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More on λ-closed sets in topological spaces

Más sobre conjuntos λ-cerrados en espacios topológicos

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Abstract. In this paper, we introduce and study topological properties of λ-derived, λ-border, λ-frontier and λ-exterior of a set using the concept of λ-open sets. We also present and study new separation axioms by using the notions of λ-open and λ-closure operator.

Key words and phrases. Topological spaces, Λ-sets, λ-open sets, λ-closed sets, λ-R_0 spaces, λ-R_1 spaces.

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Resumen. En este artículo introducimos y estudiamos propiedades topológicas de λ-derivada, λ-borde, λ-frontera y λ-exterior de un conjunto usando el concepto de λ-conjunto abierto. Presentamos un nuevo estudio de axiomas de separación usando las nociones de operador λ-abierto y λ-clausura.

Palabras y frases clave. Espacios topológicos, Λ-conjuntos, conjuntos λ-abiertos, conjuntos λ-cerrados, espacios λ-R_0, espacios λ-R_1.

1. Introduction

Maki [12] introduced the notion of Λ-sets in topological spaces. A Λ-set is a set A which is equal to its kernel(= saturated set), i.e. to the intersection of all open supersets of A. Arenas et al. [1] introduced and investigated the notion of λ-closed sets and λ-open sets by involving Λ-sets and closed sets. This enabled them to obtain some nice results. In this paper, for these sets, we introduce the notions of λ-derived, λ-border, λ-frontier and λ-exterior of a set and show that some of their properties are analogous to those for open sets. Also, we give some additional properties of λ-closure. Moreover, we offer and study new separation axioms by utilizing the notions of λ-open sets and λ-closure operator.

Throughout this paper we adopt the notations and terminology of [12] and [1] and the following conventions: (X,τ), (Y,σ) and (Z,ν) (or simply X, Y and Z) will always denote spaces on which no separation axioms are assumed unless explicitly stated.

Definition 1. *Let B be a subset of a space (X,τ). B is a Λ-set (resp. V-set)* [12] *if $B = B^\Lambda$ (resp. $B = B^V$), where:*

$$B^\Lambda = \bigcap\{U \mid U \supset B, U \in \tau\} \quad \text{and} \quad B^V = \bigcup\{F \mid B \supset F, F^c \in \tau\}.$$

Theorem 1.1 ([12]). *Let A, B and $\{B_i \mid i \in I\}$ be subsets of a space (X,τ). Then the following properties are valid:*

a) $B \subset B^\Lambda$.
b) *If* $A \subset B$ *then* $A^\Lambda \subset B^\Lambda$.
c) $B^{\Lambda\Lambda} = B^\Lambda$.
d) $\left(\bigcup_{i\in I} B_i\right)^\Lambda = \bigcup_{i\in I} B_i^\Lambda$.
e) *If* $B \in \tau$, *then* $B = B^\Lambda$ *(i.e. B is a Λ-set).*
f) $(B^c)^\Lambda = \left(B^V\right)^c$.
g) $B^V \subset B$.
h) *If* $B^c \in \tau$, *then* $B = B^V$ *(i.e. B is a V-set).*
i) $\left(\bigcap_{i\in I} B_i\right)^\Lambda \subset \bigcap_{i\in I} B_i^\Lambda$.
j) $\left(\bigcup_{i\in I} B_i\right)^V \supset \bigcup_{i\in I} B_i^V$.
k) *If B_i is a Λ-set ($i \in I$), then $\bigcup_{i\in I} B_i$ is a Λ-set.*
l) *If B_i is a Λ-set ($i \in I$), then $\bigcap_{i\in I} B_i$ is a Λ-set.*
m) *B is a Λ-set if and only if B^c is a V-set.*
n) *The subsets $\emptyset$ and X are Λ-sets.*

2. Applications of λ-closed sets and λ-open sets

Definition 2. *A subset A of a space (X,τ) is called λ-closed* [1] *if $A = B \cap C$, where B is a Λ-set and C is a closed set.*

Lemma 2.1. *For a subset A of a space (X,τ), the following statements are equivalent* [1]*:*
(a) A is λ-closed.
(b) $A = L \cap Cl(A)$, where L is a Λ-set.
(c) $A = A^\Lambda \cap Cl(A)$.

Lemma 2.2. *Every Λ-set is a λ-closed set.*

Proof. Take $A \cap X$, where A is a Λ-set and X is closed. ☑

Remark 2.3. [1]. *Since locally closed sets and λ-sets are concepts independent of each other, then a λ-closed set need not be locally closed or be a Λ-set. Moreover, in each T_0 non-T_1 space there are singletons which are λ-closed but not a Λ-set.*

Definition 3. *A subset A of a space (X, τ) is called λ-open if $A^c = X \backslash A$ is λ-closed.*

We denote the collection of all λ-open (resp. λ-closed) subsets of X by $\lambda O(X)$ or $\lambda O(X, \tau)$ (resp. $\lambda C(X)$ or $\lambda C(X, \tau)$). We set $\lambda O(X, x) = \{V \in \lambda O(X) \mid x \in V\}$ for $x \in X$. We define similarly $\lambda C(X, x)$.

Theorem 2.4. *The following statements are equivalent for a subset A of a topological space X:*
(a) A is λ-open.
(b) $A = T \cup C$, where T is a V-set and C is an open set.

Lemma 2.5. *Every V-set is λ-open.*

Proof. Take $A = A \cup \emptyset$, where A is V-set, X is Λ-set and $\emptyset = X \backslash X$. ☑

Definition 4. *Let (X, τ) be a space and $A \subset X$. A point $x \in X$ is called λ-cluster point of A if for every λ-open set U of X containing x, $A \cap U \neq \emptyset$. The set of all λ -cluster points is called the λ-closure of A and is denoted by $Cl_\lambda(A)$.*

Lemma 2.6. *Let A, B and A_i ($i \in I$) be subsets of a topological space (X, τ). The following properties hold:*
(1) If A_i is λ-closed for each $i \in I$, then $\cap_{i \in I} A_i$ is λ-closed.
(2) If A_i is λ-open for each $i \in I$, then $\cup_{i \in I} A_i$ is λ-open.
(3) A is λ-closed if and only if $A = Cl_\lambda(A)$.
(4) $Cl_\lambda(A) = \cap\{F \in \lambda C(X, \tau) \mid A \subset F\}$.
(5) $A \subset Cl_\lambda(A) \subset Cl(A)$.
(6) If $A \subset B$, then $Cl_\lambda(A) \subset Cl_\lambda(B)$.
(7) $Cl_\lambda(A)$ is λ-closed.

