} \subset \overline{U}$ and thus that $L \subset \overline{U} \cap L \subset \partial U$. We conclude that $L = \partial U$ because $\partial U \subset L$ is trivial. ☑

Corollary 9. *Let Σ be non-elementary with $L(\Sigma) \neq \emptyset$ and suppose that one of the following two conditions holds:*

(i) *Σ is not discrete and has a strictly loxodromic element.*
(ii) *Σ is discrete and $\widehat{\mathbb{C}} \setminus L(\Sigma)$ has no Σ-invariant component.*

Then there is no normal Γ-polymorphic function with $f_*(\Gamma) = \Sigma$.

Proof. Let (i) hold and suppose that f is normal. Then $f(\mathbb{D}) \neq \widehat{\mathbb{C}}$ by Theorem 8. Since Σ is non-elementary and $f(\mathbb{D})$ is Σ-invariant it follows from Proposition 4 (b) that Σ has no strictly loxodromic element, which contradicts (i). If (ii) holds then f cannot be normal by Theorem 8. ☑

The assumption that f is normal puts a rather strong condition on Σ, see for instance [1]. It follows that there are many classes of Kleinian groups for which there are no normal polymorphic functions.

In Examples 4-6, we construct various Fuchsian groups Γ, non-elementary groups Σ and Γ-polymorphic functions f. For a certain choice of the parameters the groups Σ are discrete and the functions f are normal. But for other choices of the parameters the groups are not discrete; the function f is normal in Example 5 and not normal in the other two examples.

4. Construction of examples

We use a classical method to construct functions and groups by conformal mapping and analytic continuation by repeated reflections. We use the following conventions in Examples 1-7.

Let $F \subset \mathbb{D}$ and $G \subset \mathbb{C}$ be Jordan domains that are symmetric with respect to $\mathbb{R}$ and $i\mathbb{R}$. The boundary of F consists of m pairs $A_k,\ A'_k$ of disjoint hyperbolic lines (h-lines) and the boundary of G consists of m pairs of circular arcs or line segments $B_k,\ B'_k$ that are disjoint except for their endpoints. We shall prescribe the A_k and B_k; the A'_k and B'_k are then obtained by reflection in $\mathbb{R}$ unless otherwise stated, in which case they are obtained by reflection in $i\mathbb{R}$.

Let λ_k denote the reflection in $\mathbb{R}$ or $i\mathbb{R}$ and α_k the reflection in A_k. Then

$$\gamma_k := \lambda_k \circ \alpha_k \in \text{ Möb}\,(\mathbb{D})\ , \quad (k = 1, \dots, m)\ , \tag{4.1}$$

maps A_k onto A'_k. The Klein combination theorem ([20, p. 139]) shows that $\Gamma := \langle \gamma_1, \dots, \gamma_m \rangle$ is a Fuchsian group with fundamental domain F. If β_k denotes the reflection in B_k then

$$\sigma_k := \lambda_k \circ \beta_k \in \text{ Möb}\,(\mathbb{C})\ , \quad (k = 1, \dots, m)\ , \tag{4.2}$$

map B_k onto B'_k. The group $\Sigma := \langle \sigma_1, \dots, \sigma_m \rangle$ need not even be discrete.

The constructions of F contain a parameter ζ, a point on $\mathbb{T}$ between 1 and i. Due to the high symmetry of F and G it is possible to choose ζ such that there is a conformal map f of F onto G such that $f\,(A_k) = B_k$ and $f\,(A'_k) = B'_k$ for $k = 1, \dots, m$. This means that the vertices of F are mapped onto the vertices of G.

Proposition 10. *The conformal map f from F onto G has a meromorphic continuation to $\mathbb{D}$ and f is Γ-polymorphic with $\Sigma = f_*\,(\Gamma)$. Moreover we have*

$$\sup_{z \in F} |f'\,(z)| < \infty. \tag{4.3}$$

Proof. Let $k = 1, \dots, m$. It follows from the reflection principle and from (4.1) and (4.2) that

$$f \circ \gamma_k = f \circ \lambda_k \circ \alpha_k = \lambda_k \circ f \circ \alpha_k = \lambda_k \circ \beta_k \circ f = \sigma_k \circ f\ , \tag{4.4}$$

holds on $F \cup A_k \cup \alpha_k\,(F)$. The domains $\gamma_k\,(F)$ and $\sigma_k\,(G)$ are symmetric with respect to the circular arcs $\gamma_k\,(L_k)$ and $\sigma_k\,(L_k)$ where $L_k = \mathbb{R}$ or $i\mathbb{R}$.

