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The cohomology solution and the index theorem on ring surfaces of genus g

La solución cohomológica y el teorema del índice para superficies sobre anillos de género g

Changfeng Xue[1,2,a], Wenchang Tan[2,b]

[1]Yancheng Institute of Technology, Yancheng, China
[2]Peking University, Peking, China

Abstract. In this paper, some basic properties of the cohomology solution on ring surfaces of genus g are presented, and the theorem of Dolbeault and the theorem of Serre for the operator $\bar{\partial} = \frac{\partial}{\partial \bar{z}} d\bar{z}$ are obtained. The index theorem on such ring surfaces of genus g is also discussed.

Key words and phrases. Ring surface, cohomology, genus, index.

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Resumen. En este artículo se presentan algunas propiedades básicas de la solución cohomológica para superficies sobre anillos de género g y se obtienen los teoremas de Dolbeault y Serre para el operador $\bar{\partial} = \frac{\partial}{\partial \bar{z}} d\bar{z}$. Se discute el teorema del índice para tales superficies.

Palabras y frases clave. Superficie sobre anillos, cohomología, género, índice.

1. Introduction

A lot of research results have been obtained for the study of the compact Riemann surface [1], [2], [4], [5], [7]. These results, however, focus mainly on its function-theoretic property, but rarely on its topological property. In this paper, we consider the topological property for a special complex compact Riemann surface, namely, the ring surface with genus g. First in Section 2, we discuss its cohomology group solution by presenting some basic properties of

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[b]Corresponding author.

the solution. Then in Section 3, we investigate its index property and arrive at two index theorems. And finally in Section 4, we study its spectral sequence of the Dolbeault double complex form.

2. Some properties of the cohomology group solution on T_g^2

Let T_g^2 be the ring surface of genus g, namely, a complex compact Riemann surface. Let further $\Theta\left(T_g^2\right)$ represent the sheaf of germs of holomorphic functions on T_g^2, and $\Theta^*\left(T_g^2\right)$ the sheaf of germs of holomorphic functions which are never equal to zero on T_g^2. Then, $H^1\left(T_g^2, \Theta^*\left(T_g^2\right)\right)$ represents the holomorphic line bundle group on T_g^2, called Picard group of T_g^2 and denoted by *Pic* T_g^2.

For the compact Riemann surface T_g^2, we have a sheaf exact sequence as follows

$$0 \to Z \to \Theta\left(T_g^2\right) \xrightarrow{exp} \Theta^*\left(T_g^2\right) \to 0,$$

where Z is the additive group of integers.

From the above exact sequence, we can obtain the cohomology exact sequence

$$\begin{aligned} &\to H^0\left(T_g^2, \Theta^*\left(T_g^2\right)\right) \to H^1\left(T_g^2, Z\right) \to H^1\left(T_g^2, \Theta\left(T_g^2\right)\right) \\ &\xrightarrow{\overline{exp}^*} H^1\left(T_g^2, \Theta^*\left(T_g^2\right)\right) \xrightarrow{\delta} H^2\left(T_g^2, Z\right) \to . \end{aligned}$$

Since T_g^2 is the compact Riemann surface with genus g, we have

$$H^1\left(T_g^2, \Theta\left(T_g^2\right)\right) = C^g, \qquad H^0\left(T_g^2, \Theta^*\left(T_g^2\right)\right) = Z\,.$$

$$H^0\left(T_g^2,\ Z\right) = H^2\left(T_g^2,\ Z\right), \qquad H^1\left(T_g^2,\ Z\right) = Z^{2g},$$

And since $H^1\left(T_g^2,\ \Theta^*\left(T_g^2\right)\right) = PicT_g^2$, the above exact sequence becomes

$$\to \mathrm{Z} \to \mathrm{Z}^{2g} \to C^g \xrightarrow{\overline{exp}^*} PicT_g^2 \xrightarrow{\delta} Z \to \ .$$

Let $Pic^0T_g^2$ denote the image of $\overline{exp}^*$ and $NS\left(T_g^2\right)$ the image of δ, also called Nėron-Severi group [8] of T_g^2, respectively. Then we have

