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Simplicity simplified

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Abstract. We expose the foundations of simple theories in a straightforward way including many improved proofs which can only be found scattered in the specialized literature. We start with general model theory and finish with the proof of the independence theorem.

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Resumen. Exponemos los fundamentos de las teorías simples de manera muy directa e incorporando muchas mejoras de demostraciones que están dispersas en la literatura especializada. Comenzamos con la teoría general de modelos y llegamos hasta la prueba del teorema de la independencia.

1. Introduction

Simple theories are a natural and fruitful generalization of stable theories. They were introduced by Shelah in [12] and rediscovered and fully developed by Kim and Pillay in [7], [6] and [8]. See [9] for historical background. The book [13] of Wagner is now the main reference. In the last years the foundations of simple theories have been simplified after some new ideas from Shami, Shelah and Buechler and Lessmann. Here we add further simplifications from [3] and systematize all the basic results up to the Independence Theorem and type-definability of equality of Lascar strong types. The references contain the sources for all the results. We do not give credits for every single result but we give complete proofs of everything without previous assumptions except the usual ones in Model Theory.

We use the standard notation and conventions. Everywhere T is a complete theory with infinite models in a first-order language and $\mathfrak{C}$ is its monster model. All sequences and sets considered here will be extracted from $\mathfrak{C}$ and all relations will live on $\mathfrak{C}$.

Indiscernible sequences can be usually obtained by Ramsey's Theorem. However it is very convenient to have at our disposal the following lemma based on Erdös-Rado Theorem. For a proof, the interested reader can consult [7].

Lemma 1.1. *If $\kappa \geq |T|$ is a cardinal number, $\lambda = \beth_{(2^\kappa)^+}$, $|A| \leq \kappa$ and $(a_i : i < \lambda)$ is a sequence of sequences a_i of fixed length $\alpha < \kappa^+$, then there is an A-indiscernible sequence $(b_i : i < \omega)$ such that for each $n < \omega$ there are $i_0 < \ldots < i_n < \lambda$ such that $b_0, \ldots, b_n \equiv_A a_{i_0}, \ldots, a_{i_n}$. In most of the applications α is a natural number and therefore the cardinal number λ depends only on $|T|$ and $|A|$.*

2. Lascar strong types

Definition 2.1. *A relation R is bounded if for some cardinal κ there is not a sequence $(a_i : i < \kappa)$ such that $\neg R(a_i, a_j)$ for all $i < j < \kappa$. The relation is finite if this bound κ is in fact a natural number. Observe that for definable relations finiteness is equivalent to boundedness.*

Remark 2.1. *For any cardinal number λ, any intersection of λ bounded relations is a bounded relation.*

Proof. Let $(R_l : l < \lambda)$ be a sequence of bounded relations. For all $l < \lambda$ let κ_l be a bound for R_l and let $\kappa = \lambda + \sup\{\kappa_l : l < \lambda\}$. Assume that there are $(a_i : i < (2^\kappa)^+)$ such that$\neg R(a_i, a_j)$ for all $i < j < (2^\kappa)^+$, where $R = \bigcap_{l<\lambda} R_l$. By Erdös-Rado ($(2^\kappa)^+ \to (\kappa^+)^2_\kappa$), for some $l < \lambda$ there is a subset $I \subseteq (2^\kappa)^+$ of cardinality κ^+ such that $\neg R_l(a_i, a_j)$ for all $i < j$ in I. This contradicts the choice of κ_l. ☑

Definition 2.2. *An equivalence relation E is A-invariant if it is preserved under automorphisms of the monster model fixing A pointwise, that is $E(f(a), f(b))$ whenever $E(a, b)$ and $f \in \mathrm{Aut}(\mathfrak{C}/A)$. Since every A-invariant equivalence relation is definable by a disjunction (maybe infinite) of types over A, there is a bounded number of them. Therefore the intersection of all A-invariant bounded equivalence relations is again an A-invariant bounded equivalence relation. We say that the sequences a, b have the same Lascar strong type over A and we write*

$$\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$$

if a and b are equivalent in this least A-invariant bounded equivalence relation.

Definition 2.3. *Let x, y be finite tuples of variables of the same length. We say that the formula $\theta(x, y)$ is thick if it defines a finite symmetric relation. Note that all thick formulas are reflexive. For any set A and for any sequences of variables x, y of the same length, the set of all thick formulas over A in (finite subtuples of) the variables x, y will be*

$$\mathrm{nc}_A(x, y).$$

For every natural number n, $\mathrm{nc}^n_A(x, y)$ is the type $\exists y_1 \ldots y_{n-1}(\mathrm{nc}_A(x, y_1) \wedge \mathrm{nc}_A(y_1, y_2) \wedge \ldots \wedge \mathrm{nc}_A(y_{n-1}, y))$.

Remark 2.2. (1) *Finite conjunctions and disjunctions of thick formulas are thick formulas.*

(2) *Any consequence of a thick formula is a finite formula.*
(3) *If $\varphi(x,y)$ is finite, then $\varphi(x,y) \wedge \varphi(y,x)$ is thick.*

Lemma 2.1. *For any a, b, $\models \mathrm{nc}_A(a,b)$ if and only if a, b start an infinite A-indiscernible sequence.*

Proof. If a, b start an infinite A-indiscernible sequence, then $\models \theta(a,b)$ for any thick formula $\theta(x,y)$ over A. Now assume $\models \mathrm{nc}_A(a,b)$ and let $p(x,y) = \mathrm{tp}(ab/A)$. By Ramsey's Theorem and compactness, to prove that a, b start an infinite A-indiscernible sequence it is enough to check that there is an infinite sequence $(a_i : i < \omega)$ such that $\models p(a_i, a_j)$ for all $i < j < \omega$. For this we have to prove for any $\varphi \in p$, the consistency of

$$\{\varphi(x_i, x_j) : i < j < \omega\}.$$

If this set of formulas is inconsistent, then $\neg\varphi(x,y)$ is finite and therefore $(\neg\varphi(x,y) \wedge \neg\varphi(y,x)) \in \mathrm{nc}_A(x,y)$. Hence $\models \neg\varphi(a,b)$, a contradiction. ☑

Proposition 2.4. *Equality of Lascar strong types over A is the transitive closure of the relation of starting an A-indiscernible sequence. Hence it is defined by the infinite disjunction $\bigvee_n \mathrm{nc}^n_A(x,y)$.*