We do these for every pair and then keep on reflecting the resulting domains. The domains in $\mathbb{D}$ obtained from F do not overlap whereas the domains in $\mathbb{C}$ obtained from G may overlap. As a limit we obtain a meromorphic function defined in $\mathbb{D}$. By repeated application of (4.4) we see that f is Γ-polymorphic with $f_*\,(\gamma_k) = \sigma_k$, and since the γ_k generate Γ and the σ_k generate Σ, it follows that $\Sigma = f_*\,(\Gamma)$.

Now we prove (4.3). Let h be a conformal map of $\mathbb{D}$ onto F preserving the symmetries. Then $g := f \circ h$ is a conformal map of $\mathbb{D}$ onto G. Since F and G are Jordan domains the functions h and g are continuous and injective in $\overline{\mathbb{D}}$.

By the reflection principle applied to h and g this functions are analytic in $\overline{\mathbb{D}}$ except at the vertices.

It is therefore sufficient to consider h and g near the points $s_k \in \mathbb{T}$ that correspond to the vertices of F and G; note that f maps a vertex of F to a vertex of G. Let πt_k be the angle of G at $g(s_k)$.

Let $s \to s_k,\ s \in \mathbb{D}$. Then

$$|g'(s)| \sim \begin{cases} c_k\,|s-s_k|^{t_k-1} & \text{if} \quad 0<t_k<2, \\ c_k\,|s-s_k|^{-1}\left(\log\frac{1}{|s-s_k|}\right)^{-2} & \text{if} \quad t_k=0. \end{cases}$$

See [23, Th. 3.9] for $t_k>0$ and [21, Th. 6], for $t_k=0$ using that the cusp is formed by two circular arcs. Since all vertices of F are cusps, we have

$$|h'(s)| \sim c'_k\,|s-s_k|^{-1}\left(\log\frac{1}{|s-s_k|}\right)^{-2}.$$

Hence it follows that

$$\limsup_{z\to\varphi(s_k),z\in F} |f'(z)| = \limsup_{s\to s_k,s\in\mathbb{D}} \left|\frac{g'(s)}{h'(s)}\right| < \infty.$$

☑

We choose $m=2$ in the first four examples. Let A_1 be an h-line from ζ to $-\overline{\zeta}$ and A_2 an h-line from ζ to $\overline{\zeta}$. Let A'_2 be obtained from A_2 by reflecting in $i\mathbb{R}$ instead of $\mathbb{R}$. Then the transformations γ_1 and γ_2 are hyperbolic.

Example 1. Let B_1 be an arc on a circle through ± 1 and B_2 an arc on a circle through $\pm i$; the arc B_2 is obtained by reflecting in $i\mathbb{R}$. Then σ_1 is elliptic with fixed points ± 1 and σ_2 is elliptic with fixed points $\pm i$. Hence σ_1 and σ_2 belong to the group $\mathrm{Rot}\left(\widehat{\mathbb{C}}\right)$ of rotations of the sphere and it follows that $\Sigma \subset \mathrm{Rot}\left(\widehat{\mathbb{C}}\right)$. Hence f is normal by (4.3) and Theorem 5. If the angle between B_1 and B_2 is equal to $2\pi/3$ then Σ is the group of order 6 associated with the cube and f omits the 8 vertices of the cube. If the angle ϑ between B_1 and B'_1 satisfies $\vartheta/\pi \notin \mathbb{Q}$ then Σ is not discrete and $f(\mathbb{D}) = \widehat{\mathbb{C}}$.

Example 2. Let $p,q>0$ and $B_1 = [-p+iq,\ p+iq]$, $B_2 = [p+iq,\ p-iq]$. Then G is a rectangle and $\sigma_1(w) = w-2iq$, $\sigma_2(w) = w-2p$. It follows that $\Sigma \subset \text{Möb}(\mathbb{C})$ and that Σ is discrete. We see from (4.3) and (3.7) that $f \in \mathcal{B}$. The function f omits all values $(2n_2+1)\,p + (2n_1+1)\,qi$ with $n_1, n_2 \in \mathbb{Z}$.

Example 3. Let f be the function of the previous example and $\widetilde{f} = \exp f$. With $\sigma = f_*(\gamma)$ we have

$$\widetilde{f}\circ\gamma = \exp(2n_2p + 2n_1qi)\,\widetilde{f}, \quad (n_1, n_2 \in \mathbb{Z}),$$

so that $\widetilde{\Sigma} \subset \text{Möb}(\mathbb{C} \setminus \{0\})$. The group $\widetilde{\Sigma}$ is discrete if and only if q/π is rational but the function $\widetilde{f}$ is always normal by Theorem 6. If $q = \pi$ then $\widetilde{f}$ omits all values $\exp[(2n_2 + 1)\, p]$, if $q > \pi$ then $\widetilde{f}\,(\mathbb{D}) = \mathbb{C} \setminus \{0\}$.