Theorem 2.1. *For the ring surface T_g^2 of genus* g*, we have*

$$Pic^0T_g^2 \simeq C^g/Z^{2g}, \quad NS\left(T_g^2\right) \simeq Z.$$

Now let us consider the following Dolbeault complex form on T_g^2,

$$0 \to \Theta\left(T_g^2\right) \xrightarrow{i} \Lambda^{0,0}\left(T_g^2\right) \xrightarrow{\overline{\partial}} \Lambda^{0,1}\left(T_g^2\right) \xrightarrow{\overline{\partial}} \Lambda^{0,2}\left(T_g^2\right) \to 0,$$

where $\Lambda^{0,q}\left(T_g^2\right)$ is the sheaf of germs of complex smooth $(0,q)$-forms on T_g^2 $(q = 0,1,2)$ and $\overline{\partial} = \frac{\partial}{\partial \bar{z}} d\bar{z}$. Letting $H^k(g)$ represent the k-order cohomology solution of the sheaf g, we have

Theorem 2.2. $H^1\left(\Theta\left(T_g^2\right)\right) \simeq H^0\left(\Lambda^{0,1}\left(T_g^2\right)\right)/\overline{\partial}\left(H^0\left(\Lambda^{0,0}\left(T_g^2\right)\right)\right),$

$$H^k\left(\Theta\left(T_g^2\right)\right) = 0, \qquad k > 1.$$

Proof. Consider the mapping $\overline{\partial} : \Lambda^{0,0}\left(T_g^2\right) \to \Lambda^{0,1}\left(T_g^2\right)$. Then $ker\overline{\partial} = \Theta\left(T_g^2\right)$. On the other hand, since $\overline{\partial}$ is a full mapping [6], namely, $I_m\overline{\partial} = \Lambda^{0,1}\left(T_g^2\right)$, the sequence

$$0 \to \Theta\left(T_g^2\right) \xrightarrow{i} \Lambda^{0,0}\left(T_g^2\right) \xrightarrow{\overline{\partial}} \Lambda^{0,1}\left(T_g^2\right) \to 0$$

is exact. And from this exact sequence, we can obtain the cohomology exact sequence

$$0 \to H^0\left(\Theta\left(T_g^2\right)\right) \xrightarrow{i} H^0\left(\Lambda^{0,0}\left(T_g^2\right)\right) \xrightarrow{\bar{\partial}} H^0\left(\Lambda^{0,1}\left(T_g^2\right)\right) \xrightarrow{\delta}$$
$$H^1\left(\Theta\left(T_g^2\right)\right) \to H^1\left(\Lambda^{0,0}\left(T_g^2\right)\right) \to \cdots .$$

Note that both $\Lambda^{0,0}\left(T_g^2\right)$ and $\Lambda^{0,1}\left(T_g^2\right)$ are strong sheaves. Therefore, we have

$$H^k\left(\Lambda^{0,0}\left(T_g^2\right)\right) = H^k\left(\Lambda^{0,1}\left(T_g^2\right)\right) = 0, \quad k \geq 1.$$

Consequently, the above exact sequence can be transformed to

$$0 \to H^0\left(\Theta\left(T_g^2\right)\right) \to H^0\left(\Lambda^{0,0}\left(T_g^2\right)\right) \xrightarrow{\overline{\partial}} H^0\left(\Lambda^{0,1}\left(T_g^2\right)\right) \xrightarrow{\delta} H^1\left(\Theta\left(T_g^2\right)\right) \to 0$$
$$0 \to H^k\left(\Theta\left(T_g^2\right)\right) \to 0, \qquad k > 1.$$

Therefore, we finally have

$$H^1\left(\Theta\left(T_g^2\right)\right) \simeq H^0\left(\Lambda^{0,1}\left(T_g^2\right)\right)/ker\delta \simeq H^0\left(\Lambda^{0,1}\left(T_g^2\right)\right)/\overline{\partial}\left(H^0\left(\Lambda^{0,0}\left(T_g^2\right)\right)\right),$$
$$H^k\left(\Theta\left(T_g^2\right)\right) = 0\ , \qquad k > 1.$$

