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Boundary value problems for the Vlasov-Maxwell system

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Abstract. The paper studies the special classes of the stationary and nonstationary solutions of VM system and their connection with the systems of nonlocal semilinear elliptic equations with boundary conditions. Using the proposed lower-upper solution method, we proved an existence theorem for a semilinear nonlocal elliptic boundary value problem under corresponding restrictions over the distribution function (ansatz RSS [52, 53]).

Keywords. Vlasov-Maxwell system, boundary value problem, upper-lower solution.

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Resumen. El artículo estudia clases especiales de soluciones estacionarias y no estacionarias de sistemas VM y su conexión con los sistemas semilineales elípticos no locales con condiciones de frontera. Usando el método de bajo-alto demostramos un teorema de existencia para un problema elíptico semilineal no local con valor de frontera bajo las restricciones correspondientes sobre la función de distribución (ver ansatz RSS [52, 53]) .

1. Introduction

At present, the investigation of the *Vlasov equation* goes in two different directions. The first direction is related to the existence theorems for Cauchy problem and uses an apriori estimation technique as basis for research. The second one implements the reduction of the initial problem to a simplified one, introducing a set of distribution functions (ansatz), followed by reconstruction of the characteristics for electromagnetic fields in an evident form.

This is a rather restrictive approach, since the distribution function has a special form. On the other hand, it allows us to solve a problem in an explicit form, which is important for applications.

The statement and investigation of the boundary value problem for the Vlasov equation are very difficult and have been considered in simplified cases only (see. Abdallach [1], Guo [31], Degond [21]). Reducing it to the boundary value problem for a system of nonlinear elliptic equations allows us to show a solvability in some cases. Doing the same for the initial statement of the problem is not that simple.

Nevertheless, both directions are related in terms of special structures used for studying kinetic equations. For example:

- Energy integral is applied in both cases for obtaining energy estimations in existence theorems and for construction of Lyapunov functionals;
- Virial identities in stability and instability analysis in special classes of solutions of Vlasov equation.

It is known that the solution of Vlasov equation (see Vlasov [61, 62]) is an arbitrary function of first integrals of the characteristic system (until now their smoothness remains a complicated unsolved problem), defining the trajectory of a particle motion in electromagnetic field

$$\dot{r} = V, \qquad \dot{V} = \frac{q_i}{m_i}\left(E(r,t) + \frac{1}{c}V \times B(r,t)\right), \tag{1.1}$$

where $r \triangleq (x, y, z) \in \Omega_2 \subseteq R^3$, $V \triangleq (V_x, V_y, V_z) \in \Omega_1 \subset R^3$ – position and velocity of a particle, $E \triangleq (E_x, E_y, E_z)$ – a tension of electrical field, $B \triangleq (B_x, B_y, B_z)$ – magnetic induction and m_i, q_i – mass and charge of a particle of i-th kind. For N-component distribution function, the classical Vlasov-Maxwell(VM) system has the form

$$\partial_t f_i + V \cdot \nabla_r f_i + \frac{q_i}{m_i}\left(E + \frac{1}{c}V \times B\right)\cdot \nabla_V f_i = 0, \qquad i = 1, \dots, N, \tag{1.2}$$

$$\partial_t E = c\ \mathrm{curl} B - j, \tag{1.3}$$

$$\mathrm{div} E = \rho, \tag{1.4}$$

$$\partial_t B = -c\ \mathrm{curl} E, \tag{1.5}$$

$$\mathrm{div} B = 0. \tag{1.6}$$

The charge and current densities are defined by formulae

$$\rho(r,t) = 4\pi \sum_{i=1}^{N} q_i \int_{\Omega_1} f_i \mathrm{d}V, \tag{1.7}$$

$$j(r,t) = 4\pi \sum_{i=1}^{N} q_i \int_{\Omega_1} f_i V \mathrm{d}V. \tag{1.8}$$

We impose the specular reflection condition on the boundary for the distribution function

$$f_i(t, r, v) = f_i\big(t, r, v - 2(vN_\Omega(r))N_\Omega(r)\big), \quad t \geq 0, \quad r \in \partial\Omega, \quad v \in \Omega$$

where $N_\Omega(r)$ is a normal vector to the boundary surface.

In applied problems, the impact of magnetic field is often neglected. This limit system is known as the Vlasov-Poisson (VP) one, where the Maxwell equations degenerate to the Poisson equation

$$\triangle\varphi = 4\pi \sum_{i=1}^{N} q_i \int_{\Omega_1} f_i \mathrm{d}V, \tag{1.9}$$

where $\varphi(r,t)$ – a scalar potential of the electrical field.

In general case the distribution function may be represented in the form

$$f_i = f_i(H_{i1}, H_{i2}, \ldots, H_{il}), \qquad i = 1, \ldots, N, \tag{1.10}$$

where H_{il} is the first integral (is constant along the characteristics of the equation) for (1.1).