Proof. (1) It is shown in [1], 3.3
(2) It is an immediate consequence of (1).
(3) Straightforward.
(4) Let $H = \bigcap\{F \mid A \subset F,\ F$ is λ-closed$\}$. Suppose that $x \in H$. Let U be a λ-open set containing x such that $A \bigcap U = \emptyset$. And so, $A \subset X \backslash U$. But $X \backslash U$ is λ-closed and hence $Cl_\lambda(A) \subset X \backslash U$. Since $x \notin X \backslash U$, we obtain $x \notin Cl_\lambda(A)$ which is contrary to the hypothesis.

On the other hand, suppose that $x \in Cl_\lambda(A)$, i.e., that every λ-open set of X containing x meets A. If $x \notin H$, then there exists a λ-closed set F of X

such that $A \subset F$ and $x \notin F$. Therefore $x \in X \backslash F \in \lambda O(X)$. Hence $X \backslash F$ is a λ-open set of X containing x, but $(X \backslash F) \bigcap A = \emptyset$. But this is a contradiction and thus the claim.
(5) It follows from the fact that every closed set is λ–closed. ☑

In general the converse of 2.6(5) may not be true.

Example 2.7. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. *Then* $Cl(\{a\}) = \{a, c\} \not\subset Cl_\lambda(\{a\}) = \{a\}$.

Definition 5. *Let* A *be a subset of a space* X. *A point* $x \in X$ *is said to be* λ*-limit point of* A *if for each* λ*-open set* U *containing* x, $U \cap (A \backslash \{x\}) \neq \emptyset$. *The set of all* λ*-limit points of* A *is called a* λ*-derived set of* A *and is denoted by* $D_\lambda(A)$.

Theorem 2.8. *For subsets* A, B *of a space* X, *the following statements hold:*

(1) $D_\lambda(A) \subset D(A)$ *where* $D(A)$ *is the derived set of* A.
(2) *If* $A \subset B$, *then* $D_\lambda(A) \subset D_\lambda(B)$.
(3) $D_\lambda(A) \cup D_\lambda(B) \subset D_\lambda(A \cup B)$ *and* $D_\lambda(A \cap B) \subset D_\lambda(A) \cap D_\lambda(B)$.
(4) $D_\lambda(D_\lambda(A)) \backslash A \subset D_\lambda(A)$.
(5) $D_\lambda(A \cup D_\lambda(A)) \subset A \cup D_\lambda(A)$.

Proof. (1) It suffices to observe that every open set is λ-open.
(3) it is an immediate consequence of (2).
(4) If $x \in D_\lambda(D_\lambda(A)) \backslash A$ and U is a λ -open set containing x, then $U \cap (D_\lambda(A) \backslash \{x\}) \neq \emptyset$. Let $y \in U \cap (D_\lambda(A) \backslash \{x\})$. Then since $y \in D_\lambda(A)$ and $y \in U$, $U \cap (A \backslash \{y\}) \neq \emptyset$. Let $z \in U \cap (A \backslash \{y\})$. Then $z \neq x$ for $z \in A$ and $x \notin A$. Hence $U \cap (A \backslash \{x\}) \neq \emptyset$. Therefore $x \in D_\lambda(A)$.
(5) Let $x \in D_\lambda(A \cup D_\lambda(A))$. If $x \in A$, the result is obvious. So let $x \in D_\lambda(A \cup D_\lambda(A)) \backslash A$, then for λ-open set U containing x, $U \cap (A \cup D_\lambda(A) \backslash \{x\}) \neq \emptyset$. Thus $U \cap (A \backslash \{x\}) \neq \emptyset$ or $U \cap (D_\lambda(A) \backslash \{x\}) \neq \emptyset$. Now it follows from (4) that $U \cap (A \backslash \{x\}) \neq \emptyset$. Hence $x \in D_\lambda(A)$. Therefore, in any case $D_\lambda(A \cup D_\lambda(A)) \subset A \cup D_\lambda(A)$. ☑

In general the converse of (1) may not be true and the equality does not hold in (3) of Theorem 2.8.

Example 2.9. *Let* $X = \{a, b, c\}$ *with topology* $\tau = \{\emptyset, \{a\}, \{a\}, X\}$. *Thus* $\lambda O(X, \tau) = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. *Take:*

(i) $A = \{a\}$. *We obtain* $D(A) \not\subset D_\lambda(A)$.
(ii) $C = \{a\}$ *and* $E = \{b, c\}$. *Then* $D_\alpha(C \cup E) \neq D_\alpha(C) \cup D_\alpha(E)$.

Theorem 2.10. *For any subset* A *of a space* X, $Cl_\lambda(A) = A \cup D_\lambda(A)$.

Proof. Since $D_\lambda(A) \subset Cl_\lambda(A)$, $A \cup D_\lambda(A) \subset Cl_\lambda(A)$. On the other hand, let $x \in Cl_\lambda(A)$. If $x \in A$, then the proof is complete. If $x \notin A$, then each λ-open set U containing x intersects A at a point distinct from x. Therefore $x \in D_\lambda(A)$. Thus $Cl_\lambda(A) \subset A \cup D_\lambda(A)$ which completes the proof. ☑

Definition 6. *A point $x \in X$ is said to be a λ-interior point of A if there exists a λ-open set U containing x such that $U \subset A$. The set of all λ-interior points of A is said to be λ-interior of A and is denoted by $Int_\lambda(A)$.*

Theorem 2.11. *For subsets A, B of a space X, the following statements are true:*
(1) $Int_\lambda(A)$ is the largest λ-open set contained in A.
(2) A is λ-open if and only if $A = Int_\lambda(A)$.
(3) $Int_\lambda(Int_\lambda(A)) = Int_\lambda(A)$.
(4) $Int_\lambda(A) = A \backslash D_\lambda(X \backslash A)$.
(5) $X \backslash Int_\lambda(A) = Cl_\lambda(X \backslash A)$.
(6) $X \backslash Cl_\lambda(A) = Int_\lambda(X \backslash A)$.
(7) $A \subset B$, then $Int_\lambda(A) \subset Int_\lambda(B)$.
(8) $Int_\lambda(A) \cup Int_\lambda(B) \subset Int_\lambda(A \cup B)$.
(9) $Int_\lambda(A) \cap Int_\lambda(B) \supset Int_\lambda(A \cap B)$.

