**Revista Colombiana de Matemáticas**
Volumen 42(2008)1, páginas 61-66

# Nonderogatory directed windmills

**Molinos de viento dirigidos no derogatorios**

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Abstract. A directed graph $G$ is nonderogatory if its adjacency matrix $A$ is nonderogatory, i.e., the characteristic polynomial of $A$ is equal to the minimal polynomial of $A$. Given integers $r \geq 2$ and $h \geq 3$, a directed windmill $M_h(r)$ is a directed graph obtained by coalescing $r$ dicycles of length $h$ in one vertex. In this article we solve a conjecture proposed by Gan and Koo ([3]): $M_h(r)$ is nonderogatory if and only if $r = 2$.

*Key words and phrases.* Nonderogatory matrix, characteristic polynomial of directed graphs, directed windmills.

*2000 Mathematics Subject Classification.* 05C50.

Resumen. Un grafo dirigido $G$ es no-derogatorio si su matriz de adyacencia $A$ es no-derogatoria, es decir el polinomio característico de $A$ es igual al polinomio minimal de $A$. Dados enteros $r \geq 2$ y $h \geq 3$, el molino de viento dirigido $M_h(r)$ es un grafo dirigido que se obtiene por medio de la coalescencia de $r$ diciclos de longitud $h$ en un vértice. En este artículo resolvemos una conjetura propuesta por Gan y Koo ([3]) : $M_h(r)$ es no-derogatorio si, y sólo si, $r = 2$.

*Palabras y frases clave.* matriz no-derogatoria, polinomio característico de grafos dirigidos, molinos de viento dirigidos.

## 1. Introduction

A digraph (directed graph) $G = (V, E)$ is defined to be a finite set $V$ and a set $E$ of ordered pairs of elements of $V$. The sets $V$ and $E$ are called the set of vertices and arcs, respectively. If $(u, v) \in E$ then $u$ and $v$ are adjacent and $(u, v)$ is an arc starting at vertex $u$ and terminating at vertex $v$.

Let $\mathcal{M}_n(\mathbb{C})$ denote the space of square matrices of order $n$ with entries in $\mathbb{C}$. Suppose that $\{u_1, \ldots, u_n\}$ is the set of vertices of $G$. The adjacency matrix of $G$ is the matrix $A \in \mathcal{M}_n(\mathbb{C})$ whose entry $a_{ij}$ is the number of arcs starting

[a]Financial support was received from CDCHT-ULA, Project No. C-13490505B.

at $u_i$ and terminating at $u_j$. The characteristic polynomial of $G$ is denoted by $\Phi_G(x)$ (or simply $\Phi_G$) and it is defined as the characteristic polynomial of the adjacency matrix $A$ of $G$, i.e., $\Phi_G(x) = |xI - A|$, where $I$ is the identity matrix.

The monic polynomial of least degree which annihilates $A$ is called the minimal polynomial of $G$ and is denoted by $m_G(x) = m_G$; it divides every polynomial $f \in \mathbb{C}[x]$ such that $f(A) = 0$. In particular, by the Cayley-Hamilton Theorem, $m_G(x)$ divides $\Phi_G(x)$. Moreover, $\Phi_G(x)$ and $m_G(x)$ have the same roots.

A digraph $G$ is nonderogatory if its adjacency matrix $A$ is nonderogatory, i.e., if $\Phi_G(x) = m_G(x)$; otherwise, $G$ is derogatory. For example, dipaths $P_n$, dicycles $C_n$, difans $F_n$ and diwheels $W_n$ are classes of nonderogatory digraphs. These classes of digraphs have been studied by Gan, Lam and Lim ([2],[4] and [5]). More recently ([3]), Gan and Koo considered the problem of determining when the directed windmills are nonderogatory.

Let $h, r$ be integers such that $h \geq 3$ and $r \geq 2$. A directed windmill $M_h(r)$ is the directed graph with $r(h-1)+1$ vertices obtained from the coalescence of $r$ dicycles of length $h$ in one vertex (see Figure 1).

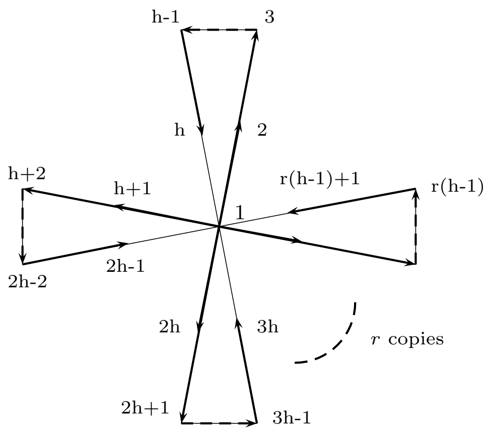


FIGURE 1. The directed windmill $M_h(r) : r$ copies of the dicycle $C_h$.