This completes the proof of Theorem 2.2. ☑

Next, let us consider the following Dolbeault complex form on T_g^2,

$$0 \to \Omega^1\left(T_g^2\right) \xrightarrow{i} \Lambda^{1,0}\left(T_g^2\right) \xrightarrow{\overline{\partial}^*} \Lambda^{1,1}\left(T_g^2\right) \xrightarrow{\overline{\partial}^*} \Lambda^{1,2}\left(T_g^2\right) = 0,$$

where $\Omega^1\left(T_g^2\right)$ is the sheaf of germs of holomorphic 1-forms on T_g^2, and $\Lambda^{1,q}\left(T_g^2\right)$ is the sheaf of germs of complex smooth $(1,q)$-forms on T_g^2 $(q = 0, 1, 2)$, respectively. Since $\bar{\partial}^* = \frac{\partial}{\partial \bar{z}} d\bar{z}$, similarly, we have

Theorem 2.3. $H^1\left(\Omega^1\left(T_g^2\right)\right) \simeq H^0\left(\Lambda^{1,1}\left(T_g^2\right)\right)/\overline{\partial}^*\left(H^0\left(\Lambda^{1,0}\left(T_g^2\right)\right)\right)$,

$$H^0\left(\Omega^1\left(T_g^2\right)\right) \simeq ker\overline{\partial}^*,\ H^k\left(\Omega^1\left(T_g^2\right)\right) = 0, \qquad k > 1.$$

For the complex differential form on the Riemann surface T_g^2, we introduce the exterior differential operator $d = \partial + \overline{\partial}$, where

$$\partial = \frac{\partial}{\partial z} dz: \quad \Lambda^{p,q}\left(T_g^2\right) \to \Lambda^{p+1,q}\left(T_g^2\right),$$
$$\overline{\partial} = \frac{\partial}{\partial \overline{z}} d\overline{z}: \quad \Lambda^{p,q}\left(T_g^2\right) \to \Lambda^{p,q+1}\left(T_g^2\right).$$

On the other hand, using the Hermite gauge, we introduce the remainder differential operator

$$\delta = - * d* = - * \left(\partial + \overline{\partial}\right) * = V + \overline{V},$$

where

$$V = -*\bar{\partial}* : \Lambda^{p,q}\left(T_g^2\right) \to \Lambda^{p-1,q}\left(T_g^2\right),$$
$$\overline{V} = -*\partial* : \Lambda^{p,q}\left(T_g^2\right) \to \Lambda^{p,q-1}\left(T_g^2\right).$$

Now let $\Delta = 2\left(\partial V + V\partial\right)$ represent the Laplace operator of ∂, and $\overline{\Delta} = 2\left(\overline{\partial}\,\overline{V} + \overline{V}\,\overline{\partial}\right)$ the Laplace operator of $\overline{\partial}$, respectively. By simple calculations, in fact we can arrive at that $\Delta = \overline{\Delta}$, $\overline{\Delta}\left(f\, d\overline{z}\right) = 0$ if and only if $\Delta\left(\bar{f} dz\right) = 0$. Therefore, we have

Lemma 2.1. *The space of Harmonic forms on T_g^2 is* $\Pi^{0,1}\left(T_g^2\right) \simeq \Pi^{1,0}\left(T_g^2\right)$.

Theorem 2.4 (Serre dual)**.** $H^1\left(\Theta\left(T_g^2\right)\right) \simeq H^0\left(\Omega^1\left(T_g^2\right)\right)$.