Proof. (4) If $x \in A \backslash D_\lambda(X \backslash A)$, then $x \notin D_\lambda(X \backslash A)$ and so there exists a λ-open set U containing x such that $U \cap (X \backslash A) = \emptyset$. Then $x \in U \subset A$ and hence $x \in Int_\lambda(A)$, i.e., $A \backslash D_\lambda(X \backslash A) \subset Int_\lambda(A)$. On the other hand, if $x \in Int_\lambda(A)$, then $x \notin D_\lambda(X \backslash A)$ since $Int_\lambda(A)$ is λ-open and $Int_\lambda(A) \cap (X \backslash A) = \emptyset$. Hence $Int_\lambda(A) = A \backslash D_\lambda(X \backslash A)$.
(5) $X \backslash Int_\lambda(A) = X \backslash (A \backslash D_\lambda(X \backslash A)) = (X \backslash A) \cup D_\lambda(X \backslash A) = Cl_\lambda(X \backslash A)$. ☑

Definition 7. *$b_\lambda(A) = A \backslash Int_\lambda(A)$ is said to be the λ-border of A.*

Theorem 2.12. *For a subset A of a space X, the following statements hold:*
(1) $b_\lambda(A) \subset b(A)$ where $b(A)$ denotes the border of A.
(2) $A = Int_\lambda(A) \cup b_\lambda(A)$.
(3) $Int_\lambda(A) \cap b_\lambda(A) = \emptyset$.
(4) A is a λ-open set if and only if $b_\lambda(A) = \emptyset$.
(5) $b_\lambda(Int_\lambda(A)) = \emptyset$.
(6) $Int_\lambda(b_\lambda(A)) = \emptyset$.
(7) $b_\lambda(b_\lambda(A)) = b_\lambda(A)$.
(8) $b_\lambda(A) = A \cap Cl_\lambda(X \backslash A)$.
(9) $b_\lambda(A) = D_\lambda(X \backslash A)$.

Proof. (6) If $x \in Int_\lambda(b_\lambda(A))$, then $x \in b_\lambda(A)$. On the other hand, since $b_\lambda(A) \subset A$, $x \in Int_\lambda(b_\lambda(A)) \subset Int_\lambda(A)$. Hence $x \in Int_\lambda(A) \cap b_\lambda(A)$ which contradicts (3). Thus $Int_\lambda(b_\lambda(A)) = \emptyset$.
(8) $b_\lambda(A) = A \backslash Int_\lambda(A) = A \backslash (X \backslash Cl_\lambda(X \backslash A)) = A \cap Cl_\lambda(X \backslash A)$.
(9) $b_\lambda(A) = A \backslash Int_\lambda(A) = A \backslash (A \backslash D_\lambda(X \backslash A)) = D_\lambda(X \backslash A)$. ☑

Definition 8. *$Fr_\lambda(A) = Cl_\lambda(A) \backslash Int_\lambda(A)$ is said to be the λ-frontier of A.*

Theorem 2.13. *For a subset A of a space X, the following statements are hold:*
(1) $Fr_\lambda(A) \subset Fr(A)$ where $Fr(A)$ denotes the frontier of A.

(2) $Cl_\lambda(A) = Int_\lambda(A) \cup Fr_\lambda(A)$.
(3) $Int_\lambda(A) \cap Fr_\lambda(A) = \emptyset$.
(4) $b_\lambda(A) \subset Fr_\lambda(A)$.
(5) $Fr_\lambda(A) = b_\lambda(A) \cup D_\lambda(A)$.
(6) A is a λ*-open set if and only if* $Fr_\lambda(A) = D_\lambda(A)$.
(7) $Fr_\lambda(A) = Cl_\lambda(A) \cap Cl_\lambda(X\backslash A)$.
(8) $Fr_\lambda(A) = Fr_\lambda(X\backslash A)$.
(9) $Fr_\lambda(A)$ *is* λ*-closed.*
(10) $Fr_\lambda(Fr_\lambda(A)) \subset Fr_\lambda(A)$.
(11) $Fr_\lambda(Int_\lambda(A)) \subset Fr_\lambda(A)$.
(12) $Fr_\lambda(Cl_\lambda(A)) \subset Fr_\lambda(A)$.
(13) $Int_\lambda(A) = A\backslash Fr_\lambda(A)$.