Example 4. Let $q > 1$ and $p \geq (q-1) / \left(4\sqrt{q}\right)$. We consider the two disjoint circles

$$C^{\pm} = \left\{ w \in \mathbb{C} : \left| w \mp p \frac{q+1}{q-1} \right| = \frac{2p\sqrt{q}}{q-1} \right\}.$$

Let G be the domain between C^-, C^+ and the two lines $\{\operatorname{Im} w = \pm i/2\}$ with B_1 on the upper line and B_2 on C^+. The four equal angles ϑ of G are given by $\cos\vartheta = (q-1) / \left(4p\sqrt{q}\right)$. Now $\sigma_1\,(w) = w - i$ is parabolic and

$$\sigma_2\,(w) = \frac{-\,(q+1)\, w + p\,(q-1)}{p^{-1}\,(q-1)\, w - (q+1)}\,, \quad \operatorname{Fix}(\sigma_2) = \{-p, p\}$$

is hyperbolic. The three standard parameters [5] of the two-generator group Σ in terms of the traces are

$$(\text{tr } \sigma_1)^2 - 4 = 0\,, \quad (\text{tr } \sigma_2)^2 - 4 = \frac{(q-1)^2}{q}\,, \quad [\sigma_1, \sigma_2] - 2 = -4\cos^2\vartheta\,.$$

Klimenko and Kopteva have shown that Σ is discrete if and only if $\vartheta = \pi/n$ with $n = 3, 4, \ldots$ or $\vartheta = 0$. See Table 1 # 4 in [10] and Table 2 # 6, # 7 in [11]. Inspection of the fundamental polyhedra [12] shows moreover that Σ has a domain of discontinuity in $\widehat{\mathbb{C}}$ if $\vartheta = \pi/n$ or $\vartheta = 0$; see also [13].

If $\vartheta = 0$ then Σ is a Schottky group with $\bigcup_{\sigma \in \Sigma} \sigma\,(G) \subset \widehat{\mathbb{C}} \setminus L\,(\Sigma)$ so that f is normal. The situation remains unclear if $\vartheta = \pi/n$; computer drawings indicate that f is normal. Now let $0 < \vartheta \neq \pi/n$. Since $\operatorname{tr}(\sigma_1 \circ \sigma_2) \notin \mathbb{R}$ and Σ is not discrete we obtain from Corollary 9 (i) that f is not normal.

We choose $m = 3$ in the next two examples. Let A_1 be again the h-line from ζ to $-\overline{\zeta}$ but now let A_2 be the h-line from 1 to ζ and A_3 the h-line from $-\overline{\zeta}$ to -1. Then γ_1 is hyperbolic whereas γ_2 and γ_3 are now parabolic.

Example 5. The domain G also lies in $\mathbb{D}$. Let B_1 be the h-line from $e^{\pi i/4}$ to $e^{3\pi i/4}$ and let B_2 and B_3 be h-arcs from $e^{i\pi/4}$ to $p > 0$ and from $e^{3\pi i/4}$ to $-p$. Now σ_1 is hyperbolic whereas σ_2 and σ_3 are elliptic. If the angle ϑ of G at p is $2\pi/n$ with $n \geq 3$ then Σ is discrete with fundamental domain G by the Klein combination theorem ([17], p. 119); in this case the elliptic fixed point p lies in $\partial f\,(\mathbb{D})$. If $\vartheta/\pi \notin \mathbb{Q}$ then σ_2 is of infinite order and Σ is not discrete; in this case $f\,(\mathbb{D}) = \mathbb{D}$ so that p lies in $f\,(\mathbb{D})$. But f is bounded and therefore always normal.

Example 6. Let $q > 0$ and let C be a circular arc from 1 through iq to -1. Let B_2 and B_3 be the two arcs of $C \setminus \{iq\}$. Furthermore let B_1 be a circle in $\mathbb{H}$ that touches C at iq. Then σ_1 is hyperbolic whereas $\sigma_2 = \sigma_3$ is elliptic. If the angle ϑ of G at 1 is $2\pi/n$, $n \geq 2$ then Σ is discontinuous with G as fundamental domain by the Klein combination theorem ([20], p. 139) so that f is normal and $f(\mathbb{D})$ is infinitely connected. If $\vartheta/\pi \notin \mathbb{Q}$ then Σ is not discrete and f is not normal because $f(0) = 0$ lies in $L(\Sigma) = i\mathbb{R}$.