In fact, it is not easy to select a structure of the distribution function (1.10) which is connected with electromagnetic potentials aiming to transform the initial system into a simplified form. Hence, in practice, we are usually restricted to energy integrals $H_i = -c_i|V|^2 + \varphi(r,t)$ or $H_i^0 = -d_i|V|^2 + \varphi(r)$ as in the stationary problem case (see Vlasov [61, 62]). Meanwhile, an introduction of the following ansatz

$$H_{il} = \varphi_{il} + (V, d_{il}) + (A_{il}V, V) + \sum_{m+k+j=3} a_{mkj}^{il} V_1^m V_2^k V_3^j \tag{1.11}$$

generalizes the form of the distribution function. Here $V \stackrel{\triangle}{=} (V_1, V_2, V_3)$ and $(A_{il}V, V)$ are quadratic forms; the following ones are the forms of higher degrees. In this case matrices A_{il} and coefficients a_{mkj}^{il} should be connected with the system (1.2)–(1.6) converting the first integrals for the characteristic system (1.1) into H_{il}.

The first statement of existence problem of classical solutions for the one-dimensional Vlasov equation has been given by Iordanskii [37], and the existence of generalized (weak) solutions for the two-dimensional problem has been proved by Arsen'ev [10].

The results of Neunzert [46], Horst [33], Batt [11], Illner, Neunzert [36], Ukai, Okabe [56], DiPerna, Lions P. [22], Wollman [64, 65], Batt, Rein [14], Pfaffelmoser [48] are devoted to existence of solutions for (1.2) and (1.9). Degond [20], Glassey, Strauss [25], Glassey, Schaeffer [26–28], Horst [34, 35], Cooper, Klimas [18], Schaeffer [54], Guo [31], Rein [50] concern its generalization to the VM system (1.2)–(1.6).

Some rigorous results obtained recently (see Guo [31], Abdallach [1], Degond [21], Abdallach, Degond and Mehats [2], Vedenyapin [58–60], Batt and Fabian [15], Braasch [16], Guo and Ragazzo [32], Dolbeault [23], Poupaud [49], Caffarelli, Dolbeault, Markowich, Schmeiser [17], Ambroso [7]) are related to analysis of (1.2)–(1.6), (1.2)–(1.9) on bounded domains with boundary conditions.

We have to mention that techniques used to prove the existence of solutions of Cauchy problem for the VM and VP systems for $\left(x \in \mathbb{R}^3, v \in \mathbb{R}^3\right)$ have limited applicability on bounded domains. Hence a necessity to study VM and VP systems with boundary conditions is valid. That is why before presenting our own results, we have to outline some already published results on VM and VP systems in bounded domains.

Existence and properties of solutions of the VM and VP systems in bounded domains. In the case of spherical symmetry rather complete results were obtained by Batt, Faltenbacher, Horst [13]. In the next paper by Batt, Berestycki, Degond, Perthame [12] a family of "local isotropic" solutions of nonstationary problem of the VP system for the distribution function

$$f(t, r, V) = \Phi\left(W(t, r) + \frac{(U - Ar)^2}{2}\right), \qquad U(t, r) = W(t, r) + \frac{(Ar)^2}{2}, \quad (1.12)$$

$$t \in \mathbb{R}, \qquad r \in D \subset \mathbb{R}^3, \qquad v \in \mathbb{R}^3, \qquad \Phi : \mathbb{R} \to [0, \infty), \qquad W : \mathbb{R}^3 \to \mathbb{R},$$

were constructed. Here U – potential and A – antisymmetric 3×3- matrix. Under this assumptions, the VP system is reduced to the Dirichlet boundary value problem for the nonlinear elliptic equation

$$\triangle W + 2|w| = 4\pi \int_{\mathbb{R}^3} \Phi\left(W + \frac{1}{2}|v|^2\right) dv, \qquad w = (w_1, w_2, w_3) \in \mathbb{R}.$$

The existence of the solution for the named problem is proved using the lower–upper solution method.

The stationary solutions of the n-component VP system for the distribution function depending on the energy integral $f_i(E)$ were studied by Vedenyapin [58–60]. He proved the existence of a solution to Dirichlet's problem

$$-\triangle u(r) = \psi(u), \qquad u(r)|_{\partial D} = u_0(r), \quad (1.13)$$

$$\psi(u) = 4\pi \sum_{k=1}^{n} q_k \int_{\mathbb{R}^3} g_k\left(\frac{1}{2} m_k |v|^2 + q_k u\right) dv$$

where an arbitrary function ψ satisfies the condition **(i)** $\frac{d}{du}\psi(u) \geq 0$. Here $u(r)$ – scalar potential, $g_k(\cdot)$ – nonnegative continuously differentiable functions, $D \subset \mathbb{R}^3$ – domain with a smooth enough boundary , $u_0(r)$ – potential given on the boundary. If $r \in D \subset \mathbb{R}^p$, $v \in \mathbb{R}^p$, then the boundary value problem (1.13) has a unique solution for arbitrary nonnegative functions g_k (Vedenyapin [58]).

Rein [51] proved the existence of a solution of (1.13) by a variational method under condition **(i)**.