Proof. Let E be this transitive closure. It is an A-invariant equivalence relation. Since the relation of starting an infinite indiscernible sequence is defined by a set of finite formulas, it is bounded. Hence its transitive closure E is also bounded. From this it follows that equality of Lascar strong types is contained in E. For the other direction it suffices to show that if a, b start an infinite A-indiscernible sequence then $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$. Let κ be a strict bound for the number of classes in the relation of equality of Lascar strong types over A. Choose an A-indiscernible sequence starting with a, b of length κ. If $\mathrm{Lstp}(a/A) \neq \mathrm{Lstp}(b/A)$ then by A-invariance $\mathrm{Lstp}(a'/A) \neq \mathrm{Lstp}(b'/A)$ for any two different a', b' in the indiscernible sequence, which contradicts the choice of κ. ☑

Lemma 2.2. (1) *If $\mathrm{nc}_A(a,b)$, then there is a model $M \supseteq A$ such that $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$.*
(2) *If for some model $M \supseteq A$ $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$, then $\mathrm{nc}^2_A(a,b)$.*

Proof. For *1* fix a model $M \supseteq A$ and an infinite A-indiscernible sequence I starting with a, b. By Ramsey's Theorem and compactness we can obtain another infinite A-indiscernible sequence $(a_i : i < \omega)$ where $\mathrm{tp}(ab/A) = \mathrm{tp}(a_0a_1/A)$ and $\mathrm{tp}(a_i/M) = \mathrm{tp}(a_j/M)$ for all $i < j < \omega$. This implies that $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$. For *2* we assume that a, b have the same type over some model $M \supseteq A$ and we show that for any thick formula $\theta(x,y)$ over A, $\models \exists z(\theta(a,z) \wedge \theta(b,z))$. Let n be the maximal length of a sequence $a_1, \ldots, a_n$ such that $\models \neg\theta(a_i, a_j)$ for all $i < j \leq n$. We can find such $a_1, \ldots, a_n$ in M. For some $i, j \leq n$, $\models \theta(a, a_i)$ and $\models \theta(b, a_j)$. Since a, b have the same type over M we may take $i = j$. ☑

Proposition 2.5. *Equality of Lascar strong types over A is the transitive closure of the relation of having the same type over a model containing A.*

Proof. Clear by Proposition 2.4 and Lemma 2.2. ☑

Definition 2.6. *The group* $\mathrm{Autf}(\mathfrak{C}/A)$ *of* strong automorphisms over A *of the monster model* $\mathfrak{C}$ *is the subgroup of* $\mathrm{Aut}(\mathfrak{C}/A)$ *generated by the automorphisms fixing a small submodel containing A:*

$$\mathrm{Autf}(\mathfrak{C}/A) = \langle \bigcup_{M \supseteq A} \mathrm{Aut}(\mathfrak{C}/M) \rangle$$

Corollary 2.1. $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ *if and only if* $f(a) = b$ *for some* $f \in \mathrm{Autf}(\mathfrak{C}/A)$.

Proof. It follows from Proposition 2.5. ☑

Corollary 2.2. *If* $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ *then for any c there is some d such that* $\mathrm{Lstp}(ac/A) = \mathrm{Lstp}(bd/A)$

Proof. Choose $f \in \mathrm{Autf}(\mathfrak{C}/A)$ such that $f(a) = b$ and put $d = f(c)$. ☑

3. Dividing and forking

Definition 3.1. *The formula $\varphi(x, a)$ divides over the set A with respect to $k < \omega$ if there is an infinite sequence $I = (a_i : i < \omega)$ of realizations of* $\mathrm{tp}(a/A)$ *such that $\{\varphi(x, a_i) : i < \omega\}$ is k-inconsistent, i.e, every subset of k elements is inconsistent. We may always assume that I is A-indiscernible. We may also assume that $a = a_0$. Finally, in place of ω we may choose any infinite linear ordering. The formula $\varphi(x, a)$ divides over A if it divides over A with respect to some k.*

Remark 3.1.
1. *If $\varphi(x, a)$ divides over A with respect to k and $\psi(x, b) \vdash \varphi(x, a)$, then $\psi(x, b)$ divides over A with respect to k too.*
2. *If $a \in$* $\mathrm{acl}(A)$ *and $\varphi(x, a)$ is consistent, then $\varphi(x, a)$ does not divide over A.*

Definition 3.2. *The set of formulas $\pi(x, a)$ divides over the set A if it implies a formula $\varphi(x, b)$ which divides over A. We may always assume that $b = a$ and that $\varphi(x, y)$ is a conjunction of formulas in $\pi(x, y)$.*

Remark 3.2.
1. *If $\pi(x, a)$ is inconsistent, it divides over A.*
2. *$\pi(x, a)$ divides over A iff for some infinite A-indiscernible sequence $(a_i : i < \omega)$ with $a_0 = a$, the set of formulas $\bigcup_{i<\omega} \pi(x, a_i)$ is inconsistent.*
3. *$\varphi(x, a)$ divides over A iff the set $\{\varphi(x, a)\}$ divides over A.*
4. $\mathrm{acl}(A) = \{a :$ $\mathrm{tp}(a/Aa)$ *does not divide over* $A\}$

Definition 3.3. *The set of formulas $\pi(x, a)$ forks over A if for some n there are formulas $\varphi_1(x, a_1), \ldots, \varphi_n(x, a_n)$ such that $\pi(x, a) \vdash \varphi_1(x, a_1) \vee \ldots \vee \varphi_n(x, a_n)$ and every $\varphi_i(x, a_i)$ divides over A. The formula $\varphi(x, a)$ forks over A if the set $\{\varphi(x, a)\}$ forks over A.*

The advantage of forking over dividing lies in that we can make sure that nonforking types can be extended while it is not clear whether we can do it in the case of dividing.

Remark 3.3. (1) *If $\pi(x,a)$ divides over A, then it forks over A.*
(2) *If $\pi(x,a)$ is finitely satisfiable in A, then it does not fork over A.*
(3) *$\pi(x,a)$ forks over A iff a conjunction of formulas in $\pi(x,a)$ forks over A.*
(4) *If $\pi(x,a)$ does not fork over A, then it can be extended to a complete type over a which does not fork over A. Any complete type over a extending the partial type $\pi(x,a) \cup \{\neg\varphi(x,a) : \varphi(x,a)$ forks over $A\}$ does the job.*

Next lemma turns out to be very useful. From it we can prove a result on pairs which anticipates some version of transitivity for nondividing (Proposition 3.4).

