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Algebraic closure in continuous logic

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Abstract. We study the algebraic closure construction for metric structures in the setting of continuous first order logic. We give several characterizations of algebraicity, and we prove basic properties analogous to ones that algebraic closure satisfies in classical first order logic.

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Resumen. Estudiamos la construcción de la clausura algebraica para estructuras métricas en el contexto de la lógica continua de primer orden. Damos varias caracterizaciones de algebricidad y probamos propiedades básicas análogas a aquellas que satisface la clausura algebraica en lógica clásica de primer orden.

1. Definition and characterizations

We use the $[0,1]$-valued version of continuous logic presented by Ben Yaacov, Berenstein, Henson, and Usvyatsov in [1], [3]. Throughout this paper we fix a metric signature L. For simplicity we assume that L is 1-sorted.

Recall [1] that, given an L-structure $\mathcal{M}$ and $A \subset M$, a set $S \subset M^n$ is A-definable if and only if $\operatorname{dist}(x,S)$ is an A-definable predicate. In turn, this means that $\operatorname{dist}(x,S)$ is the uniform limit on M^n of a sequence of the interpretations in $\mathcal{M}$ of $L(A)$-formulas $(\varphi_n(x) \mid n \in \mathbb{N})$.

Definition 1.1. *[Algebraic closure] Let $\mathcal{M}$ be an L-structure and $A \subset M$. The* algebraic closure *of A in $\mathcal{M}$, denoted* $\operatorname{acl}_{\mathcal{M}}(A)$*, is the union of all compact subsets of M that are A-definable in $\mathcal{M}$. An element $a \in \operatorname{acl}_{\mathcal{M}}(A)$ is said to be* algebraic over A in $\mathcal{M}$ *(or simply* algebraic in $\mathcal{M}$ *in the case $A = \emptyset$).*

For many proofs in this section we will take $A = \emptyset$, for simplicity of notation. This is done without loss of generality, since $\operatorname{acl}_{\mathcal{M}}(A) = \operatorname{acl}_{\mathcal{M}(A)}(\emptyset)$, where $\mathcal{M}(A)$ is the $L(A)$-structure $(\mathcal{M}, a)_{a \in A}$. In continuous logic, structures are taken to be complete for their metrics. We will denote by $\bar{A}$ the closure of a set A in the metric topology.

The following result about compact 0-definable sets will prove useful. It is suggested by the analogy between compact sets in continuous logic and finite sets in classical first order logic.

Lemma 1.2. *Let $\mathcal{M}$ be a metric structure, and let K be a compact subset of M, 0-definable in $\mathcal{M}$. Let $\mathcal{M}' \succcurlyeq \mathcal{M}$ and let $Q\colon M' \to [0,1]$ be a predicate such that $(\mathcal{M}', Q(x)) \succcurlyeq (\mathcal{M}, \mathrm{dist}(x,K))$. Then the zero set of Q in $\mathcal{M}'$ is K.*

Proof. Let $n \geq 1$ be arbitrary. Since K is compact, it has a finite $\frac{1}{n}$-dense set for each n, say of size k_n. Extend the language with a set of new constants $(c_j^{(n)} | 1 \leq j \leq k_n)$ for each $n \geq 1$, to be interpreted in $\mathcal{M}$ by a $\frac{1}{n}$-dense subset of K. Then, in $\mathcal{M}$ it is true that for all $x \in M$, $0 = \mathrm{dist}(x,K)$ implies that there exists $1 \leq j \leq k_n$ such that $\mathrm{d}(x, c_j^{(n)}) \leq \frac{1}{n}$. By the triangle inequality, this implies that for every $x \in M$, if $\mathrm{dist}(x,K) < \frac{1}{n}$ then there exists $1 \leq j \leq k_n$ such that $\mathrm{d}(x, c_j^{(n)}) \leq \frac{2}{n}$. This can be expressed in continuous logic by the condition $0 = \sup_x(\min(\frac{1}{n} \dot{-} \mathrm{dist}(x,K), \mathrm{d}(x, c_j^{(n)}) \dot{-} \frac{2}{n} | j = 1, \ldots, k_n))$. This holds in $\mathcal{M}'$; hence, for each $x \in M'$, $Q(x) < \frac{1}{n}$ implies that for some $1 \leq j \leq k_n$, $\min(\mathrm{d}(x, c_j^{(n)})) \leq \frac{2}{n}$. Let K' be the zero set of Q in $\mathcal{M}'$. Then every element of K' is the limit of some sequence from $\{c_j^{(n)} | n \geq 1,\, 1 \leq j \leq k_n\}$, which is a subset of K. So $K' \subset \bar{K} = K$. Hence $K' = K$. ☑

