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Unmixed bipartite graphs

Grafos bipartitos sin mezcla

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Abstract. In this note we give a combinatorial characterization of all the unmixed bipartite graphs.

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Resumen. En esta nota nosotros presentamos una caracterización combinatoria de todos los grafos bipartitos no-mezcladas.

Palabras y frases clave. Grafos no-mezclados, cubrimiento de vértices mínimo, grafos bipartitos, teorema de König.

1. Unmixed graphs

In the sequel we use [3] as a reference for standard terminology and notation on graph theory.

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A subset $C \subset V(G)$ is a *minimal vertex cover* of G if: (1) every edge of G is incident with one vertex in C, and (2) there is no proper subset of C with the first property. If C satisfies condition (1) only, then C is called a *vertex cover* of G. Notice that C is a minimal vertex cover if and only if $V(G) \setminus C$ is a maximal independent set. A graph G is called *unmixed* if all the minimal vertex covers of G have the same number of elements and it is called *well covered* [6] if all the maximal independent sets of G have the same number of elements.

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The notion of unmixed graph is related to some other graph theoretical and algebraic properties. The following implications hold for any graph without isolated vertices [1, 3, 8]:

$$\text{Cohen-Macaulay} \implies \text{unmixed} \implies B\text{-graph} \implies \text{vertex-critical.}$$

Structural aspects of Cohen-Macaulay bipartite graphs were first studied in [2]. In loc. cit. it is shown that G is Cohen-Macaulay if and only if the simplicial complex Δ_G generated by the maximal independent sets of G is shellable. The main result that we present in this note is the following combinatorial characterization of all the unmixed bipartite graphs. Our result is inspired by a criterion of Herzog and Hibi [4, Theorem 3.4] that describe all Cohen-Macaulay bipartite graphs in combinatorial terms.

Theorem 1.1. *Let G be a bipartite graph without isolated vertices. Then G is unmixed if and only if there is a bipartition $V_1 = \{x_1, \dots, x_g\}$, $V_2 = \{y_1, \dots, y_g\}$ of G such that:* (a) *$\{x_i, y_i\} \in E(G)$ for all i, and* (b) *if $\{x_i, y_j\}$ and $\{x_j, y_k\}$ are in $E(G)$ and i, j, k are distinct, then $\{x_i, y_k\} \in E(G)$.*

Proof. $\Rightarrow$) Since G is bipartite, there is a bipartition (V_1, V_2) of G, i.e., $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, and every edge of G joins V_1 with V_2. Let g be the vertex covering number of G, i.e., g is the number of elements in any minimal vertex cover of G. Notice that V_1 and V_2 are both minimal vertex covers of G, hence $g = |V_1| = |V_2|$. By König theorem [3, Theorem 10.2, p. 96] g is the maximum number of independent edges of G. Therefore after permutation of the vertices we obtain that $V_1 = \{x_1, \dots, x_g\}$, $V_2 = \{y_1, \dots, y_g\}$, and that $\{x_i, y_i\} \in E(G)$ for $i = 1, \dots, g$. Thus we have proved that (a) holds. To prove (b) take $\{x_i, y_j\}$ and $\{x_j, y_k\}$ in $E(G)$ such that i, j, k are distinct. Assume that x_i is not adjacent to y_k. Then there is a maximal independent set of vertices A containing x_i and y_k. Notice that $|A| = g$ because G is unmixed. Hence $C = V(G) \setminus A$ is a minimal vertex cover of G with g vertices. Since x_i and y_k are not on C, we get that y_j and x_j are both in C. As C intersects $\{x_\ell, y_\ell\}$ in at least one vertex for $\ell \neq j$, we obtain that $|C| \geq g + 1$, a contradiction.

$\Leftarrow$) Let C be a minimal vertex cover of G. It suffices to prove that C intersects $\{x_j, y_j\}$ in exactly one vertex for $j = 1, \dots, g$. Assume that x_j and y_j belong to C for some j. If $v \in V(G)$, we denote the neighbor set of v by $N_G(v)$. Thus there are $x_i \in N_G(y_j) \setminus \{x_j\}$ and $y_k \in N_G(x_j) \setminus \{y_j\}$ such that $x_i \notin C$ and $y_k \notin C$. Notice that i, j, k are distinct. Indeed if $i = k$, then $\{x_i, y_i\}$ is an edge of G not covered by C, which is impossible. Therefore using (b) we get that $\{x_i, y_k\}$ is an edge of C, a contradiction. ☑

Ravindra [7] has shown a characterization of well covered bipartite graphs. Namely, G is well covered if and only if for every edge $\{x, y\}$ in the perfect matching, the induced subgraph $\langle N_G(x) \cup N_G(y) \rangle$ is a complete bipartite graph. The advantage of our characterization is that it admits a natural possible extension to hypergraphs and clutters with a perfect matching of König type [5].

As a consequence of Theorem 1.1 we recover the following result on the structure of unmixed trees.

Corollary 1.1. [8, Theorem 2.4, Corollary 2.5] *Let G be a tree with at least three vertices. Then G is unmixed if and only if there is a bipartition* $V_1 = \{x_1, \ldots, x_g\}$, $V_2 = \{y_1, \ldots, y_g\}$ *of G such that:* (a) $\{x_i, y_i\} \in E(G)$ *for all i, and* (b) *for each i either* $\deg(x_i) = 1$ *or* $\deg(y_i) = 1$.

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