Lemma 3.1. *Those following are equivalent.*
(1) *$\mathrm{tp}(a/Ab)$ does not divide over A.*
(2) *For every infinite A-indiscernible sequence I such that $b \in I$, there is $a' \equiv_{Ab} a$ such that I is Aa'-indiscernible.*
(3) *For every infinite A-indiscernible sequence I such that $b \in I$, there is $J \equiv_{Ab} I$ such that J is Aa-indiscernible.*

Proof. The equivalence of *2* and *3* follows by conjugation. It is clear that *3* implies *1*. We prove that *1* implies *2*. We may assume that A is empty, that $I = (b_i : i < \omega)$ and that $b = b_0$. Let $p(x,b) = \mathrm{tp}(a/b)$ and let $\Gamma(x,(x_i : i < \omega))$ be a set of formulas expressing that $(x_i : i < \omega)$ is x-indiscernible. It will be enough to prove that $p(x,b) \cup \Gamma(x,(b_i : i < \omega))$ is consistent. By *1* $q(x) = \bigcup_{i<\omega} p(x,b_i)$ is consistent. Let $c \models q$ and let Γ_0 a finite subset of Γ. By Ramsey's Theorem, there is an order preserving $f : \omega \to \omega$ such that $\models \Gamma_0(c,(b_{f(i)} : i < \omega))$. By indiscernibility $(b_i : i < \omega) \equiv (b_{f(i)} : i < \omega)$ and therefore we can find c' such that $c'(b_i : i < \omega) \equiv c(b_{f(i)} : i < \omega)$. Clearly $c' \models q(x) \cup \Gamma_0(x,(b_i : i < \omega))$. ☑

Proposition 3.4. *If $\mathrm{tp}(a/B)$ does not divide over $A \subseteq B$ and $\mathrm{tp}(b/Ba)$ does not divide over Aa, then $\mathrm{tp}(ab/B)$ does not divide over A.*

Proof. It is an easy application of Lemma 3.1. ☑

Proposition 3.5. *If $\varphi(x,a)$ divides over A with respect to k and $\mathrm{tp}(b/Aa)$ does not divide over A, then $\varphi(x,a)$ divides over Ab with respect to k.*

Proof. Let $I = (a_i : i < \omega)$ be an infinite A-indiscernible sequence such that $a = a_0$ and $\{\varphi(x,a_i) : i < \omega\}$ is k-inconsistent. By Lemma 3.1 there is $J \equiv_{Aa} I$ which is Ab-indiscernible. Then J witnesses that $\varphi(x,a)$ divides over Ab with respect to k. ☑

4. **The tree property and simplicity**

This section can be skipped if one chooses to define simple theories as the theories where every complete type does not fork over a subset of cardinality at most $|T|$ of its domain (Proposition 4.1). With this definition it is straightforward that a type does not fork over its domain. Here we present the proofs of the equivalence of this definition with some others, like not having the tree property, types not dividing over a small subset of its domain or finiteness of rank $D(p, \varphi, k)$.

Definition 4.1. *$\varphi(x, y)$ has the tree property with respect to $k < \omega$ if there is a tree $(a_s : s \in \omega^{<\omega})$ such that for all $\eta \in \omega^\omega$, the branch $\{\varphi(x, a_{\eta\restriction n}) : n < \omega\}$ is consistent and for all $s \in \omega^{<\omega}$ the set $\{\varphi(x, a_{s^\frown i}) : i < \omega\}$ is k-inconsistent. By compactness it is easy to obtain a corresponding tree $(a_s : s \in \kappa^{<\lambda})$ for any cardinals κ, λ.*

Lemma 4.1. *Let α be an ordinal number, $\pi(x, a)$ a partial type, $(\varphi_i(x, y_i) : i < \alpha)$ a sequence of formulas and $(k_i : i < \alpha)$ a sequence of natural numbers. The following are equivalent.*

(1) *There is a tree $(a_s : s \in \omega^{<\alpha})$ such that for all $\eta \in \omega^\alpha$, the branch $\pi(x, a) \cup \{\varphi_i(x, a_{\eta\restriction i+1}) : i < \alpha\}$ is consistent and for all $i < \alpha$ and $s \in \omega^i$, the set $\{\varphi_i(x, a_{s^\frown j}) : j < \omega\}$ is k_i-inconsistent.*

(2) *There is a sequence $(a_i : i < \alpha)$ such that $\pi(x, a) \cup \{\varphi_i(x, a_i) : i < \alpha\}$ is consistent and for every $i < \alpha$, $\varphi_i(x, a_i)$ divides over $a \cup \{a_j : j < i\}$ with respect to k_i.*

Moreover in 1 we can add that all the branches $(a_{\eta\restriction i} : i < \alpha)$ have the same type over a.

Proof. Assume first that the tree is given. By compactness we may obtain a corresponding tree $(a_s : s \in \lambda^{<\alpha})$ for a very big cardinal λ. By induction on $\beta \leq \alpha$ we can show then that there is such a tree with the additional property that for all $i < \beta$ all $\eta \in \lambda^i$ and all $j, l < \lambda$, $\mathrm{tp}(a_{\eta^\frown j}/a(a_{\eta\restriction h} : h \leq i)) = \mathrm{tp}(a_{\eta^\frown l}/a(a_{\eta\restriction h} : h \leq i))$. Apply this with $\beta = \alpha$. Any branch in the resulting tree is a dividing sequence as required. For the other direction, fix the dividing sequence $(\varphi_i(x, a_i) : i < \alpha)$ and construct the tree inductively with the additional property that every branch is isomorphic over a to the initial segment of the chain of the same level. It is enough to show how to extend a given branch and it is clear how to do it using the fact that the formulas in the sequence always divide. Observe that the parameters a_s in the tree play no role at all if $s \in \omega^i$ and i is either 0 or a limit ordinal. ☑

Definition 4.2. *A dividing chain for $\varphi(x, y)$ is a sequence $(a_i : i < \alpha)$ such that $\{\varphi(x, a_i) : i < \alpha\}$ is consistent and for every $i < \alpha$, $\varphi(x, a_i)$ divides over $\{a_j : j < i\}$. If it divides with respect to k_i, we say that it is a dividing chain with respect to $(k_i : i < \alpha)$. We say that $\varphi(x, y)$ divides α times (with respect* *to $(k_i : i < \alpha)$) if there is a dividing chain of length α for $\varphi(x, y)$ (with respect to $(k_i : i < \alpha)$).*