Corollary 1.3. *Let $\mathcal{M} \preccurlyeq \mathcal{M}'$ be L-structures. If $K \subset M$ is compact and 0-definable in $\mathcal{M}$, then K is 0-definable in $\mathcal{M}'$.*

Proof. Let $Q\colon M' \to [0,1]$ be a predicate 0-definable in $\mathcal{M}'$ such that

$$(\mathcal{M}', Q(x)) \succcurlyeq (\mathcal{M}, \mathrm{dist}(x,K)).$$

Q is in fact the uniform limit in $\mathcal{M}'$ of a sequence of formulas that converges uniformly to $\mathrm{dist}(x,K)$ in $\mathcal{M}$. By the previous lemma we have that the zero set of Q in $\mathcal{M}'$ is K. On the other hand, in $\mathcal{M}$ the following conditions are true:

$$0 = \sup_x \inf_y \max(\mathrm{dist}(y,K), |\,\mathrm{dist}(x,K) - \mathrm{d}(x,y)|); \qquad (E_1)$$

$$0 = \sup_x |\,\mathrm{dist}(x,K) - \inf_y \min(\mathrm{dist}(y,K) + \mathrm{d}(x,y), 1)|. \qquad (E_2)$$

Hence the same is valid in $\mathcal{M}'$. So Q satisfies (E_1) and (E_2). Therefore, by [1, Section 9], $Q(x) = \mathrm{dist}(x,K)$ for all $x \in M'$. Therefore K is 0-definable in $\mathcal{M}'$. ☑

Corollary 1.4. *Let $\mathcal{M} \preccurlyeq \mathcal{N}$ be L-structures, and A a set in $\mathcal{M}$. Then $\mathrm{acl}_{\mathcal{M}}(A) = \mathrm{acl}_{\mathcal{N}}(A)$.*

Proof. As mentioned above, without loss of generality we may set $A = \emptyset$.
($\subseteq$) Let K be a compact subset of M, 0-definable in $\mathcal{M}$. Then K is compact in $\mathcal{N}$, and 0-definable in $\mathcal{N}$ by the previous corollary, so $\mathrm{acl}_{\mathcal{M}}(\emptyset) \subseteq \mathrm{acl}_{\mathcal{N}}(\emptyset)$.
($\supseteq$) Let K be a compact subset of N, 0-definable in $\mathcal{N}$. Then $\mathrm{dist}(x, K)$ is 0-definable in $\mathcal{N}$. The restriction of $\mathrm{dist}(x, K)$ to $\mathcal{M}$ is $\mathrm{dist}(x, K \cap M)$, so by [1, Section 9],

$$(\mathcal{N}, \mathrm{dist}(x, K)) \succcurlyeq (\mathcal{M}, \mathrm{dist}(x, K \cap M)). \tag{1.1}$$

The 0-definability of $\mathrm{dist}(x, K)$ in $\mathcal{N}$ also implies that there is a sequence of formulas $(\varphi_n | n \geq 1)$ and a continuous function $u\colon [0,1]^{\mathbb{N}} \to [0,1]$ such that for all $x \in N$, $\mathrm{dist}(x, K) = u(\varphi_n^{\mathcal{N}}(x) | n \geq 1)$. By (1.1), $\mathrm{dist}(x, K \cap M) = u(\varphi_n^{\mathcal{M}}(x) | n \geq 1)$ for all $x \in M$. Therefore $\mathrm{dist}(x, K \cap M)$ is 0-definable in $\mathcal{M}$. Moreover, since $\mathcal{M}$ is complete, $K \cap M$ is compact. So, by Lemma 1.2 $K \cap M = K \subset \mathrm{acl}_{\mathcal{M}}(\emptyset)$. ☑

The following is a useful characterization of algebraicity, inspired by the usual characterization of algebraicity in first order model theory.