Proof. Using the Dolbeault Theorem [4], we have

$$H^1\left(\Theta\left(T_g^2\right)\right) \simeq H_{\partial}^{0,1}\left(T_g^2\right), \ H^0\left(\Omega^1\left(T_g^2\right)\right) \simeq H_{\overline{\partial}}^{0,1}\left(T_g^2\right).$$

On the other hand, using the Hodge Theorem [4], we have

$$H_{\partial}^{0,1}\left(T_g^2\right) \simeq \Pi^{0,1}\left(T_g^2\right), \ H_{\overline{\partial}}^{0,1}\left(T_g^2\right) \simeq \Pi^{1,0}\left(T_g^2\right).$$

Then using Lemma 2.1, we have $H^1\left(\Theta\left(T_g^2\right)\right) \simeq H^0\left(\Omega^1\left(T_g^2\right)\right)$.

☑

Theorem 2.5. $H^0\left(\Lambda^{0,1}\left(T_g^2\right)\right)/\bar{\partial}\left(H^0\left(\Lambda^{0,0}\left(T_g^2\right)\right)\right) \simeq ker\bar{\partial}^*$.

Proof. From Theorem 2.2, we have

$$H^0\left(\Lambda^{0,1}\left(T_g^2\right)\right)/\overline{\partial}\left(H^0\left(\Lambda^{0,0}\left(T_g^2\right)\right)\right) \simeq H^1\left(\Theta\left(T_g^2\right)\right).$$

On the other hand, from Theorem 2.4, we have

$$H^1\left(\Theta\left(T_g^2\right)\right) \simeq H^0\left(\Omega^1\left(T_g^2\right)\right).$$

Then from Theorem 2.3, we have $H^0\left(\Omega^1\left(T_g^2\right)\right) \simeq ker\overline{\partial}^*$. ☑

3. The index theorem on the ring surface of genus *g*

Now let us consider the Dolbeault complex form on T_g^2

$$0 \to \Theta\left(T_g^2\right) \xrightarrow{i} \Lambda^{0,0}\left(T_g^2\right) \underset{\overline{V}}{\overset{\bar{\partial}}{\longleftrightarrow}} \Lambda^{0,1}\left(T_g^2\right) \underset{\overline{V}}{\overset{\bar{\partial}}{\longleftrightarrow}} \Lambda^{0,2}\left(T_g^2\right) \to 0,$$

where $\overline{V}$ is the dual of $\overline{\partial}$. Since the Laplace operator of $\overline{\partial}$: $\overline{\Delta} = 2\left(\bar{\partial}\,\overline{V} + \overline{V}\,\bar{\partial}\right)$ is an elliptic differential operator, the above complex form is an elliptic complex form with index being [6]

$$\begin{aligned} Ind\left(\overline{\partial}\right) &= \sum (-1)^p \dim H_{\partial}^{0,p}\left(T_g^2\right) \\ &= \sum_{p=0}^{1} (-1)^p \dim \Pi^{0,p}\left(T_g^2\right) = \dim \Pi^{0,0}\left(T_g^2\right) - \dim \Pi^{0,1}\left(T_g^2\right) \\ &= 1 - g, \end{aligned} \tag{3.1}$$

where g is the genus of T_g^2 and $\dim \Pi^{0,1}\left(T_g^2\right) = g$ [3].

The above index property can be generalized. Let $\mu\left(T_g^2\right)$ be the set of all meromorphic functions on T_g^2. For a divisor D given on T_g^2, we can define a divisor sheaf $\Theta_D = \left\{f \in \mu\left(T_g^2\right) | (f) \geq -D\right\}$, where (f) represents the principal divisor of f. If $D = P$ is a point divisor, then Θ_D is the set of all meromorphic functions which have at most a single pole at P. If $D = 0$, then $\Theta_D = \Theta\left(T_g^2\right)$. Furthermore, let $\mu^*\left(T_g^2\right)$ represent the sheaf of germs of the meromorphic functions which are never equal to zero on T_g^2. Then, $\mu^*\left(T_g^2\right)/\Theta^*\left(T_g^2\right)$ represents a divisor presheaf, and the divisor $D \in H^0\left(\mu^*\left(T_g^2\right)/\Theta^*\left(T_g^2\right)\right)$ represents a family of $D = (\mu_i, \alpha_i)$, where $\alpha_i \in \mu^*(\mu_i)$, $\alpha_i/\alpha_j \in \Theta^*(U_i \cap U_j)$, $\underset{i}{\cup} U_i = T_g^2$. The exact sequence of sheaves becomes