Remark 4.1. (1) *$\varphi(x, y)$ divides ω times with respect to k iff it has the tree property with respect to k.*
(2) *If $\varphi(x, y)$ divides n times with respect to k for every $n < \omega$, then it divides α times with respect to k for every ordinal α.*
(3) *If $\varphi(x, y)$ divides ω_1 times, then for some $k < \omega$, $\varphi(x, y)$ divides ω times with respect to k.*

Definition 4.3. *T is simple if in T there are no formulas with the tree property. This is clearly equivalent to the non existence of formulas which divide ω times with respect to some k and also to the non existence of formulas which divide ω_1 times.*

Proposition 4.4. *The following conditions are equivalent to the simplicity of T.*

(1) *For every type $p \in S(A)$ in finitely many variables there is a $B \subseteq A$ such that $|B| \leq |T|$ and p does not divide over B.*
(2) *There is some cardinal κ such that for every type $p \in S(A)$ in finitely many variables there is a $B \subseteq A$ such that $|B| \leq \kappa$ and p does not divide over B.*
(3) *There is no increasing chain $(p_i(x) : i < |T|^+)$ of types $p_i(x) \in S(A_i)$ in finitely many variables such that for every $i < |T|^+$, p_{i+1} divides over A_i.*
(4) *For some cardinal κ there is no increasing chain $(p_i(x) : i < \kappa)$ of types $p_i(x) \in S(A_i)$ in finitely many variables such that for every $i < \kappa$, p_{i+1} divides over A_i.*

Proof. Simplicity implies *1*, since if $p \in S(A)$ divides over every subset of A of cardinality $\leq |T|$, then we can inductively construct a sequence of formulas $(\varphi_i(x, y_i) : i < |T|^+)$ and a sequence $(a_i : i < |T|^+)$ of parameters $a_i \in A$ such that $\varphi_i(x, a_i) \in p$ and $\varphi(x, a_i)$ divides over $\{a_j : j < i\}$. Clearly one formula $\varphi(x, y)$ appears ω_1 times in the sequence and this contradicts simplicity. It is clear that *1* implies *2* and that *3* implies *4*. To show that *1* implies *3*, observe that if the increasing chain $(p_i(x) : i < |T|^+)$ is given and we set $A = \bigcup A_i$ and $p = \bigcup p_i$, then $p(x) \in S(A)$ divides over every subset of A of cardinality $\leq |T|$. The same argument proves *4* from *2*. It remains only to show simplicity from *4*. If T is not simple, then some formula $\varphi(x, y)$ divides κ times. Let $(a_i : i < \kappa)$ be a witness of this. Let a be a realization of $\{\varphi(x, a_i) : i < \kappa\}$, let $A_i = \{a_j : j < i\}$ and let $p_i = \mathrm{tp}(a/A_i)$. The chain $(p_i : i < \kappa)$ contradicts point *4*. ☑

Definition 4.5. *Let $\Delta = \{\varphi_i(x, y_i) : i = 1, \ldots, n\}$ and let $k < \omega$. For any partial type $\pi(x, a)$ we define the rank $D(\pi(x, a), \Delta, k)$ as the supremum (an ordinal or ∞) of the ordinals α such that $D(\pi(x, a), \Delta, k) \geq \alpha$ according to the* *following recursive definition. If we want we can agree that inconsistent sets have rank* -1*.*

(1) $D(\pi(x,a),\Delta,k) \geq 0$ *iff* $\pi(x,a)$ *is consistent.*
(2) $D(\pi(x,a),\Delta,k) \geq \alpha+1$ *iff for some* $\varphi(x,y) \in \Delta$ *there is a sequence* $(a_i : i < \omega)$ *such that* $\{\varphi(x,a_i) : i < \omega\}$ *is* k*-inconsistent and for every* $i < \omega$, $D(\pi(x,a) \cup \{\varphi(x,a_i)\},\Delta,k) \geq \alpha$*.*
(3) *For limit* β, $D(\pi(x,a),\Delta,k) \geq \beta$ *iff* $D(\pi(x,a),\Delta,k) \geq \alpha$ *for all* $\alpha < \beta$*.*

Remark 4.2. (1) *If* $\pi(x,a) \vdash \pi'(x,a')$, $\Delta \subseteq \Delta'$, *and* $k \leq k'$, *then* $D(\pi(x,a),\Delta,k) \leq D(\pi'(x,a'),\Delta',k')$
(2) *If* $\pi(x,a)$, $\sigma(x,b)$ *are equivalent partial types, then* $D(\pi(x,a),\Delta,k) = D(\sigma(x,b),\Delta,k)$*.*

Proof. *2* follows from *1* and to prove *1* one shows by induction on α that

$$D(\pi(x,a),\Delta,k) \geq \alpha \Rightarrow D(\pi'(x,a'),\Delta',k') \geq \alpha.$$

☑

Lemma 4.2. $D(\pi(x,a),\Delta,k) \geq n$ *iff there is a sequence* $(\varphi_i(x,y_i) : i < n)$ *of formulas in* Δ *and parameters* $(a_i : i < n)$ *such that* $\pi(x,a) \cup \{\varphi_i(x,a_i) : i < n\}$ *is consistent and for all* $i < n$, $\varphi_i(x,a_i)$ *divides over* $a \cup \{a_j : j < i\}$ *with respect to* k*.*

Proof. By Lemma 4.1. ☑

Proposition 4.6. (1) *If* $D(\pi(x,a),\Delta,k) \geq \omega$, *then* $D(\pi(x,a),\Delta,k) = \infty$*.*
(2) T *is simple iff for all finite* Δ *and all* k, $D(x=x,\Delta,k) < \omega$*.*

Proof. *1.* Assume $D(\pi(x,a),\Delta,k) \geq \omega$. By compactness, Lemma 4.2, and Lemma 4.1, for some $\varphi(x,y) \in \Delta$, for every ordinal α there is a sequence $(a_i : i < \alpha)$ such that $\pi(x,a) \cup \{\varphi(x,a_i) : i < \alpha\}$ is consistent and for every $i < \alpha$, $\varphi(x,a_i)$ divides over $a \cup \{a_j : j < i\}$ with respect to k. By induction on α one easily sees that this implies $D(\pi(x,a),\Delta,k) \geq \alpha$.