Proof. (2) $Int_\lambda(A) \cup Fr_\lambda(A) = Int_\lambda(A) \cup (Cl_\lambda(A)\backslash Int_\lambda(A)) = Cl_\lambda(A)$.
(3) $Int_\lambda(A) \cap Fr_\lambda(A) = Int_\lambda(A) \cap (Cl_\lambda(A)\backslash Int_\lambda(A)) = \emptyset$.
(5) Since $Int_\lambda(A) \cup Fr_\lambda(A) = Int_\lambda(A) \cup b_\lambda(A) \cup D_\lambda(A)$; $Fr_\lambda(A) = b_\lambda(A) \cup D_\lambda(A)$.
(7) $Fr_\lambda(A) = Cl_\lambda(A)\backslash Int_\lambda(A) = Cl_\lambda(A) \cap Cl_\lambda(X\backslash A)$.
(9) $Cl_\lambda(Fr_\lambda(A)) = Cl_\lambda(Cl_\lambda(A) \cap Cl_\lambda(X\backslash A)) \subset Cl_\lambda(Cl_\lambda(A)) \cap Cl_\lambda(Cl_\lambda(X\backslash A)) = Fr_\lambda(A)$. Hence $Fr_\lambda(A)$ is λ-closed.
(10) $Fr_\lambda(Fr_\lambda(A)) = Cl_\lambda(Fr_\lambda(A)) \cap Cl_\lambda(X\backslash Fr_\lambda(A)) \subset Cl_\lambda(Fr_\lambda(A)) = Fr_\lambda(A)$.
(12) $Fr_\lambda(Cl_\lambda(A)) = Cl_\lambda(Cl_\lambda(A))\backslash Int_\lambda(Cl_\lambda(A)) = Cl_\lambda((A))\backslash Int_\lambda(Cl_\lambda(A)) = \mathrm{Cl}_\lambda(A)\backslash Int_\lambda(A) = Fr_\lambda(A)$.
(13) $A\backslash Fr_\lambda(A) = A\backslash(Cl_\lambda(A)\backslash Int_\lambda(A)) = Int_\lambda(A)$. ☑

The converses of (1) and (4) of the Theorem 2.13 are not true in general as are shown by Example 2.14.

Example 2.14. *Consider the topological space* (X, τ) *given in Example 2.7 . If* $A = \{a\}$*. Then* $Fr(A) \not\subseteq Fr_\lambda(A)$ *and if* $B = \{a, c\}$*, then* $Fr_\lambda(B) \not\subseteq b_\lambda(B)$*.*

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is said to be λ-continuous [1] if $f^{-1}(V) \in \lambda C(X)$ for every closed subset V of Y.

Theorem 2.15. *For a function* $f : X \to Y$*, the following are equivalent:*
(1) f *is* λ*-continuous;*
(2) for every open subset V *of* Y*,* $f^{-1}(V) \in \lambda O(X)$*;*
(3) for each $x \in X$ *and each* $V \in O(Y, f(x))$*, there exists* $U \in \lambda O(X, x)$ *such that* $f(U) \subset V$*.*

Proof. (1) → (2) : This follows from $f^{-1}(Y\backslash V) = X\backslash f^{-1}(V)$.
(1) → (3) : Let $V \in O(Y)$ and $f(x) \in V$. Since f is λ-continuous $f^{-1}(V) \in \lambda O(X)$ and $x \in f^{-1}(V)$. Put $U = f^{-1}(V)$. Then $x \in U$ and $f(U) \subset V$.
(3) → (1) : Let V be an open set of Y and $x \in f^{-1}(V)$. Then $f(x) \in V$. Therefore by (3) there exists a $U_x \in \lambda O(X)$ such that $X \in U_x$ and $f(U_x) \subset V$. Therefore $X \in U_x \subset f^{-1}(V)$. This implies that $f^{-1}(V)$ is a union of λ-open sets of X. Consequently $f^{-1}(V) \subset \lambda O(X)$. Hence f is λ-continuous. ☑

In the following theorem $\sharp\Lambda.c.$ denotes the set of points x of X for which a function $f:(X,\tau)\to(Y,\sigma)$ is not λ-continuous.

Theorem 2.16. *$\sharp\Lambda.c.$ is identical with the union of the λ-frontiers of the inverse images of λ-open sets containing $f(x)$.*

Proof. Suppose that f is not λ-continuous at a point x of X. Then there exists an open set $V\subset Y$ containing $f(x)$ such that $f(U)$ is not a subset of V for every $U\in\lambda O(X)$ containing x. Hence we have $U\cap\big(X\setminus f^{-1}(V)\big)\neq\emptyset$ for every $U\in\lambda O(X)$ containing x. It follows that $x\in Cl_\lambda\big(X\setminus f^{-1}(V)\big)$. We also have $x\in f^{-1}(V)\subset Cl_\lambda\big(f^{-1}(V)\big)$. This means that $x\in Fr_\lambda\big(f^{-1}(V)\big)$.
Now, let f be λ-continuous at $x\in X$ and $V\subset Y$ be any open set containing $f(x)$. Then $x\in f^{-1}(V)$ is a λ-open set of X. Thus $x\in Int_\lambda(f^{-1}(V))$ and therefore $x\notin Fr_\lambda\big(f^{-1}(V)\big)$ for every open set V containing $f(x)$. ☑

Definition 9. *$Ext_\lambda(A)=Int_\lambda(X\backslash A)$ is said to be a λ-exterior of A.*

Theorem 2.17. *For a subset A of a space X, the following statements are hold:*
(1) $Ext(A)\subset Ext_\lambda(A)$ where $Ext(A)$ denotes the exterior of A.
(2) $Ext_\lambda(A)$ is λ-open.
(3) $Ext_\lambda(A)=Int_\lambda(X\backslash A)=X\backslash Cl_\lambda(A)$.
(4) $Ext_\lambda(Ext_\lambda(A))=Int_\lambda(Cl_\lambda(A))$.
(5) If $A\subset B$, then $Ext_\lambda(A)\supset Ext_\lambda(B)$.
(6) $Ext_\lambda(A\cup B)\subset Ext_\lambda(A)\cup Ext_\lambda(B)$.
(7) $Ext_\lambda(A\cap B)\supset Ext_\lambda(A)\cap Ext_\lambda(B)$.
(8) $Ext_\lambda(X)=\emptyset$.
(9) $Ext_\lambda(\emptyset)=X$.
(10) $Ext_\lambda(A)=Ext_\lambda(X\backslash Ext_\lambda(A))$.
(11) $Int_\lambda(A)\subset Ext_\lambda(Ext_\lambda(A))$.
(12) $X=Int_\lambda(A)\cup Ext_\lambda(A)\cup Fr_\lambda(A)$.