Lemma 1.5. *Let $\mathcal{M}$ be an L-structure, $A \subset M$ and $a \in M$. Then $a \in \mathrm{acl}_{\mathcal{M}}(A)$ if and only if there is some predicate $P\colon M \to [0,1]$, A-definable in $\mathcal{M}$, such that $P(a) = 0$ and $\{u \in N | Q(u) = 0\}$ is compact for all $(\mathcal{N}, Q) \succcurlyeq (\mathcal{M}, P)$.*

Proof. As before, we set $A = \emptyset$ without loss of generality.
($\Rightarrow$) By definition, $a \in \mathrm{acl}_{\mathcal{M}}(\emptyset)$ implies that a is in a compact set K, 0-definable in $\mathcal{M}$. This implies that $\mathrm{dist}(x, K)$ is 0-definable in $\mathcal{M}$. Let $P(x) = \mathrm{dist}(x, K)$. By Lemma 1.2, if $(\mathcal{N}, Q) \succcurlyeq (\mathcal{M}, P)$, the zero set of Q in $\mathcal{N}$ is K, so it is compact.
($\Leftarrow$) Let $P\colon M \to [0,1]$, 0-definable in $\mathcal{M}$, such that $P(a) = 0$ and $K_{\mathcal{N}} = \{u \in N | Q(u) = 0\}$ is compact for all $(\mathcal{N}, Q) \succcurlyeq (\mathcal{M}, P)$. In particular, choose $\mathcal{N}$ to be ω_1-saturated. Then by [1, Section 10], $K_{\mathcal{N}}$ is 0-definable in $\mathcal{N}$, and hence $K_{\mathcal{N}} \cap M$ is 0-definable in $\mathcal{M}$, and compact since $\mathcal{M}$ is complete. Therefore $a \in \mathrm{acl}_{\mathcal{M}}(\emptyset)$. ☑

Note that for the proof from right to left the full strength of the condition was not needed; in fact, we have the following:

Corollary 1.6. *Let $\mathcal{M}$ be an L-structure, $A \subset M$ and $a \in M$. Then $a \in \mathrm{acl}_{\mathcal{M}}(A)$ if and only if there is some predicate $P\colon M \to [0,1]$, A-definable in $\mathcal{M}$, such that $P(a) = 0$ and $\{u \in N | Q(u) = 0\}$ is compact for some $(\mathcal{N}, Q) \succcurlyeq (\mathcal{M}, P)$ where $\mathcal{N}$ is ω_1-saturated.*

Another natural notion of closure is one related to the boundedness of the set of realizations of a type. It is in fact a common notion not only in classic first order model theory, but also in research related to simplicity and stability of cats in [2].

Definition 1.7 (Bounded closure)**.** *Let $\mathcal{M}$ be an L-structure, and $A \subset M$. The* bounded closure *of A in $\mathcal{M}$, denoted* $\mathrm{bdd}_{\mathcal{M}}(A)$*, is the collection of all*

$a \in M$ for which there is some cardinal τ such that for any $\mathcal{N} \succcurlyeq \mathcal{M}$, the set of realizations of $\mathrm{tp}(a/A)$ *in $\mathcal{N}$ has cardinality less than or equal to τ.*

In the setting of metric structures, bounded closure and algebraic closure are in fact the same:

Lemma 1.8. *Let $\mathcal{M}$ be an L-structure, and $A \subset M$. Then $\mathrm{acl}_{\mathcal{M}}(A) = \mathrm{bdd}_{\mathcal{M}}(A)$.*

Proof. Without loss of generality, we set $A = \emptyset$.
($\subseteq$) Let $a \in \mathrm{acl}_{\mathcal{M}}(\emptyset)$. Let P be as in Lemma 1.5. Let $(\mathcal{N}, Q) \succcurlyeq (\mathcal{M}, P)$, and A the set of realizations of $\mathrm{tp}(a)$ in $\mathcal{N}$. Then $A \subset \{u \in N | Q(u) = 0\}$; since this latter set is compact, $|\{u \in N|Q(u) = 0\}| \leq 2^{\aleph_0}$. Therefore $|A| \leq 2^{\aleph_0}$, and hence $a \in \mathrm{bdd}_{\mathcal{M}}(\emptyset)$.
($\supseteq$) Without loss of generality, we may assume that $\mathcal{M}$ is ω_1-saturated. Let $a \in M \setminus \mathrm{acl}_{\mathcal{M}}(\emptyset)$.

Claim 1.9. There exists $n \geq 1$ such that for all L-formulas $\varphi(x)$ such that $\varphi^{\mathcal{M}}(a) = 0$, the zero set of φ in $\mathcal{M}$ has no finite $\frac{1}{n}$-dense set.