2. By Lemma 4.2 it is clear that it holds if we state it for sets Δ consisting in only one formula. The general case can be established by a standard coding of Δ-types in φ-types or by noticing that $D(x=x,\Delta,k) \leq \sum_{\varphi\in\Delta} D(x=x,\varphi,k)$. ☑

Remark 4.3. *For all* $\pi(x,a)$, Δ, k *and* $(\varphi(x,a_i) : 1 \leq i \leq n)$*,*

$$D(\pi(x,a) \cup \{\bigvee_{i=1}^{n} \varphi(x,a_i)\},\Delta,k) = \max_{1\leq i\leq n} D(\pi(x,a) \cup \{\varphi(x,a_i)\},\Delta,k)$$

Proof. Let $\alpha = D(\pi(x,a) \cup \{\bigvee_{i=1}^{n} \varphi(x,a_i)\},\Delta,k)$ and let $\alpha_i = D(\pi(x,a) \cup \{\varphi(x,a_i)\},\Delta,k)$. By Remark 4.2 $\alpha \geq \max_{1\leq i\leq n} \alpha_i$. By Lemma 4.2 if $\alpha \geq m$ then for some i, $\alpha_i \geq m$. Hence $\max_{1\leq i\leq n} \alpha_i \geq \alpha$. ☑

Lemma 4.3. *Let $\Delta = \{\varphi_1(x, y_1), \ldots, \varphi_n(x, y_n)\}$, $D(\pi(x, a) \upharpoonright A, \Delta, k) < \omega$ and $\pi(x, a) \vdash \varphi_1(x, a_1) \vee \ldots \vee \varphi_n(x, a_n)$ where every $\varphi(x, a_i)$ divides over A with respect to k. Then $D(\pi(x, a), \Delta, k) < D(\pi(x, a) \upharpoonright A, \Delta, k)$.*

Proof. By Remark 4.3 and 4.2, for some i

$$D(\pi(x, a), \Delta, k) \leq D(\pi(x, a) \upharpoonright A \cup \{\varphi_i(x, a_i)\}, \Delta, k)$$

Let $m = D(\pi(x, a) \upharpoonright A \cup \{\varphi_i(x, a_i)\}, \Delta, k)$. By Lemma 4.2 there is a sequence $(\psi_j(x, z_j) : j < m)$ of formulas in Δ and a sequence $(b_j : j < m)$ such that $\pi(x, a) \upharpoonright A \cup \{\varphi_i(x, a_i)\} \cup \{\psi_j(x, b_j) : j < m\}$ is consistent and every $\psi_j(x, b_j)$ divides over $A \cup \{a_i\} \cup \{a_l : l < j\}$ with respect to k for all $j < m$. Again by Lemma 4.2, the sequence $\varphi_i(x, a_i), \psi_0(x, b_0), \ldots, \psi_{m-1}(x, b_{m-1})$ witnesses that $D(\pi(x, a) \upharpoonright A, \Delta, k) \geq m + 1$. ☑

Proposition 4.7. *Simplicity is also equivalent to the conditions in Proposition 4.4 if we replace forking for dividing.*

Proof. Point *4* from Proposition 4.4 stated for forking implies its original version with dividing. The arguments in the proof of Proposition 4.4 showing that *1* implies *2* and *3* and that any of *2* and *3* implies *4* adapt to its version with forking. Moreover it is pretty clear that *3* implies *1* in any version. Hence it will be enough to prove that simple theories verify point *3* in this new version for forking. Here is where we use $D(\pi, \Delta, k)$-rank. Assume $(p_i(x) : i < |T|^+)$ is a increasing chain of types $p_i(x) \in S(A_i)$ such that p_{i+1} forks over A_i for all $i < |T|^+$. This means that for all $i < |T|^+$ we can find some $\varphi^i_1(x), \ldots, \varphi^i_{n_i}(x)$ such that $p_{i+1}(x) \vdash \varphi^i_1(x) \vee \ldots \vee \varphi^i_{n_i}(x)$ and each $\varphi^i_j(x)$ divides over A_i with respect to some $k_{ji} < \omega$. Clearly we may assume that there are $n, k < \omega$ and some $\varphi_1(x, y_1), \ldots, \varphi_n(x, y_n)$ such that for all $i < |T|^+$ there are tuples $a^i_1, \ldots, a^i_n \in A_{i+1}$ for which $\varphi_i(x, a^i_j) = \varphi^i_j(x)$ and moreover $k = k_{j,i}$. Let $\Delta = \{\varphi_1(x, y_1), \ldots, \varphi_n(x, y_n)\}$. By Lemma 4.3 $D(p_i(x), \Delta, k) > D(p_{i+1}(x), \Delta, k)$ for all $i < |T|^+$, which is a contradiction. ☑

Corollary 4.1. *If T is simple and $p(x) \in S(A)$, then p does not fork over A.*

Proof. By Proposition 4.7. ☑

5. Independence and Morley sequences

Definition 5.1. *We say that A is independent of B over C (written $A \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.7em\lower0.1ex\hbox{$|$}}_C B$) if for every finite sequence $a \in A$, $\mathrm{tp}(a/BC)$ does not fork over C.*

Remark 5.1.
(1) *$A \perp_C B$ iff $A \perp_C CB$.*
(2) *If $a \perp_C b$ for all finite $a \in A$, $b \in B$, then $A \perp_C B$.*
(3) *If $A \perp_C B$ and $B' \subseteq B$, then $A \perp_{CB'} B$.*
(4) *If $A \perp_C B$, $A' \subseteq A$ and $B' \subseteq B$, then $A' \perp_C B'$.*

Definition 5.2. *Let X be a linearly ordered set. The sequence $(a_i : i \in X)$ is A-independent if for every $i \in X$, $a_i \mathop{\smile\!\!\!\!\!\!\mid}_A \{a_j : j < i\}$. A Morley sequence over A is a sequence $(a_i : i \in X)$ which is A-independent and A-indiscernible. It is said to be a Morley sequence in the type p if every a_i realizes p.*

Remark 5.2. *Let $(a_i : i \in X)$ be A-independent. If Y, Z are subsets of X such that $Y < Z$, then $\mathrm{tp}((a_i : i \in Z)/A(a_i : i \in Y))$ does not divide over A.*

Proof. It can be assumed that Z is finite. An induction on $|Z|$ using Lemma 3.4 gives easily the result. ☑