Proof. (4) $Ext_\lambda(Ext_\lambda(A)) = Ext_\lambda(X\backslash Cl_\lambda(A)) = Int_\lambda(X\backslash(X\backslash Cl_\lambda(A))) = Int_\lambda(Cl_\lambda(A))$.
(10) $Ext_\lambda(X\backslash Ext_\lambda(A))=Ext_\lambda(X\backslash Int_\lambda(X\backslash A))=Int_\lambda(X\backslash(X\backslash Int_\lambda(X\backslash A)))=Int_\lambda(Int_\lambda(X\backslash A))=Int_\lambda(X\backslash A)=Ext_\lambda(A)$.
(11) $Int_\lambda(A)\subset.Int_\lambda(Cl_\lambda(A))=Int_\lambda(X\backslash Int_\lambda(X\backslash A))=Int_\lambda(X\backslash Ext_\lambda(A))=Ext_\lambda(Ext_\lambda(A))$. ☑

Example 2.18. *Consider the topological space (X,τ) given in Example 2.7. Hence, if $A=\{a\}$ and $B=\{b\}$, Then $Ext_\lambda(A)\nsubseteq Ext(A)$, $Ext_\lambda(A\cap B)\neq Ext_\lambda(A)\cap Ext_\lambda(B)$ and $Ext_\lambda(A\cup B)\neq Ext_\lambda(A)\cup Ext_\lambda(B)$.*

3. Some new separation axioms

We recall with the following notions which will be used in the sequel:

A space (X,τ) is said to be R_0 [3] (resp. λ-R_0 [2]) if every open set contains the closure of each of its singletons. A space (X,τ) is said to be R_1 [3] (resp. λ-R_1 [2]) if for x, y in X with $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $Cl(\{x\})$ is a subset of U and $Cl(\{y\})$ is a subset of V. A space is T_0 if for $x, y \in X$ such that $x \neq y$ there exists a open set U of X containing x but not y or an open set V of X containing y but not x. A space (X,τ) is T_1 if to each pair of distinct points x and y of X, there exists a pair of open sets one containing x but not y and the other containing y but not x. A space is (X,τ) is T_2 if to each pair of distinct points x and y of X, there exists a pair of disjoint open sets, one containing x and the other containing y. Recall that a space (X,τ) is called a $T_{\frac{1}{2}}$-space [11] if every generalized closed subset of X is closed or equivalently if every singleton is open or closed [6]. In [1], Arenas et al. have shown that a space (X,τ) is called a $T_{\frac{1}{2}}$-space if and only if every subset of X is λ-closed.

Definition 10. *Let X be a space. A subset $A \subset X$ is called a λ-Difference set (in short λ-D-set) if there are two λ-open sets U, V in X such that $U \neq X$ and $A = U \setminus V$.*

It is true that every λ-open set $U \neq X$ is a λ-D-set since $U = U \setminus \emptyset$.

Definition 11. *A space (X,τ) is said to be:*

- (i) *λ-D_0 (resp. λ-D_1) if for $x, y \in X$ such that $x \neq y$ there exists a λ-D-set of X containing x but not y or (resp. and) a λ-D-set containing y but not x.*
- (ii) *A topological space (X,τ) is λ-D_2 if for $x, y \in X$ such that $x \neq y$ there exist disjoint λ-D-sets G and E such that $x \in G$ and $y \in E$.*
- (iii) *λ-T_0 (resp. λ-T_1) if for $x, y \in X$ such that $x \neq y$ there exists a λ-open set U of X containing x but not y or (resp. and) a λ-open set V of X containing y but not x.*
- (iv) *λ-T_2 if for $x, y \in X$ such that $x \neq y$ there exist disjoint λ-open sets U and V such that $x \in U$ and $y \in V$.*

Remark 3.1.

- (i) *If (X,τ) is λ-T_i, then it is λ-T_{i-1}, $i = 1, 2$.*
- (ii) *Obviously, if (X,τ) is λ-T_i, then (X,τ) is λ-D_i, $i = 0, 1, 2$.*
- (iii) *If (X,τ) is λ-D_i, then it is λ-D_{i-1}, $i = 1, 2$.*

Theorem 3.2. *For a space (X,τ) the following statements are true:*
(1) (X,τ) is λ-D_0 if and only if (X,τ) is λ-T_0.
(2) (X,τ) is λ-D_1 if and only if , (X,τ) is λ-D_2.

Proof. The sufficiency for (1) and (2) follows from the Remark 3.1.
Necessity condition for (1). Let (X,τ) be λ-D_0 so that for any distinct pair of points x and y of X at least one belongs to a λ-D set O. Therefore we choose $x \in O$ and $y \notin O$. Suppose $O = U \setminus V$ for which $U \neq X$ and U and V are λ-open sets in X. This implies that $x \in U$. For the case that $y \notin O$ we have

(i) $y \notin U$, (ii) $y \in U$ and $y \in V$. For (i), the space X is λ-T_0 since $x \in U$ and $y \notin U$. For (ii), the space X is also λ-T_0 since $y \in V$ but $x \notin V$.

The necessity condition for (2). Suppose that X is λ-D_1. It follows from the definition that for any distinct points x and y in X there exist λ-D sets G and E such that G containing x but not y and E containing y but not x. Let $G = U \setminus V$ and $E = W \setminus D$, where U, V, W and D are λ-open sets in X. By the fact that $x \notin E$, we have two cases, i.e. either $x \notin W$ or both W and D contain x. If $x \notin W$, then from $y \notin G$ either (i) $y \notin U$ or(ii) $y \in U$ and $y \in V$. If (i) is the case, then it follows from $x \in U \setminus V$ that $x \in U \setminus (V \cup W)$, and also it follows from $y \in W \setminus D$ that $y \in (U \cup D)$. Thus we have $U \setminus (V \cup W)$ and $W \setminus (U \cup D)$ which are disjoint. If (ii) is the case, it follows that $x \in U \setminus V$ and $y \in V$ since $y \in U$ and $y \in V$. Therefore $(U \setminus V) \cap V = \emptyset$. If $x \in W$ and $x \in D$, we have $y \in W \setminus D$ and $x \in D$. Hence $(W \setminus D) \cap D = \emptyset$. This shows that X is λ-D_2. ☑

Theorem 3.3. *If (X, τ) is λ-D_1, then it is λ-T_0.*

Proof. Remark 3.1(iii) and Theorem 3.2. ☑

We give an example which shows that the converse of Theorem 3.3 is false.