$$0 \to \Theta^*\left(T_g^2\right) \to \mu^*\left(T_g^2\right) \to \mu^*\left(T_g^2\right)/\Theta^*\left(T_g^2\right) \to 0,$$

from which we can obtain the long exact sequence

$$\begin{aligned} 0 &\to H^0\left(U^*\left(T_g^2\right)\right) \to H^0\left(U^*\left(T_g^2\right)/\Theta^*\left(T_g^2\right)\right) \xrightarrow{\delta} H^1\left(\Theta^*\left(T_g^2\right)\right) \\ &\to H^1\left(U^*\left(T_g^2\right)\right) \to H^1\left(U^*\left(T_g^2\right)/\Theta^*\left(T_g^2\right)\right) \to H^2\left(\Theta^*\left(T_g^2\right)\right) = 0. \end{aligned} \tag{3.2}$$

Recall that $\Theta^*\left(T_g^2\right)$ is the sheaf of germs of the holomorphic functions which are never equal to zero on T_g^2. Using a result from [6], we have that $H^1\left(\Theta^*\left(T_g^2\right)\right)$ is the first-order cohomology group of the holomorphic line bundles which are never equal to zero on T_g^2.

If the divisor $D \in H^0\left(U^*\left(T_g^2\right)/\Theta^*\left(T_g^2\right)\right)$, then $\delta D = [D]$ represents a holomorphic line bundle whose connectivity function is $g_{ij} = \alpha_i/\alpha_j \in \Theta^*(U_i \cap U_j)$. Here, $[D]$ is also called a holomorphic line bundle generated by the divisor D. Since $[D]$ is a complex line bundle, the first churn class $C_1([D]) \in H^2\left(T_g^2, Z\right)$, and dim $H^1\left(\Theta^*\left(T_g^2\right)\right)$ can be measured by the first churn number [6]. Since any divisor D can be constructed by the point divisors, we have

Lemma 3.1. *Suppose that $D = P$ is a point divisor. Then $[-P] = L$, where* L *is a natural line bundle on the Riemann sphere $CP(1) \simeq S^2$: $L = \left\{\left(x, z^0, z^1\right) \middle| \left(z^0, z^1\right)\right.$ are the homogeneous coordinates of x on $\left.CP(1)\right\}$.*

Proof. Since $D = P$ is a point divisor, we can choose the Riemann surface

$$W = CP(1) = S^2 = CU\{\infty\}$$

as the Riemann sphere. Note that $CP(1)$ can be covered by two open sets

$$U_0 = \left\{\left[z^0, z^1\right] \in P(1) \middle| z^0 \neq 0\right\}$$

and

$$U_1 = \left\{\left[z^0, z^1\right] \in CP(1) \middle| z^1 \neq 0\right\}.$$

Choose the point P as $P = [1, 0]$. Then, on U_0, construct a meromorphic function $\alpha_0\left(\left[z^0, z^1\right]\right) = z^1/z^0$, which has a single pole exactly at the point P, and the divisor $D = P$. Moreover, let $\alpha_1 = 1$ on U_1. Then the connectivity function of the bundle $[P]$, defined on $U_0 \cap U_1$, is $g_{01} = \alpha_0/\alpha_1 = z^1/z^0$.

On the other hand, $CP\,(1)$ has a natural line bundle L, whose partial cross sections on the two domains U_0 and U_1 are $\sigma_0 = \left(1, z^1/z^0\right)$ and $\sigma_1 = \left(z^0/z^1, 1\right)$ [3], respectively. And the connectivity function of the bundle L is defined by $\sigma = \overline{g}_{01}\sigma_0 = \left(z^0/z^1\right)\sigma_0$, namely, $\tilde{g}_{01} = z^0/z^1$. By comparing the connectivity function of the bundle $[P]$ to that of the bundle L, we have that the bundle $[P]$ is the conjugate bundle of the bundle L, that is to say, for the point divisor P^{-1} or $-P$ connected to the single pole P, we can have $[-P] = L$. This completes the proof of Lemma 3.1. ☑