Lemma 5.1. *If $p(x) \in S(B)$ does not fork over $A \subseteq B$, there is a Morley sequence $(a_i : i < \omega)$ in p over A which is moreover B-indiscernible.*

Proof. Using the extension Lemma (item 4 in Remark 3.3) one can construct a large sequence $(b_i : i < \lambda)$ of realizations b_i of p such that $b_i \mathop{\smile\!\!\!\!\!\!\mid}_A B\{b_j : j < i\}$. Lemma 1.1 gives us a B-indiscernible sequence $(a_i : i < \omega)$ such that for each $n < \omega$ there are $i_0 < \ldots < i_n < \lambda$ such that $a_0, \ldots, a_n \equiv_B b_{i_0}, \ldots, b_{i_n}$. It is easy to check that it is a Morley sequence over A in p. ☑

Proposition 5.3. *Let T be simple. The formula $\varphi(x, a)$ divides over A iff for every infinite Morley sequence $(a_i : i < \omega)$ over A in $\mathrm{tp}(a/A)$, $\{\varphi(x, a_i) : i < \omega\}$ is inconsistent.*

Proof. Without loss of generality $A = \emptyset$. Assume that $\varphi(x, a)$ divides over $\emptyset$ but for some infinite Morley sequence the inconsistency fails. Let X be a linearly ordered set isomorphic to the reverse order of the cardinal $|T|^+$. By compactness there is an infinite Morley sequence $a_X = (a_i : i \in X)$ in $\mathrm{tp}(a)$ such that $\{\varphi(x, a_i) : i \in X\}$ is consistent. Let c realize this type. By simplicity there is $Y \subseteq X$ of cardinality at most $|T|$ such that $\mathrm{tp}(c/a_X)$ does not fork over $a_Y = (a_i : i \in Y)$. By choice of the order of X we can find $i \in X$ such that $i < Y$. By Lemma 5.2 $\mathrm{tp}(a_Y/a_i)$ does not divide over $\emptyset$. Since $\varphi(x, a_i)$ divides over $\emptyset$, by Proposition 3.5 it divides over a_Y. But $\mathrm{tp}(c/a_X)$ contains $\varphi(x, a_i)$ and hence it divides (and forks) over a_Y, a contradiction. ☑

Proposition 5.4. *Let T be simple. A partial type $\pi(x, a)$ divides over A iff it forks over A.*

Proof. Assume $\varphi(x, a)$ does not divide over A but it implies a disjunction $\varphi_1(x, a_1) \vee \ldots \vee \varphi_n(x, a_n)$ where every $\varphi(x, a_i)$ divides over A. Let $(a^j a_1^j \ldots a_n^j : j < \omega)$ be an infinite Morley sequence over A in $\mathrm{tp}(aa_1 \ldots a_n/A)$. Then $(a^j : j < \omega)$ is an A-indiscernible sequence of realizations of $\mathrm{tp}(a/A)$. By definition of dividing, there exists a realization c of $\{\varphi(x, a^j) : j < \omega\}$. For every $j < \omega$ there is some i such that c realizes some $\varphi_i(x, a_i^j)$. By the pigeonhole principle, there is some i such that for an infinite subset $X \subseteq \omega$, c realizes every $\varphi_i(x, a_i^j)$ with $j \in X$. By indiscernibility, $\{\varphi_i(x, a_i^j) : j < \omega\}$ is consistent and then by Proposition 5.3 $\varphi_i(x, a_i)$ does not divide over A since $(a_i^j : j < \omega)$ is an infinite Morley sequence over A in $\mathrm{tp}(a_i/A)$. ☑

Proposition 5.5. *In a simple theory independence is a symmetric relation, i.e,* $A \downfree_C B$ *implies* $B \downfree_C A$.

Proof. It is enough to prove that if $\mathrm{tp}(a/Cb)$ does not fork over C, then $\mathrm{tp}(b/Ca)$ does not divide over C. By Lemma 5.1 there is an infinite Morley sequence $I = (a_i : i < \omega)$ in $\mathrm{tp}(a/C)$ which is Cb-indiscernible and starts with $a_0 = a$. Let $\varphi(x, y, z)$ be a formula and $c \in C$ such that $\models \varphi(a, b, c)$. We will show that $\varphi(a, y, c)$ does not divide over C. By indiscernibility of I over Cb we know that $\models \varphi(a_i, b, c)$ for all $i < \omega$. Hence $\{\varphi(a_i, y, c) : i < \omega\}$ is consistent. Since $(a_i c : i < \omega)$ is a Morley sequence in $\mathrm{tp}(ac/C)$, by Proposition 5.3 we conclude that $\varphi(a, y, c)$ does not divide over C. ☑

Proposition 5.6. *In a simple theory independence is a transitive relation, i.e, whenever* $B \subseteq C \subseteq D$, $A \downfree_B C$ *and* $A \downfree_C D$, *then* $A \downfree_B D$.

Proof. It is a direct consequence of Proposition 5.5, Lemma 3.4 and Proposition 5.4. ☑

Corollary 5.1. *Let T be simple. If I is an ordered set and $(a_i : i \in I)$ is an A-independent sequence, then* $a_i \downfree_A \{a_j : j \neq i\}$ *for all* $i \in I$.

Proof. By induction on n it is easy to show that for all different $i_1, \ldots, i_{n+1} \in I$, $a_{i_{n+1}} \downfree_A a_{i_1}, \ldots, a_{i_n}$. For the inductive case one uses symmetry and Lemma 3.4. ☑

6. The independence theorem

Lemma 6.1. *Let T be simple. If $(a_i : i < \omega + \omega)$ is an infinite A-indiscernible sequence, then $(a_i : \omega \leq i < \omega + \omega)$ is a Morley sequence over $A\{a_i : i < \omega\}$.*

Proof. Let $I = (a_i : i < \omega)$. Clearly $(a_i : \omega \leq i < \omega + \omega)$ is AI-indiscernible. It suffices to show that it is AI-independent. Let X be a finite subset of $\{i : \omega \leq i < \omega + \omega\}$ an let $i < \omega + \omega$ be greater than every element in X. By symmetry it will be enough to check that $a_X \downfree_{AI} a_i$, where $a_X = (a_j : j \in X)$. But this is clear since by A-indiscernibility $\mathrm{tp}(a_X/AIa_i)$ is finitely satisfiable in I. ☑

Proposition 6.1. *Let T be simple. If $\varphi(x, a)$ does not divide over A and a, b start an infinite A-indiscernible sequence, then $\varphi(x, a) \wedge \varphi(x, b)$ does not divide over A.*

Proof. Let us first assume that a, b start an infinite Morley sequence $(a_i : i < \omega)$ over A. Since $\varphi(x, a)$ does not divide over A, $\{\varphi(x, a_i) : i < \omega\}$ is consistent. Let $b_i = a_{2i}a_{2i+1}$. Then $(b_i : i < \omega)$ is an infinite Morley sequence in $\mathrm{tp}(ab/A)$ over A. Since $\{\varphi(x, a_{2i}) \wedge \varphi(x, a_{2i+1}) : i < \omega\}$ is consistent, by Proposition 5.3, $\varphi(x, a) \wedge \varphi(x, b)$ does not divide over A.