Gan and Koo showed that $M_3(r)$ is nonderogatory if and only if $r = 2$. Moreover, they conjectured that for every $h \geq 3$

$$M_h(r) \text{ is nonderogatory } \Leftrightarrow r = 2\,.$$

In this paper we show that this conjecture is true.

## 2. Nonderogatory directed windmills

Recall that a linear directed graph is a digraph in which each indegree and each outdegree is equal to 1 (i.e. it consists of cycles). The coefficient theorem for digraphs ([1, Theorem 1.2]) relates the coefficients of the characteristic polynomial with the structure of the digraph.

**Theorem 2.1.** *Let*

$$\Phi_G(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$$

*be the characteristic polynomial of the digraph $G$. Then for each $i = 1, \ldots, n$*

$$a_i = \sum_{L \in \mathcal{L}_i} (-1)^{p(L)} ,$$

*where $\mathcal{L}_i$ is the set of all linear directed subgraphs $L$ of $G$ with exactly $i$ vertices; $p(L)$ denotes the number of components of $L$.*

**Lemma 2.2.** *The characteristic polynomial of $M_h(r)$ is*

$$\Phi_{M_h(r)} = x^{r(h-1)+1} - r x^{r(h-1)+1-h} = x^{r(h-1)+1-h}\left[x^h - r\right] .$$

*Proof.* This is an immediate consequence of Theorem 2.1. ☑

Let $G$ be a directed graph and $A = (a_{ij})$ its adjacency matrix. By a walk of length $k$ in $G$ we mean a sequence of vertices $v_0 v_1 \cdots v_k$ in which each $(v_{i-1}, v_i)$ is an arc of $G$. It is well known that the number of walks of length $k$ between two vertices $v_i$ and $v_j$ of $G$ is $a_{ij}^{(k)}$, the entry $ij$ of the power matrix $A^k$ ([1, Theorem 1.9]).

**Theorem 2.3.** *$M_h(r)$ is nonderogatory if and only if $r = 2$.*

*Proof.* The characteristic polynomial of $M_h(2)$ is

$$\Phi_{M_h(2)} = x^{h-1}\left(x^h - 2\right) .$$

Let $f(x) = x^{h-2}\left(x^h - 2\right)$ and $A = (a_{ij})$ the adjacency matrix of $M_h(2)$. From the structure of $M_h(2)$ it can be easily seen that $a_{h+1,h}^{(2h-2)} = 1$ and $a_{h+1,h}^{(h-2)} = 0$. Consequently $f(A) \neq 0$, which implies that $\Phi_{M_h(2)} = m_{M_h(2)}$ and $M_h(2)$ is nonderogatory.

We next show that if $r \geq 3$ then $M_h(r)$ is derogatory. For $i = 1, \ldots, h-1$, we denote by $e_i$ the canonical row vector of $\mathbb{C}^{h-1}$ and $f_i$ the canonical column vector of $\mathbb{C}^{h-1}$. Labeling the vertices of $M_h(r)$ as shown in Figure 1, the adjacency matrix $A$ of $M_h(r)$ has the form

$$A = \begin{pmatrix} 0 & e_1 & e_1 & \cdots & e_1 \\ f_{h-1} & X & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & \ddots & & \vdots \\ f_{h-1} & \mathbf{0} & \cdots & X & \mathbf{0} \\ f_{h-1} & \mathbf{0} & \mathbf{0} & \cdots & X \end{pmatrix}$$

where $\mathbf{0} \in \mathcal{M}_{h-1}(\mathbb{C})$ is the zero matrix and $X = (x_{ij}) \in \mathcal{M}_{h-1}(\mathbb{C})$ is the matrix such that $x_{i,i+1} = 1$ for $i = 1, \ldots, h-2$, and the rest of the entries of $X$ are zero. Set $Y_1 = X$ , $Z_1 = \mathbf{0}$ and for $j = 2, \ldots, h-1$ define recursively

$$Y_j = f_{h+1-j} e_1 + Y_{j-1} X \tag{1}$$

and

$$Z_j = f_{h+1-j}\, e_1 + Z_{j-1} X \,. \tag{2}$$

We next show that for every $j = 1, \ldots, h-1$

$$A^j = \begin{pmatrix} 0 & e_j & e_j & \cdots & e_j \\ f_{h-j} & Y_j & Z_j & \cdots & Z_j \\ \vdots & & \ddots & & \vdots \\ f_{h-j} & Z_j & \cdots & Y_j & Z_j \\ f_{h-j} & Z_j & Z_j & \cdots & Y_j \end{pmatrix} . \tag{3}$$