Example 3.4. *Let $X = \{a, b\}$ with topology $\tau = \{\emptyset, \{a\}, X\}$. Then (X, τ) is λ-T_0, but not λ-D_1 since there is not a λ-D-set containing a but not b.*

Example 3.5. *Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, \{c\}, \{b\}, \{b, c\}, \{b, c, d\}, X\}$. Then we have that $\{a\}, \{a, d\}, \{a, b, d\}$ and $\{a, c, d\}$ are λ-open and (X, τ) is a λ-D_1, since $\{a\}$, $\{b\} = \{a, b, d\} \setminus \{a, d\}$, $\{c\} = \{a, c, d\} \setminus \{a, d\}$, $\{d\} = \{a, d\} \setminus \{a\}$. But (X, τ) is not λ-T_2.*

Example 3.6.

(1) As a consequence of the Example 3.4, we obtain that (X, τ) is λ-T_0, but not λ-T_1.
(2) As a consequence of the Example 3.5, we obtain that (X, τ) is λ-T_0, but not λ-T_2.

A subset B_x of a space X is said to be a λ-neighbourhood of a point $x \in X$ if and only if there exists a λ-open set A such that $x \in A \subset B_x$.

Definition 12. *Let x be a point in a space X. If x does not have a $\lambda-$neighborhood other than X, then we call x a λ-neat point. neigtbourhood*

Theorem 3.7. *For a λ-T_0 space (X, τ) the following are equivalent:*
(1) (X, τ) is λ-D_1;
(2) (X, τ) has no λ-neat point.

Proof. (1) $\rightarrow$ (2) : If X is λ-D_1 then each point $x \in X$ belongs to a λ-D-set $A = U \setminus V$; hence $x \in U$. Since $U \neq X$, thus x is not a λ-neat point.
(2) $\rightarrow$ (1) : If X is λ-T_0, then for each distinct pair of points $x, y \in X$, at least one of x, y, say x has a λ -neighborhood U such that $x \in U$ and $y \notin U$.

Hence $U \neq X$ is a λ-D-set. If X does not have a λ-neat point, then y is not a λ-neat point. So there exists a λ-neighbourhood V of y such that $V \neq X$. Now $y \in V \backslash U$, $x \notin V \backslash U$ and $V \backslash U$ is a λ-D-set. Therefore X is λ-D_1. ☑

Corollary 3.8. *A λ-T_0 space X is not λ-D_1 if and only if there is a unique λ-neat point in X.*

Proof. We only prove the uniqueness of the λ-neat point. If x and y are two λ-neat points in X, then since X is λ-T_0, at least one of x and y, say x, has a λ-neighborhood U such that $x \in U$, $y \notin U$. Hence $U \neq X$. Therefore x is not a λ-neat point which is a contradiction. ☑

Theorem 3.9. *A space X is λ-T_0 if and only if for each pair of distinct points x, y of X, $Cl_\lambda(\{x\}) \neq Cl_\lambda(\{y\})$.*

Proof. Sufficiency. Suppose that $x, y \in X$, $x \neq y$ and $Cl_\lambda(\{x\}) \neq Cl_\lambda(\{y\})$. Let z be a point of X such that $z \in Cl_\lambda(\{x\})$ but $z \notin Cl_\lambda(\{y\})$. We claim that $x \notin Cl_\lambda(\{y\})$. For, if $x \in Cl_\lambda(\{y\})$, then $Cl_\lambda(\{x\}) \subset Cl_\lambda(\{y\})$. This contradicts the fact that $z \notin Cl_\lambda(\{y\})$. Consequently x belongs to the λ-open set $[Cl_\lambda(\{y\})]^c$ to which y does not belong.
Necessity. Let X be a λ-T_0 space and x, y be any two distinct points of X. There exists a λ-open set G containing x or y, say x but not y. Then G^c is a λ-closed set which does not contain x but contains y. Since $Cl_\lambda(\{y\})$ is the smallest λ-closed set containing y (Lemma 2.6), $Cl_\lambda(\{y\}) \subset G^c$, and so $x \notin Cl_\lambda(\{y\})$. Consequently $Cl_\lambda(\{x\}) \neq Cl_\lambda(\{y\})$. ☑

Theorem 3.10. *A space X is λ-T_1 if and only if the singletons are λ-closed sets.*

Proof. Suppose X is λ-T_1 and x is any point of X. Let $y \in \{x\}^c$. Then $x \neq y$. So there exists a λ-open set A_y such that $y \in A_y$ but x$\notin A_y$. Consequently $y \in A_y \subset \{x\}^c$ i.e., $\{x\}^c = \bigcup\{A_y / y \in \{x\}^c\}$ which is λ -open.
Conversely, let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in \{x\}^c$. Hence$\{x\}^c$ is a λ-open set containing y but not x. Similarly $\{y\}^c$ is a λ-open set containing x but not y. Accordingly X is a λ-T_1 space. ☑

Theorem 3.11. *A topological space X is λ-T_1 if and only if X is T_0.*

Proof. This is proved by Theorem 3.10 and [1][Theorem 2.5.] ☑

Example 3.12. *The Khalimsky line or the so-called digital line (*[8], [9]*) is the set of the integers, $\mathbf{Z}$, equipped with the topology $\mathbf{K}$, having $\{\{2n-1, 2n, 2n+1\} : n \in \mathbf{Z}\}$ as a subbase. This space is of great importance in the study of applications of point-set topology to computer graphics. In the digital line $(\mathbf{Z}, K)$, every singleton is open or closed, that is, the digital line is T_0. Thus by Theorem 3.11, the digital line is λ-T_1 which is not T_1.*

Remark 3.13. *From Example 3.4, Example 3.5, Example 3.6 and Example 3.12 we have the following diagram:*

$$\begin{array}{ccc} \lambda\text{-}T_0 & & \\ \uparrow & \nwarrow & \\ \lambda\text{-}T_1 & \rightarrow & \lambda\text{-}D_1 \\ \uparrow & \nearrow & \\ \lambda\text{-}T_2 & & \end{array}$$

(1) $T_1 \Longrightarrow \lambda$-$T_1$ and $T_2 \Longrightarrow \lambda$-$T_2$. The converses are not true:

Example 3.14. *Let (X,τ) be a topological space such that $X = \{a,b,c\}$ and $\tau = \{\emptyset, \{a\}, \{a,b\}, X\}$. Then we have that*

$$\lambda O(X,\tau) = \{\emptyset, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}\,.$$

Therefore:

(i) (X,τ) is λ-T_1 but it is not T_1. (see also as another example the Khalimsky line i.e., the digital line which is given in Example 3.12).