Note that the first churn class of the natural line bundle L of $CP\,(1)$ is the Kähler form on the complex manifold $CP\,(1)$, namely,

$$C_1\,(L) = \frac{-i}{2\pi}\partial\overline{\partial}ln\,(1 + z\overline{z}) = \frac{-i}{2\pi}\frac{dz\Lambda d\overline{z}}{(1 + z\overline{z})^2} = \frac{-1}{\pi}\frac{dx\Lambda dy}{(1 + x^2 + y^2)^2}\,,$$

and the first churn number (again denoted by $C_1\,(L)$) is

$$C_1\,(L) = \frac{-1}{\pi}\int\limits_{s^2}\frac{dx\Lambda dy}{(1 + x^2 + y^2)^2} = \frac{-1}{\pi}\int\limits_0^\infty\int\limits_0^{2\pi}\frac{rdrd\theta}{(1 + r^2)^2} = -1.$$

Then $C_1\,([-P]) = C_1\,(L) = -1$, from which we have

Corollary 3.1. *Suppose that the Riemann surface W has only a point divisor $D = P$. Then dim $H^1\,(\Theta^*\,(W)) = C_1\,([P]) = -deg\,(P) = 1$.*

Corollary 3.2. *For any divisor D on W, dim $H^1\,(\Theta^*\,(W)) = C_1\,([D]) = -$ $deg[D]$, where $deg[D]$ represents the degree of the divisor D.*

From Corollaries 3.1 and 3.2, we obtain

Theorem 3.1. *Suppose that D is the divisor on the compact Riemann surface T_g^2 of genus g. Then*

$$dim\ H^0\left(U^*\left(T_g^2\right)/\Theta^*\left(T_g^2\right)\right) - dim\ H^1\left(U^*\left(T_g^2\right)/\Theta^*\left(T_g^2\right)\right) = deg\,(D) - g + 1.$$

Proof. From [6], we have the Euler number of the exact sequence in (3.2) is 0, i.e.,

$$\chi\left(H\left(U^*\left(T_g^2\right)\right)\right) - \chi\left(H\left(U^*\left(T_g^2\right)/\Theta^*\left(T_g^2\right)\right)\right) - \text{dim}\ H^1\left(\Theta^*\left(T_g^2\right)\right) = 0.$$

From Corollaries 3.1 and 3.2, we have

$$\begin{aligned}\text{dim}\ H^0\left(U^*\left(T_g^2\right)\right) - \text{dim}\ H^1\left(U^*\left(T_g^2\right)\right) - \text{dim}\ H^0\left(U^*\left(T_g^2\right)/\Theta^*\left(T_g^2\right)\right)\\ + \text{dim}\ H^1\left(U^*\left(T_g^2\right)/\Theta^*\left(T_g^2\right)\right) + deg\,(D) = 0.\end{aligned} \tag{3.3}$$

Then, inserting dim $H^0\left(U^*\left(T_g^2\right)\right) = 1$ and dim $H^1\left(U^*\left(T_g^2\right)\right) = g$ [5] in (3.3) leads to

$$\text{dim}\ H^0\left(U^*\left(T_g^2\right)/\Theta^*\left(T_g^2\right)\right) - \text{dim}\ H^1\left(U^*\left(T_g^2\right)/\Theta^*\left(T_g^2\right)\right) = \text{deg}\,(D) - g + 1.$$

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In particular, when the divisor $D = 0$, Theorem 3.1 reduces to the index theorem represented by (3.1). Moreover, when the divisor $D = 0$, and the genus $g = 1$, Theorem 3.1 reduces to Theorem 5 in [9].