In fact, this is clear for $j = 1$. Assume (3) holds for $1 \le i \le h-2$. Note that

$$e_i f_{h-1} = \mathbf{0} \text{ and } e_i X = e_{i+1} \,. \tag{4}$$

On the other hand, since $X f_j = f_{j-1}$ for every $j = 2, \ldots, h-1$ then

$$Y_i f_{h-1} = f_{h+1-i} e_1 f_{h-1} + Y_{i-1} X f_{h-1} = Y_{i-1} f_{h-2}$$

and after $i-1$ steps we deduce

$$Y_i f_{h-1} = Y_{i-1} f_{h-2} = Y_{i-2} f_{h-3} = \cdots = Y_1 f_{h-i} \,.$$

But recall that $Y_1 = X$ and so

$$Y_i f_{h-1} = f_{h-(i+1)} \,. \tag{5}$$

Similarly,

$$Z_i f_{h-1} = Z_{i-1} f_{h-2} = \cdots = Z_1 f_{h-i} \,,$$

but $Z_1 = 0$ implies

$$Z_i f_{h-1} = 0 \,. \tag{6}$$

Also we know that

$$f_{h-i} e_1 + Y_i X = f_{h+1-(i+1)} e_1 + Y_{(i+1)-1} X = Y_{i+1} \tag{7}$$

and

$$f_{h-i} e_1 + Z_i X = Z_{i+1} \,. \tag{8}$$

Consequently, it follows from equations (4)-(8) that

$$A^{i+1} = A^i A = \begin{pmatrix} 0 & e_{i+1} & e_{i+1} & \cdots & e_{i+1} \\ f_{h-(i+1)} & Y_{i+1} & Z_{i+1} & \cdots & Z_{i+1} \\ \vdots & & \ddots & & \vdots \\ f_{h-(i+1)} & Z_{i+1} & \cdots & Y_{i+1} & Z_{i+1} \\ f_{h-(i+1)} & Z_{i+1} & Z_{i+1} & \cdots & Y_{i+1} \end{pmatrix} ,$$

hence (3) holds for every $j = 1, \ldots, h-1$.

On the other hand,

$$e_{h-1}f_{h-1} = 1,\ e_{h-1}X = \mathbf{0},$$

$$Y_{h-1}f_{h-1} = \mathbf{0} = Z_{h-1}f_{h-1},$$

and from repeated use of (1) and the fact that $X^h = \mathbf{0}$,

$$\begin{aligned}
f_1e_1 + Y_{h-1}X &= f_1e_1 + (f_2e_1 + Y_{h-2}X)\,X \\
&= f_1e_1 + f_2e_2 + Y_{h-2}X^2 = \cdots \\
&= \sum_{k=1}^{h-2} f_ke_k + Y_2X^{h-2} = \sum_{k=1}^{h-2} f_ke_k + (f_{h-1}e_1 + Y_1X)\,X^{h-2} \\
&= \sum_{k=1}^{h-2} f_ke_k + f_{h-1}e_1X^{h-2} + X^h = \sum_{k=1}^{h-1} f_ke_k = I\,.
\end{aligned}$$

Similarly, using (2) it can be shown that $f_1e_1 + Z_{h-1}X = I$. It follows from these relations and (3) that

$$A^h = A^{h-1}A = \begin{pmatrix} r & 0 & \cdots & 0 \\ 0 & I & \cdots & I \\ \vdots & \vdots & & \vdots \\ 0 & I & \cdots & I \end{pmatrix}, \tag{9}$$

where the $0's$ in the first row are the zero vectors in $\mathbb{C}^{h-1}$, the $0's$ in the first column are the zero column vectors of $\mathbb{C}^{h-1}$ and $I \in \mathcal{M}_{h-1}(\mathbb{C})$ is the identity.

Relation (9) implies that for every integer $k \geq 2$

$$A^{kh} = \begin{pmatrix} r^k & 0 & \cdots & 0 \\ 0 & r^{k-1}I & \cdots & r^{k-1}I \\ \vdots & \vdots & & \vdots \\ 0 & r^{k-1}I & \cdots & r^{k-1}I \end{pmatrix} = rA^{(k-1)h}\,. \tag{10}$$

Now consider the polynomial $g \in \mathbb{C}[x]$ defined as

$$g(x) = x^{rh-r-h}\left(x^h - r\right)\,,$$

we will show that $g(A) = 0$. To see this, note that since $r \geq 3$ and $h \geq 3$, by the division algorithm, we can find integers $q \geq 2$ and $0 \leq s \leq h-1$ such that

$$rh - r = qh + s\,.$$

From relation (10) we deduce that

$$A^{rh-r} = A^{qh+s} = rA^{(q-1)h+s} = rA^{qh+s-h} = rA^{rh-r-h}$$

which implies $g(A) = 0$ and so $M_h(r)$ is derogatory. ☑

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(Recibido en octubre de 2007. Aceptado en febrero de 2008)

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