(ii) (X,τ) is λ-T_2 but it is not T_2.

(2) T_0 implies λ-T_0 But the converse is not true as it is shown in the following example.

Example 3.15. *Let $X = \{a,b\}$ with topology $\tau = \{\emptyset, \{a\}, X\}$. Then (X,τ) is λ-T_0. Observe that (X,τ) is not T_0.*

(3) λ-T_1 implies λ-T_0 and λ-T_2 implies λ-T_0. The converses are not true (Example 3.6).

(4) λ-R_1 implies λ-R_0. The converse is not true (Example 3.15).

(5) λ-T_1 does not imply R_0 and λ-T_0 does not imply R_0. (Example 3.14).

(6) R_1 implies R_0 [3]. The converse is not true as it is shown by the following example.

Example 3.16. *Let $X = \{a,b\}$ with indiscrete topology τ. Then (X,τ) is R_0 but it is not R_1.*

(7) (i) λ-$R_0 \not\Longrightarrow R_0$ and (ii) λ-$R_1 \not\Longrightarrow R_1$ (Example 3.14).

(8) (8) (i) $T_{\frac{1}{2}}$ implies T_0 which is equivalent with λ-T_1 (see Theorem 3.11) and (ii) $T_{\frac{1}{2}}$ implies λ-$T_{\frac{1}{2}}$. The converses are not true. For case (i), it is well known and for case (ii), it follows form the fact that every λ-T_1 is λ-$T_{\frac{1}{2}}$ (where a topological space is λ-$T_{\frac{1}{2}}$ [2] if every singleton is λ-open or λ-closed).

(9) λ-$T_1 \not\Longrightarrow T_{\frac{1}{2}}$. It is shown in the following example.

Example 3.17. *[*[1]*[Example 3.2]] Let X be the set of non-negative integers with the topology whose open sets are those which contain 0 and have finite complement. This space is not $T_{\frac{1}{2}}$, but it is T_0 is equivalent with λ-T_1 (see Theorem 3.11). Therefore also λ-$T_{\frac{1}{2}}$ does not imply $T_{\frac{1}{2}}$.*

(10) X is a $T_{\frac{1}{4}}$-space [1] if and only if every finite subset of X is λ-closed. We see that $T_{\frac{1}{4}}$-space is strictly placed between $T_{\frac{1}{2}}$ and λ-T_1. On the other hand, the space $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ is λ-T_1 but not $T_{\frac{1}{4}}$. Example 3.17 is a example of a space $T_{\frac{1}{4}}$ which is not $T_{\frac{1}{2}}$.

In what follows, we refer the interested reader to [10] for the basic definitions and notations. Recall that a representation of a C^*-algebra $\mathcal{A}$ consists of a Hilbert space $\mathcal{H}$ and a *-morphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the C^*-algebra of bounded operators on $\mathcal{H}$. A subspace $\mathcal{I}$ of a C^*-algebra $\mathcal{A}$ is called a primitive ideal if $\mathcal{A}= ker(\pi)$ for some irreducible representation $(\mathcal{H}, \pi)$ of $\mathcal{A}$. The set of all primitive ideals of a C^*-algebra $\mathcal{A}$ plays a very important role in noncommutative spaces and its relation to particle physics. We denote this set by Prim $\mathcal{A}$. As Landi [10] mentions, for a noncommutative C^*-algebra, there is more than one candidate for the analogue of the topological space X:

1. The structure space of $\mathcal{A}$ or the space of all unitary equivalence classes of irreducible *-representations and
2. The primitive spectrum of $\mathcal{A}$ or the space of kernels of irreducible *-representations which is denoted by Prim $\mathcal{A}$. Observe that any element of Prim $\mathcal{A}$ is a two-sided *-ideal of $\mathcal{A}$.

It should be noticed that for a commutative C^*-algebra, 1 and 2 are the same but this is not true for a general C^*-algebra $\mathcal{A}$. Natural topologies can be defined on spaces of 1 and 2. But here we are interested in the Jacobsen (or hull-kernel) topology defined on Prim $\mathcal{A}$ by means of closure operators. The interested reader may refer to [4] for basic properties of Prim $\mathcal{A}$. It follows from Theorem 3.11 that Prim $\mathcal{A}$ is also a λ-T_1-space. Jafari [7] has shown that T_1-spaces are precisely those which are both R_0 and λ-T_1.

Theorem 3.18. *A space X is λ-T_2 if and only if the intersection of all λ-closed λ-neighborhoods of each point of the space is reduced to that point.*

Proof. Let X be λ-T_2 and $x \in X$. Then for each $y \in X$, distinct from x, there exist λ-open sets G and H such that $x \in G$, $y \in H$ and $G \cap H = \emptyset$. Since $x \in G \subset H^c$, then H^c is a λ-closed λ-neighborhood of x to which y does not belong. Consequently, the intersection of all λ-closed λ -neighborhood of x is reduced to $\{x\}$.
Conversely, let x, $y \in X$ and $x \neq y$. Then by hypothesis, there exists a λ-closed λ-neighbourhood U of x such that $y \notin U$. Now there is a λ-open set G such that $x \in G \subset U$. Thus G an U^c are disjoint λ-open sets containing x and y, respectively. Hence X is λ-T_2. ☑

Definition 13. *A space (X, τ) will be termed λ-symmetric if for any x and y in X, $x \in Cl_\lambda(\{y\})$ implies $y \in Cl_\lambda(\{x\})$.*

Definition 14. *A subset A of a space (X, τ) is called a λ-generalized closed set (briefly λ-g-closed) if $Cl_\lambda(A) \subset U$ whenever $A \subset U$ and U is λ-open in (X, τ).*

Lemma 3.19. *Every λ-closed set is λ-g-closed.*

Example 3.20. *In Example 3.6, if $A = \{a\}$, then A is a λ-g-closed set, but it is not a λ-closed set (hence it is not a closed set).*

Theorem 3.21. *Let (X, τ) be a space. Then,*
(i) (X, τ) is λ-symmetric if and only if $\{x\}$ is λ–g-closed for each x in X.
(ii) If (X, τ) is a λ-T_1 space, then (X, τ) is λ-symmetric.
(iii) (X, τ) is λ-symmetric and λ-T_0 if and only if (X, τ) is λ-T_1.