From Theorem 3.1 we have

Theorem 3.2. *Suppose that D is the divisor on T_g^2. If $deg\,(D) > 2g-2$, then $dim\ H^1\left(U^*\left(T_g^2\right)/\Theta^*\left(T_g^2\right)\right) = 0$.*

4. The spectral sequence of the Dolbeault double complex form on the n dimensional complex ring surface T^n

Combining the operators ∂ and $\overline{\partial}$ on the complex ring surface T^n, we can obtain the $\partial - \bar{\partial}$ double complex form

$$\begin{array}{lll}
\Lambda^{0,0}\left(T^n\right) \overset{\overline{\partial}}{\rightarrow} & \Lambda^{0,1}\left(T^n\right) \overset{\overline{\partial}}{\rightarrow} & \Lambda^{0,2}\left(T^n\right) \overset{\overline{\partial}}{\rightarrow} \\
\downarrow\partial & \downarrow\partial & \downarrow\partial \\
\Lambda^{1,0}\left(T^n\right) \overset{\overline{\partial}}{\rightarrow} & \Lambda^{1,1}\left(T^n\right) \overset{\overline{\partial}}{\rightarrow} & \Lambda^{1,2}\left(T^n\right) \overset{\overline{\partial}}{\rightarrow} \\
\downarrow\partial & \downarrow\partial & \downarrow\partial \\
\Lambda^{2,0}\left(T^n\right) \overset{\overline{\partial}}{\rightarrow} & \Lambda^{2,1}\left(T^n\right) \overset{\overline{\partial}}{\rightarrow} & \Lambda^{2,2}\left(T^n\right) \overset{\overline{\partial}}{\rightarrow} \\
\downarrow\partial & \downarrow\partial & \downarrow\partial
\end{array}.$$

Define

$$\begin{aligned}
&Z^{i,j}\left(T^n\right) = \left\{\xi \in \Lambda^{i,j}\left(T^n\right) \mid\ \bar{\partial}\xi = 0,\ \partial\xi = 0,\ i, j \geq 1\right\},\\
&B^{i,j}\left(T^n\right) = \left\{\partial\overline{\partial}\Lambda^{i-1,j-1}\left(T^n\right) | i, j \geq 1\right\},\\
&H^{i,j}\left(T^n\right) = Z^{i,j}\left(T^n\right)/B^{i,j}\left(T^n\right).
\end{aligned}$$

Then, obviously we have

Theorem 4.1. *Given $r \in N$ and $r \leq n$. Then*

$$H\left(\underset{i+j=r}{\oplus}\Lambda^{i,j}\left(T^n\right)\right) = \underset{i+j=r}{\oplus} H\left(\Lambda^{i,j}\left(T^n\right)\right) = \underset{i+j=r}{\oplus} H^{i,j}\left(T^n\right).$$

Given $r \in N$. Let

$$T_r\left(T^n\right) = \underset{i+j=r}{\oplus}\Lambda^{i,j}\left(T^n\right),$$

and

$$F^p T_r\left(T^n\right) = \underset{0\leq i\leq p}{\oplus}\Lambda^{i,r-i}\left(T^n\right), 1 \leq p \leq r.$$

Then we have

$$F^p T_r\left(T^n\right) \leq F^{p+1} T_r\left(T^n\right) \leq T_r\left(T^n\right), \quad (*)$$

namely, F is a filter of $T_r(T^n)$. Then we can obtain the commutative diagram as follows

$$\begin{array}{ccc} F^{p-1}T_r(T^n) & \xrightarrow{\partial} & F^pT_{r+1}(T^n) \\ \downarrow\eta & & \downarrow\eta \\ F^pT_r(T^n) & \xrightarrow{\partial} & F^{p+1}T_{r+1}(T^n) \\ \downarrow\pi & & \downarrow\pi \\ \Lambda^{p,r-p}(T^n) & \xrightarrow{\partial} & \Lambda^{p+1,r-p}(T^n) \end{array}$$

where η is an imbedding mapping, and π a natural surjection homomorphism, respectively. From the short exact sequence

$$0 \to F^{p-1}T_r(T^n) \xrightarrow{\eta} F^pT_r(T^n) \xrightarrow{\pi} \Lambda^{p,r-p}(T^n) \to 0,$$

we can obtain the cohomology exact sequence

$$\to \bigoplus_{0\le i\le p-1} H^{i,r-i}(T^n) \xrightarrow{\bar{\eta}^*} \bigoplus_{0\le i\le p} H^{i,r-i}(T^n) \xrightarrow{\bar{\pi}^*}$$

$$H^{p,r-p}(T^n) \xrightarrow{\bar{\delta}^*} \bigoplus_{0\le i\le p} H^{i,r-i+1}(T^n) \to .$$