Suppose this is not the case. Then for each n there is a φ_n such that $0 = \varphi_n^{\mathcal{M}}(a)$ and the zero set of φ_n in $\mathcal{M}$, C_n, has a finite $\frac{1}{n}$-dense set. Let $K = \bigcap_n C_n$. Then, by [1, Section 9], K is also a zero set, and it is clearly compact; therefore, by [1, Section 10], it is 0-definable. But this contradicts the assumption that $a \notin \mathrm{acl}_{\mathcal{M}}(\emptyset)$, thus proving the claim.
Fix n as in the claim. Let τ be any cardinal. Take a collection $(x_\alpha | \alpha < \tau)$ of new variables and let

$$\Sigma = \left\{0 = \varphi(x_\alpha) | \alpha < \tau,\, \varphi^{\mathcal{M}}(a) = 0\right\} \cup \left\{0 = \frac{1}{n} \dot{-} \mathrm{d}(x_\alpha, x_\beta) | \alpha < \beta < \tau\right\}$$

By the claim above, Σ is finitely satisfied in $\mathcal{M}$. Let $\mathcal{M}'$ be a κ-saturated elementary extension of $\mathcal{M}$, with $\kappa > \tau$. Then Σ is realized in $\mathcal{M}'$, say by $(a_\alpha | \alpha < \tau)$. But clearly for any $\alpha < \tau$, a_α is a realization of $\mathrm{tp}(a)$ in $\mathcal{M}'$, so the set of realizations of $\mathrm{tp}(a)$ in $\mathcal{M}'$ has cardinality greater than τ. Therefore $a \notin \mathrm{bdd}_{\mathcal{M}} \mathcal{M}(\emptyset)$. ☑

This last proof gives an interesting dichotomy for sets defined by a complete type in a saturated structure, which we restate:

Proposition 1.10. *Let $\mathcal{M}$ be a κ-saturated L-structure, with $\kappa > 2^{\aleph_0}$. Let X be the set of realizations in $\mathcal{M}$ of a complete type, say* $\mathrm{tp}(a/A)$*, with $|A| < \kappa$. Then either $|X| \leq 2^{\aleph_0}$ and X is an algebraic set (i.e. a is algebraic over A), or $|X| \geq \kappa$.*

2. **Basic properties**

We first check that $\mathrm{acl}_{\mathcal{M}}$ actually does define a closure operation. So we fix a κ-saturated, strongly κ-homogeneous metric structure $\mathcal{M}$ (for κ sufficiently large) and subsets A and B of M of cardinality less than κ. When there is no confusion, we omit the subscript $\mathcal{M}$ from $\mathrm{acl}_{\mathcal{M}}$.

Proposition 2.1. *$A \subset \mathrm{acl}(A)$.*

Proof. Suppose $a \in A$. Then $\{a\}$ is compact and $\mathrm{dist}(x, \{a\}) = d(x, a)$ is A-definable. ☑

Proposition 2.2. *$A \subset B$, then $\mathrm{acl}(A) \subset \mathrm{acl}(B)$.*

Proof. Since $A \subset B$, if $K \subset M$ is A-definable, then it is of course B-definable. ☑

Proposition 2.3. *If $A \subseteq \mathrm{acl}_{\mathcal{M}}(B)$, then $\mathrm{acl}_{\mathcal{M}}(A) \subseteq \mathrm{acl}_{\mathcal{M}}(B)$.*

Proof. Let $a \in \mathrm{acl}_{\mathcal{M}}(A)$. Note that, by homogeneity of $\mathcal{M}$, for any element b, $\mathrm{tp}(b/B) = \mathrm{tp}(a/B)$ if and only if there exists $\sigma \in \mathrm{Aut}_B(\mathcal{M})$ such that $\sigma(a) = b$. For each b with the same type as a over B, fix such an automorphism, and call it σ_b. Define the following equivalence relation in X, the set of realizations of $\mathrm{tp}(a/B)$ in $\mathcal{M}$: $b_1 \sim b_2$ if for all $x \in A$ $\sigma_{b_1}(x) = \sigma_{b_2}(x)$ (the fact that it is an equivalence relation is easy to check, and left to the reader). Notice that if $b_1 \sim b_2$, then $\mathrm{tp}(b_1/\sigma_{b_1}(A)) = \mathrm{tp}(b_2/\sigma_{b_1}(A))$. Therefore, the number of equivalence classes $|X/\sim|$ is less than or equal to the number of possible images of A under an automorphism of $\mathcal{M}$ that fixes B. However, since every element of A is algebraic over B, by Lemma 1.8 and Proposition 1.10 it can have at most $2^{\aleph_0}$ distinct images under such automorphisms. Therefore $|X/\sim| \leq \left(2^{\aleph_0}\right)^{|A|}$. On the other hand, for any given $b \in X$, the size of its equivalence class is bounded, since $[b]_\sim$ is a subset of the set of realizations of $\mathrm{tp}(\sigma_b(a)/\sigma_b(A))$, which is of the same size as the set of realizations of $\mathrm{tp}(a/A)$, and this set has cardinality $\leq 2^{\aleph_0}$ because a is algebraic over A, by Proposition 1.10. Thus $|[b]_\sim| \leq 2^{\aleph_0}$, and therefore $|X| \leq \left(2^{\aleph_0}\right)^{|A|} 2^{\aleph_0} = 2^{\aleph_0 |A|}$. In other words, X is bounded, which by Proposition 1.10 implies that in fact $|X| \leq 2^{\aleph_0}$. Therefore $a \in \mathrm{acl}_{\mathcal{M}}(B)$, by Lemma 1.8. ☑