Now let us consider the general case, where a, b start an A-indiscernible sequence $(a_i : i < \omega)$. Choose $J = (b_i : i < \omega)$ such that $(b_i : i < \omega)^\frown(a_i : i <$

ω) is A-indiscernible. By Lemma 6.1 $(a_i : i < \omega)$ is a Morley sequence over $A \cup J$. Let $p(x,a)$ be a complete type over AJa which does not fork over A and contains $\varphi(x,a)$. By the first case and compactness, $p(x,a) \cup p(x,b)$ does not divide over AJ. Let $c \models p(x,a) \cup p(x,b)$ be such that $c \mathop{\smile\!\!\!\!\!\!\,|}_{AJ} ab$. Since $p(x,a)$ does not fork over A, $c \mathop{\smile\!\!\!\!\!\!\,|}_{A} Ja$. It follows that $c \mathop{\smile\!\!\!\!\!\!\,|}_{A} Jab$ and hence that $p(x,a) \cup p(x,b)$ does not fork over A. In particular $\varphi(x,a) \wedge \varphi(x,b)$ does not divide over A. ☑

Remark 6.1. *The proof of Proposition 6.1 generalizes to show that in a simple theory, if $\pi(x,a)$ does not divide over A and $I \ni a$ is an infinite A-indiscernible sequence, then $\bigcup_{b \in I} \pi(x,b)$ does not divide over A.*

Lemma 6.2. *Let T be simple. If a, b start an infinite A-indiscernible sequence and $c \mathop{\smile\!\!\!\!\!\!\,|}_{Aa} b$, then for some d, the extended sequences ac, bd start an infinite A-indiscernible sequence also.*

Proof. Assume $A = \emptyset$. Let $c \mathop{\smile\!\!\!\!\!\!\,|}_{a} b$ and assume $I = (a_i : i < \omega)$ is an infinite indiscernible sequence with $a = a_0$ and $b = a_1$. Since $(a_n : n \geq 1)$ is a-indiscernible and $c \mathop{\smile\!\!\!\!\!\!\,|}_{a} b$, by Lemma 3.1 there is an ac-indiscernible sequence $(a'_n : n \geq 1)$ such that $(a_n : n \geq 1) \equiv_{ab} (a'_n : n \geq 1)$. Thus we may assume that $a_n = a'_n$ for all $n \geq 1$. Let $c_0 = c$ and choose for $n \geq 1$ some c_n such that

$$ca_0a_1 \ldots \equiv c_n a_n a_{n+1} \ldots$$

Since $(a_n : n \geq 1)$ is ac-indiscernible, $cab \equiv caa_m$. Hence $cab \equiv c_n a_n a_{n+m}$, i.e., in the sequence $(c_n a_n : n < \omega)$ all triangles $c_n a_n a_{n+m}$ have the same type $p(x,y,z) = \mathrm{tp}(cab)$. By Ramsey's Theorem there is an indiscernible sequence $(d_n b_n : n < \omega)$ where all triangles $d_n b_n b_{n+m}$ satisfy $p(x,y,z)$. Clearly we may assume that $c = d_0$, $a = b_0$ and $b = b_1$. Take $d = d_1$. ☑

Proposition 6.2. *Let T be simple and assume that $\varphi(x,a) \wedge \psi(x,b)$ does not fork over A. If b, b' start an infinite A-indiscernible sequence and $a \mathop{\smile\!\!\!\!\!\!\,|}_{Ab} b'$, then $\varphi(x,a) \wedge \psi(x,b')$ does not fork over A.*

Proof. Apply Lemma 6.2 finding c such that $ba, b'c$ start an infinite A-indiscernible sequence. By Proposition 6.1, $\varphi(x,a) \wedge \psi(x,b) \wedge \varphi(x,c) \wedge \psi(x,b')$ does not fork over A. In particular $\varphi(x,a) \wedge \psi(x,b')$ does not fork over A. ☑

Corollary 6.1. *Let T be simple and assume that $\varphi(x,a) \wedge \psi(x,b)$ does not fork over A. If $\mathrm{Lstp}(b/A) = \mathrm{Lstp}(b'/A)$ and $a \mathop{\smile\!\!\!\!\!\!\,|}_{A} bb'$, then $\varphi(x,a) \wedge \psi(x,b')$ does not fork over A.*

Proof. Assume $A = \emptyset$. Find $b_1, \ldots, b_n$ such that $b = b_1$, $b' = b_n$ and b_i, b_{i+1} start an infinite indiscernible sequence. Let a' be such that $a' \equiv_{bb'} a$ and $a' \mathop{\smile\!\!\!\!\!\!\,|}_{bb'} b_1, \ldots b_n$. By Proposition 6.2 we see that $\varphi(x,a') \wedge \psi(x,b_i)$ does not fork over the empty set for all $i \leq n$. Hence $\varphi(x,a) \wedge \psi(x,b')$ does not fork over the empty set . ☑

Lemma 6.3. *Let T be simple. Let κ be a cardinal number bigger than $|T|+|A|$. If $(a_i : i < \kappa)$ is A-independent and the length of every a_i is smaller than κ, then for any a of length smaller than κ there is some $i < \kappa$ such that $a \mathop{\smile\hskip-0.9em\raise0.35ex\hbox{$|$}}_A a_i$.*