Furthermore, from the above exact sequence, using the Massey method [10] we can obtain the Lerry spectral sequence

$$\left\{E^s_{p,r-p}(T^n),\ \frac{V}{\partial^s_{p,r-p}}\right\},\quad s\ge 1\,,$$

where

$$E^1_{p,r-p}(T^n) = H^{p,r-p}(T^n)\,,$$

$$\frac{V}{\partial^1_{p,r-p}}H^{p,r-p}(T^n) \to H^{p,r-p+1}(T^n)$$

is the cohomology mapping induced by

$$\overline{\partial}_{p,r-p:}\Lambda^{p,r-p}(T^n) \to \Lambda^{p,r-p+1}(T^n)\,,$$

and

$$E^2_{p,r-p}(T^n) = H\left(E^1_{p,r-p}\right) = \ker\frac{V}{\partial^1_{p,r-p}}/Im\frac{V}{\partial^1_{p,r-p-1}}$$

$$\cdots, E^s_{p,r-p}(T^n) = H\left(E^{s-1}_{p,r-p}\right) = \ker\frac{V}{\partial^{s-1}_{p,r-p}}/Im\frac{V}{\partial^{s-1}_{p,r-p-1}}\ .$$

Here H is the cohomology functor.

Note that the filter $(*)$ is finite, namely,

$$0 = F^{-1}T_r(T^n) \subseteq F^0T_r(T^n) \subseteq \cdots \subseteq F^pT_r(T^n) \subseteq \cdots \subseteq F^rT_r(T^n) = T_r(T^n)\,.$$

Hence, from [10], we have that for $1 \le p \le r \le n \in N$, there exists a certain number s such that

(1)

$$E^s_{p,r-p}\left(T^n\right) = E^{s+1}_{p,r-p}\left(T^n\right) = \cdots .$$

(2)

$$\begin{aligned}
E^s_{p,r-p}\left(T^n\right) &= F^p H\left(T_r\left(T^n\right)\right)/F^{p-1}H\left(T_r\left(T^n\right)\right)\\
&= F^p H\left(\bigoplus_{i+j=r}\Lambda^{i,j}\left(T^n\right)\right)/F^{p-1}H\left(\bigoplus_{i+j=r}\Lambda^{i,j}\left(T^n\right)\right)\\
&= F^p \bigoplus_{i+j=r} H^{i,j}\left(T^n\right)/F^{p-1}\bigoplus_{i+j=r} H^{i,j}\left(T^n\right)\\
&= \bigoplus_{0\le i\le p} H^{i,r-i}\left(T^n\right)/\bigoplus_{0\le i\le p-1} H^{i,r-i}\left(T^n\right)\\
&= H^{p,r-p}\left(T^n\right)\\
&= E^1_{p,r-p}\left(T^n\right).
\end{aligned}$$

Therefore, we finally have

Theorem 4.2. *For the $\partial - \overline{\partial}$ double complex form on the n dimensional complex ring surface T^n, the Lerry spectral sequence*

$$\left\{E^s_{p,r-p}\left(T^n\right), \frac{V}{\partial^s_{p,r-p}}\right\}, \quad s \geq 1$$

converges to $E^1_{p,r-p}\left(T^n\right) = H^{p,r-p}\left(T^n\right)$.

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Department of Fundamental Sciences
Yancheng Institute of Technology
Yancheng, Jiangsu 224003
P.R. China
e-mail: chfxue@gmail.com

State Key Laboratory of Turbulence and Complex Systems
College of Engineering
Peking University
Beijing 100871, P.R. China
e-mail: tanwch@mech.pku.edu.cn