Proof. (i) Sufficiency. Suppose $x \in Cl_\lambda(\{y\})$, but $y \notin Cl_\lambda(\{x\})$. Then $\{y\} \subset [Cl_\lambda(\{x\})]^c$ and thus $Cl_\lambda(\{y\}) \subset [Cl_\lambda(\{x\})]^c$. Then $x \in [Cl_\lambda(\{x\})]^c$, a contradiction.
Necessity. Suppose $\{x\} \subset E \in \lambda O(X, \tau) = \{B \subset X \mid B \text{ is } \lambda\text{-open}\}$, but $Cl_\lambda(\{x\}) \nsubseteq E$. Then $Cl_\lambda(\{x\}) \cap E^c \neq \emptyset$; take $y \in Cl_\lambda(\{x\}) \cap E^c$. Therefore $x \in Cl_\lambda(\{y\}) \subset E^c$ and $x \notin E$, a contradiction.
(ii) In a λ-T_1 space, singleton sets are λ-closed (Theorem 3.10) and therefore λ-g-closed (Lemma 3.19). By (i), the space is λ-symmetric.
(iii) By (ii) and Remark 3.1(i) it suffices to prove only the necessity condition. Let $x \neq y$. By λ-T_0, we may assume that $x \in E \subset \{y\}^c$ for some $E \in \lambda O(X, \tau)$. Then $x \notin Cl_\lambda(\{y\})$ and hence $y \notin Cl_\lambda(\{x\})$. There exists a $F \in \lambda O(X, \tau)$ such that $y \in F \subset \{x\}^c$ and thus (X, τ) is a λ-T_1 space. ☑

Theorem 3.22. *Let (X, τ) be a λ-symmetric space. Then the following are equivalent.*
(i) (X, τ) is λ-T_0,
(ii) (X, τ) is λ-D_1,
(iii) (X, τ) is λ-T_1.

Proof. $(i) \to (iii)$: Theorem 3.21.
$(iii) \to (ii) \to (i)$: Remark 3.1 and Theorem 3.3. ☑

A function $f : (X, \tau) \to (Y, \sigma)$ is called λ-irresolute if $f^{-1}(V)$ is λ-open in (X, τ) for every λ-open set V of (Y, σ).

Example 3.23. *Let (X, τ) be as Example 3.14 and $f : (X, \tau) \to (X, \tau)$ such that $f(a) = c$, $f(b) = c$ and $f(a) = a$. Then f is λ-irresolute, but it is not irresolute.*

Example 3.24 ([1])**.** *Consider the classical Dirichlet function $f : \mathbf{R} \to \mathbf{R}$, where $\mathbf{R}$ is the real line with the usual topology:*

$$f(x) = \begin{cases} 1 & \text{if} \quad x \text{ is rational} \\ \\ 0 & \text{if} \quad x \text{ is otherwise} \end{cases}$$

Therefore f is λ-continuous, but it is not continuous.

Theorem 3.25. *If $f : (X, \tau) \to (Y, \sigma)$ is a λ-irresolute surjective function and S is a λ-D-set in Y, then $f^{-1}(A)$ is a λ-D-set in X.*

Proof. Let A be a λ-D-set in Y. Then there are λ-open sets U and V in Y such that $A = U\backslash V$ and $U \neq Y$. By the λ-irresoluteness of f, $f^{-1}(U)$ and $f^{-1}(V)$ are λ-open in X. Since $U \neq Y$, we have $f^{-1}(U) \neq X$. Hence $f^{-1}(A) = f^{-1}(U)\backslash f^{-1}(V)$ is a λ-D-set. ☑

Theorem 3.26. *If (Y, σ) is λ-D_1 and $f : (X, \tau) \to (Y, \sigma)$ is λ-irresolute and bijective, then (X, τ) is λ-D_1.*

Proof. Suppose that Y is a λ-D_1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is λ-D_1, there exist λ-D-sets A_x and B_y of Y containing $f(x)$ and $f(y)$ respectively, such that $f(y) \notin A_x$ and $f(x) \notin B_y$. By Theorem 3.25, $f^{-1}(A_x)$ and $f^{-1}(B_y)$ are $\lambda - D-$ sets in X containing x and y, respectively. This implies that X is a λ-D_1 space. ☑

We now prove another characterization of λ-D_1 spaces.

Theorem 3.27. *A space X is $\lambda - D_1$ if and only if for each pair of distinct points x and y in X, there exists a λ-irresolute surjective function f of X onto a λ-D_1 space Y such that $f(x) \neq f(y)$.*

Proof. Necessity. For every pair of distinct points of X, it suffices to take the identity function on X.
Sufficiency. Let x and y be any pair of distinct points in X. By hypothesis, there exists a λ-irresolute, surjective function f of a space X onto a λ-D_1 space Y such that $f(x) \neq f(y)$. Therefore, there exist disjoint λ-D-sets A_x and B_y in Y such that $f(x) \in A_x$ and $f(y) \in B_y$. Since f is λ-irresolute and surjective, by Theorem 3.25, $f^{-1}(A_x)$ and $f^{-1}(B_y)$ are disjoint λ-D-sets in X containing x and y, respectively. Hence by Theorem 3.2(2), X is λ-D_1 space. ☑

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