Proposition 2.4. *$\mathrm{acl}(\bar{A}) = \mathrm{acl}(A)$.*

Proof. ($\supseteq$) is a corollary of Proposition 2.3.
($\subseteq$) Let $K \subset M$ be compact, $\bar{A}$-definable in $\mathcal{M}$. By definition, this means that $\mathrm{dist}(x, K)$ is the uniform limit of a sequence of $L(\bar{A})$-formulas $(\varphi_n(x) | n \geq 1)$. Say a_n is the tuple of elements of $\bar{A}$ that occur in φ_n, that is $\varphi_n(x)$ is $\varphi_n(x, a_n)$. Since each a_n is a tuple of elements of $\bar{A}$, for each n there is a sequence $(a_n^{(k)} | k \geq 1)$ of tuples of elements of A that converges to a_n. Without loss of generality (by taking subsequences), we may assume that for every $n \geq 1$, $|\varphi_n^{\mathcal{M}}(x, a_n) - \mathrm{dist}(x, K)| < \frac{1}{2n}$ for all x in M. Let Δ_n be the modulus of uniform continuity of $\varphi_n^{\mathcal{M}}$. By taking a subsequence of $(a_n^{(k)})$, if necessary, we may assume that for each $n \geq 1$, $\mathrm{d}(a_n, a_n^{(k)}) < \Delta_n(\frac{1}{2n})$ for $k \geq n$. This implies that $|\varphi_n^{\mathcal{M}}(x, a_n) - \varphi_n^{\mathcal{M}}(x, a_n^{(n)})| < \frac{1}{2n}$ for all x in M. Therefore $|\,\mathrm{dist}(x, K) -$

$\varphi_n^{\mathcal{M}}(x, a_n^{(n)})| \leq |\operatorname{dist}(x, K) - \varphi_n^{\mathcal{M}}(x, a_n)| + |\varphi_n^{\mathcal{M}}(x, a_n) - \varphi_n^{\mathcal{M}}(x, a_n^{(n)})| \leq \frac{1}{n}$ for every x in $\mathcal{M}$. So $(\varphi_n(x, a_n^{(n)})|n \geq 1)$ converges uniformly to $\operatorname{dist}(x, K)$, proving that K is A-definable. ☑

Lemma 2.5 (Local character)**.** *If $a \in \operatorname{acl}(A)$, then there exists a countable subset A_0 of A such that $a \in \operatorname{acl}(A_0)$.*

Proof. Let $a \in \operatorname{acl}(A)$. By definition, this means that $a \in K$ for some A-definable compact K. Therefore, the predicate $\operatorname{dist}(x, K)$ is the uniform limit of a sequence $(\varphi_n(x, a_n)|n \geq 1)$ of $L(A)$-formulas (here, a_n is a tuple of elements of A, and $\varphi_n(x, y)$ is an L-formula.) If we let $A_0 = \{a_n|n \geq 1\}$, then it is clear that K is A_0-definable, so $a \in \operatorname{acl}(A_0)$. ☑

Proposition 2.6. $|\operatorname{acl}(A)| \leq |L(A)|^{\aleph_0}$

Proof. The number of A-definable predicates is bounded by the number of sequences of $L(A)$-formulas, $|L(A)|^{\aleph_0}$. Therefore the number of A-definable compact sets is bounded by this same value. Each compact set has at most $2^{\aleph_0}$ elements, so $\operatorname{acl}(A)$ (the union of all the A-definable compact sets) has at most $|L(A)|^{\aleph_0} \cdot 2^{\aleph_0} = |L(A)|^{\aleph_0}$ elements. ☑