Proof. By choice of κ, there is a proper subset $B \subseteq \{a_i : i < \kappa\}$ such that $a \mathop{\smile\hskip-0.9em\raise0.35ex\hbox{$|$}}_B \{a_i : i < \kappa\}$. Take $a_i \notin B$. Then $a \mathop{\smile\hskip-0.9em\raise0.35ex\hbox{$|$}}_B a_i$ and, by Corollary 5.1, $a_i \mathop{\smile\hskip-0.9em\raise0.35ex\hbox{$|$}}_A B$. By symmetry and transitivity, $a \mathop{\smile\hskip-0.9em\raise0.35ex\hbox{$|$}}_A a_i$. ☑

Lemma 6.4. *Let T be simple. For any a, A and $B \supseteq A$ there is a' such that* $\mathrm{Lstp}(a'/A) = \mathrm{Lstp}(a/A)$ *and $a' \mathop{\smile\hskip-0.9em\raise0.35ex\hbox{$|$}}_A B$.*

Proof. Let κ be a cardinal bigger than $|T| + |B|$ and bigger than the length of a. We may assume that $\mathrm{tp}(a/A)$ is not algebraic. Let $(a_i : i < \kappa)$ be a Morley sequence in $\mathrm{tp}(a/A)$. By Lemma 6.3 there is some $i < \kappa$ such that $B \mathop{\smile\hskip-0.9em\raise0.35ex\hbox{$|$}}_A a_i$. Clearly, $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(a_i/A)$. ☑

Lemma 6.5. *Let T be simple and* $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$. *For any c, B there is some d such that* $\mathrm{Lstp}(ac/A) = \mathrm{Lstp}(bd/A)$ *and $d \mathop{\smile\hskip-0.9em\raise0.35ex\hbox{$|$}}_{Ab} B$.*

Proof. By Corollary 2.2 there is some d' such that $\mathrm{Lstp}(ac/A) = \mathrm{Lstp}(bd'/A)$ and by Corollary 2.1, there is a strong automorphism $f \in \mathrm{Autf}(\mathfrak{C}/A)$ such that $f(ac) = bd'$. By Lemma 6.4 there is some d such that $\mathrm{Lstp}(d/Ab) = \mathrm{Lstp}(d'/Ab)$ and $d \mathop{\smile\hskip-0.9em\raise0.35ex\hbox{$|$}}_{Ab} B$. Again by Corollary 2.1 there is some $g \in \mathrm{Autf}(\mathfrak{C}/Ab)$ such that $g(d') = d$. It follows that $g \circ f \in \mathrm{Autf}(\mathfrak{C}/A)$ and $g \circ f(ac) = bd$. Hence $\mathrm{Lstp}(ac/A) = \mathrm{Lstp}(bd/A)$. ☑

Corollary 6.2 (Independence Theorem)**.** *Let T be simple and $a \mathop{\smile\hskip-0.9em\raise0.35ex\hbox{$|$}}_A b$. If there are c, d such that $\models \varphi(c, a)$, $c \mathop{\smile\hskip-0.9em\raise0.35ex\hbox{$|$}}_A a$, $\models \psi(d, b)$, $d \mathop{\smile\hskip-0.9em\raise0.35ex\hbox{$|$}}_A b$, and* $\mathrm{Lstp}(c/A) = \mathrm{Lstp}(d/A)$, *then $\varphi(x, a) \wedge \psi(x, b)$ does not fork over A.*

Proof. By Lemma 6.5, choose $b' \mathop{\smile\hskip-0.9em\raise0.35ex\hbox{$|$}}_{Ac} ab$ such that $\mathrm{Lstp}(cb'/A) = \mathrm{Lstp}(db/A)$. Then $\models \varphi(c, a) \wedge \psi(c, b')$ and $c \mathop{\smile\hskip-0.9em\raise0.35ex\hbox{$|$}}_A ab'$. Therefore $\varphi(x, a) \wedge \psi(x, b')$ does not fork over A. Since $a \mathop{\smile\hskip-0.9em\raise0.35ex\hbox{$|$}}_A bb'$ by Corollary 6.1, $\varphi(x, a) \wedge \psi(x, b)$ does not fork over A. ☑

Remark 6.2.

(1) *The version of the Independence Theorem for partial types $\pi(x, a)$, $\sigma(x, b)$ in place of formulas $\varphi(x, a)$, $\psi(x, b)$ follows in a straightforward way. It can be also generalized easily to any ordered sequence of types $(\pi_i(x, a_i) : i \in I)$ if the corresponding sequence $(a_i : i \in I)$ of parameters is independent over A and* $\mathrm{Lstp}(c/A) = \mathrm{Lstp}(d/A)$ *whenever $c \models \pi_i(x, a_i)$ and $d \models \pi_j(x, a_j)$.*

(2) *The Independence Theorem for types over a model follows also easily from the stated version.*

Proposition 6.3. *Let T be simple. If* $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ *and $a \mathop{\smile\hskip-0.9em\raise0.35ex\hbox{$|$}}_A b$, then a, b start a Morley sequence over A.*

Proof. Let $p = \mathrm{tp}(ab/A)$. We prove first that for each cardinal κ there is an infinite A-independent sequence $(a_i : i < \kappa)$ such that $\models p(a_i, a_j)$ for all $i < j < \kappa$. Note that this implies $\mathrm{Lstp}(a_i/A) = \mathrm{Lstp}(a_j/A)$. The sequence is constructed inductively starting with $a_0 = a$ and $a_1 = b$. To get a_α we need to prove that $\bigcup_{i<n} p(a_i, x)$ does not fork over A. But this is clear by the generalized version of the Independence Theorem (point (1) of Remark 6.2) since for all $c \models p(a_i, x)$ and $d \models p(a_j, x)$ we have $\mathrm{Lstp}(c/A) = \mathrm{Lstp}(d/A)$. Now, once we have this A-independent sequence we still need to make it A-indiscernible. But this can be done by Lemma 1.1. ☑

Proposition 6.4. *If T is simple, then* $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ *if and only if there is some c such that a, c start an infinite A-indiscernible sequence and b, c start an infinite indiscernible sequence.*

Proof. Assume $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ and find with Lemma 6.4 some c such that $\mathrm{Lstp}(c/A) = \mathrm{Lstp}(a/A)$ and $c \mathop{\smile\!\!\!\!\!\!\mid}_A ab$. By Proposition 6.3 a, c start an infinite Morley sequence and b, c start an infinite Morley sequence. ☑

Corollary 6.3. *If T is simple, then equality of Lascar strong types over A is type definable over A by* $\exists z(\mathrm{nc}_A(x, z) \wedge \mathrm{nc}_A(y, z))$.

Proof. Clear, by Proposition 6.4. ☑

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