The group Σ leaves invariant the right halfplane, where Σ acts as a Fuchsian group of the second kind. Hence Σ is not "truly spatial" ([11], Th. 1.1) and therefore does not appear in Table 2 of that paper.

The final three examples are somewhat different. The second example twice uses the construction process described at the begin of the section, whereas the third example uses the uniformisation theorem.

Example 7. Let A_1 and A_2 be the h-lines from ± 1 to i. Then γ_1 and γ_2 are parabolic. Let $B_1 = [1, iq]$ and $B_2 = [-1, iq]$, where q is chosen such that the angle ϑ between B_1 and B_1' at 1 satisfies $\vartheta/\pi \notin \mathbb{Q}$. Then σ_1 and σ_2 are elliptic of infinite order. We have $f(\mathbb{D}) = \mathbb{C}$ but f is normal by Theorem 6(i) and Proposition 10.

Example 8. (a) We again consider the domain of Example 7 which we now call F_0. Let $G_0 \subset \mathbb{D}$ be bounded by four h-segments C_k, C_k' from $\pm i$ to p and $-p$ where $p > 0$ is chosen such that the angle at p is $\pi/2$. Let g be the conformal map of F_0 onto G_0 mapping vertices to vertices. We obtain parabolic λ_1 and λ_2 for F_0 and elliptic σ_1, σ_2 of order 4 for G_0. Then $\Lambda := \langle \lambda_1, \lambda_2 \rangle$ and $\Sigma := \langle \sigma_1, \sigma_2 \rangle$ are Fuchsian groups and g is Λ-polymorphic with $g(\mathbb{D}) = \mathbb{D}$ and $g_*(\Lambda) = \Sigma$.

(b) Let H_1 and H_2 be disjoint symmetric horodisks that touch $\mathbb{T}$ at ± 1. First we construct F. Let B_1 and B_2 be the h-lines from ζ to i and from $-\overline{\zeta}$ to i, and let B_1', B_2' be their reflections in $\mathbb{R}$. Furthermore let L_1 and L_2 be the arcs of $\mathbb{T}$ from ζ to $\overline{\zeta}$ and from $-\zeta$ to $-\overline{\zeta}$. The domain F is bounded by B_k, B_k' and L_k $(k = 1, 2)$. Now we make repeated reflections starting with the arcs B_k and B_k' in $\mathbb{D}$ but not using the arcs L_k on $\mathbb{T}$. Then we stay in $\mathbb{D}$ and finally obtain a Fuchsian group Γ.

Let h be a conformal map of F onto $F_1 = F_0 \setminus (H_1 \cup H_2)$ preserving the symmetries. Reflecting in the arcs $A_k \cap F_1$ and $A_k' \cap F_1$ we extend h analytically to $\mathbb{D}$. The function h is Γ-polymorphic with $h_*(\Gamma) = \Lambda$ where Λ was constructed in (a). We have

$$h(\mathbb{D}) = \mathbb{D} \setminus \bigcup_{\lambda \in \Lambda} \lambda \left(\overline{H_1} \cup \overline{H_2} \right). \tag{4.5}$$

The function $f := g \circ h$ is Γ-polymorphic with $f_* = g_* \circ h_*$ and therefore $f_*(\Gamma) = g_*(\Lambda) = \Sigma$. We see from (4.5) that $f(\mathbb{D})$ consists of $\mathbb{D}$ minus countably many closed Jordan domains which contain the elliptic fixed points $\sigma(\pm p)$ in their interior. It follows that $\mathrm{Fix}(\sigma) \cap f\left(\overline{\mathbb{D}}\right) = \emptyset$. Note furthermore that $\partial f(\mathbb{D})$ is larger than $L(\Sigma) = \mathbb{T}$. The function f is bounded and therefore normal.

Example 9. Let f be the universal covering map of $\mathbb{D}$ onto

$$V = \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} \{n + i, n - i\},$$

such that $f(0) = 0$, $f'(0) > 0$. Then V is conformally equivalent to $\mathbb{D}/\Lambda$ for some infinitely generated Fuchsian group Λ. We have $V + 1 \subset V$. Since

f is locally univalent it follows that $\gamma := f^{-1} \circ (f+1)$ is locally defined in $\mathbb{D}$ and is therefore globally defined in $\mathbb{D}$ by the monodromy theorem. Since γ is locally univalent and maps $\mathbb{D}$ onto $\mathbb{D}$ it follows that $\gamma \in \text{Möb}(\mathbb{D})$ and f is Γ-polymorphic where $\Gamma = \langle \Lambda, \gamma \rangle$; the group $\Sigma = f_*(\Gamma)$ is generated by $w \longmapsto w+1$. The function f is normal but $f \notin \mathcal{B}$; compare Theorem 7(i).

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