Proposition 2.7. *Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures, $A \subset M$ and $B \subset N$. If $f\colon A \to B$ is an elementary map, then there exists an elementary map $g\colon \operatorname{acl}_{\mathcal{M}}(A) \to \operatorname{acl}_{\mathcal{N}}(B)$ extending f. Moreover, if f is onto, then so is g.*

Proof. There exists an L-structure $\mathcal{M}'$ sufficiently saturated and strongly homogeneous, where $\mathcal{M}$ and $\mathcal{N}$ embed elementarily. By homogeneity of $\mathcal{M}'$, f extends to an automorphism g of $\mathcal{M}'$. Now, if $K \subset M$ is compact and A-definable in $\mathcal{M}$, then by continuity of g, $g(K)$ is compact. And if $\operatorname{dist}(x, K) = u(\varphi_n(x, a_n)|n \geq 1)$ for some connective u and some sequence of $L(A)$-formulas $(\varphi_n(x, a_n)|n \geq 1)$, then $\operatorname{dist}(x, g(K)) = u(\varphi_n(x, f(a_n))|n \geq 1)$, so $g(K)$ is B-definable in $\mathcal{N}$. Furthermore, if $f(A) = B$, then for any B-definable compact C, $\operatorname{dist}(x, C)$ can be written as $u(\varphi_n(x, f(a_n))|n \geq 1)$ for some connective u, some sequence of L-formulas $(\varphi_n(x, y)|n \geq 1)$ and some sequence of tuples (a_n) in A. This implies that $C = g(K)$ for some A-definable $K \subset M$. Since g^{-1} is continuous, K is also compact. Therefore $g(\operatorname{acl}_{\mathcal{M}}(A)) = \operatorname{acl}_{\mathcal{N}}(B)$ ☑

All the definitions and properties above are valid when a denotes a tuple of elements, in which case the compact sets in question would be subsets of the appropriate cartesian product of M, and the predicates would be of the corresponding arity. We now verify that such definitions are well behaved.

Proposition 2.8. *Let $\mathcal{M}$ be an L-structure and $A \subset M$.*

(1) Let $a = (a_1, \ldots, a_n) \in M^n$. Then $a \in \operatorname{acl}(A)$ if and only if $a_i \in \operatorname{acl}(A)$ for all $i = 1, \ldots, n$.

(2) Let $a = (a_i|i \geq 1) \in M^\omega$. Then $a \in \operatorname{acl}(A)$ if and only if $a_i \in \operatorname{acl}(A)$ for all $i \geq 1$. Here we consider the product topology in M^ω, with the metric $\mathrm{d}(x, y) = \sum_{i \geq 1} 2^{-i}\, \mathrm{d}(x_i, y_i)$.

Proof. As before, we may assume $A = \emptyset$ without loss of generality.

(1) ($\Leftarrow$) Let $\pi_i \colon M^n \to M$ be the projection on the i^{th} coordinate. If $K \subset M^n$ is compact and 0-definable in $\mathcal{M}$, then $\pi_i(K)$ is also compact as π_i is continuous, and 0-definable as

$$\operatorname{dist}(x_i, \pi_i(K)) = \inf_{y \in K}(\mathrm{d}(x_i, y_i)) = \inf_y(\mathrm{d}(x_i, y_i) + \operatorname{dist}(y, K)).$$

($\Rightarrow$) If $K_i \subset M$ is compact and 0-definable for each $i = 1, \ldots, n$, then $K = \prod_{1 \le i \le k} K_i$ is compact, and 0-definable as

$$\operatorname{dist}(x, K) = \max(\operatorname{dist}(x_i, K_i) | i = 1, \ldots, n).$$

(2) ($\Leftarrow$) Same argument as above, except here

$$\operatorname{dist}(x_i, \pi_i(K)) = \inf_y(\mathrm{d}(x_i, y_i) + 2^i \operatorname{dist}(y, K)).$$

($\Rightarrow$) Tychonoff's theorem guarantees that if $(K_i | i \ge 1)$ is a family of compact subsets of M, then $K = \prod_{i \ge 1} K_i$ is compact. And if each K_i is 0-definable in $\mathcal{M}$, then so is K, as $\operatorname{dist}(x, K) = \sum_{i \ge 1} 2^{-i} \operatorname{dist}(x_i, K_i)$. ☑

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