In the paper [15] Batt and Fabian studied a transformation of the stationary VP system into (1.13) in general case, considering distribution functions depending on energy $f_i(E)$ and on the sum of energy and momentum $f_i(E+P)$. Using a lower–upper solution method (Pao [47]), they proved the existence of the solutions of (1.13) under condition **(i)**. Therefore the condition **(i)** became a primary condition to prove the existence theorems for the problem (1.13). The general weak global solution of the VP system has been presented by Weckler in [63].

Dolbeault [23] proved the existence and uniqueness of Maxwellian solutions

$$f(t,x,v) = \frac{1}{(2\pi T)N/2}\rho(x)e^{\frac{-|v|^2}{2T}}, \qquad (x,v) \in \Omega \times \mathbb{R}^N \tag{1.14}$$

using variational methods.

A new direction in the study of the VP system is connected with the free boundary problems for semiconductor modeling. Caffarelli, Dolbeault, Markowich, Schmaiser [17] considered a semilinear elliptic integro-differential equation with Neumann boundary condition

$$\epsilon\triangle\phi = q(n-p-C) \qquad \Omega, \tag{1.15}$$

$$\frac{\partial\phi}{\partial\nu} = 0 \qquad \partial\Omega,$$

where local densities of electrons $n(x)$ and holes $p(x)$ in insulated semiconductor are given by Boltzmann-Maxwellian statistics

$$n(x) = \frac{N\exp\big(q\phi(x)/(kT)\big)}{\int_\Omega \exp\big(q\phi/(kT)\big)dx}, \qquad p(x) = \frac{P\exp\big(-q\phi(x)/(kT)\big)}{\int_\Omega \exp\big(-q\phi/(kT)\big)dx}.$$

$C(x)$ – is given background, $x \in \Omega$, $\Omega \subset \mathbb{R}^d$ a bounded domain. Using a variational problem statement they proved the existence and uniqueness of the solutions and showed that the limit potential is a solution of the free boundary problem.

Concerning a study of the nonlocal problem (1.15), we recommend the paper by Maslov [42].

Existence and properties of solutions of the VM system in the bounded domains. If we change velocity v by its relativistic analogue $\hat{v} = \frac{v}{\sqrt{1+|v|^2}}$ we have to face another complicated problem, since the classical VM system is not invariance in the sense of Galilei and Lorentz.

Adding boundary conditions

$$E(t,x) \times N_\Omega(x) = 0, \qquad B(t,x)N_\Omega(x) = 0, \qquad t \geq 0, \qquad x \in \partial\Omega \tag{1.16}$$

to the system (1.2)–(1.8) we obtain a different problem statement. Here N_Ω is the unit normal vector to $\partial\Omega$ and reflection condition

$$f_k(t,x,v) = f_k\big(t,x,\tilde{v}(x,v)\big), \qquad t \geq 0, \qquad x \in \partial\Omega, \qquad v \in \mathbb{R}^3, \tag{1.17}$$

where $\tilde{v} : \mathbb{R}^3 \to \mathbb{R}^3$ – bijective mapping for $x \in \partial\Omega$. One of the most known reflection mechanisms is a specular reflection condition of the form

$$\tilde{v}(x,v) = v - 2\big(vN_\Omega(x)\big)N_\Omega(x), \qquad x \in \partial\Omega, \qquad v \in \mathbb{R}^3, \tag{1.18}$$

or invertible reflection condition

$$\tilde{v}(x,v) = -v, \qquad x \in \partial\Omega, \qquad v \in \mathbb{R}^3. \tag{1.19}$$

At present, only a few number of papers study the VM system in bounded domains. For the first time the boundary value problem for one-dimensional VM system has been considered by Cooper, Klimas [18].

In the paper of Rudykh, Sidorov, Sinitsyn [52] stationary classical solutions $(f_1, \dots, f_n, E, B)$ for the VM system of the special form (RSS ansatz)

$$
\begin{aligned}
f_k(x,v) &= \psi_k\left(-\alpha_k v^2 + \mu_{1k} U_1(x), vd + \mu_{2k} U_2(x)\right), \\
E(x) &= \frac{1}{\alpha_1 q_1} \nabla U_1(x), \\
B(x) &= -\frac{1}{q_1 d^2}\left(d \times \nabla U_2(x)\right)
\end{aligned}
$$

were constructed. Here functions $\psi_k : \mathbb{R}^2 \to [0, \infty)$ and parameters $d \in \mathbb{R}^3 \backslash \{0\}$, $\alpha_k > 0$, $\mu_{ik} \neq 0$ ($k \in \{1, \dots, n\}, i \in \{1, 2\}$) – are given; Functions U_1, U_2 have to be defined. This approach (RSS ansatz) is closely connected with the paper of Degond [20].

Batt, Fabian [15] applied RSS ansatz technique for the VM system with distribution functions $\psi(\mathbb{E})$, $\psi(\mathbb{E}, F)$, $\psi(\mathbb{E}, F, P)$, where functions $\mathbb{E}(x, v)$, $F(x, v)$ and $P(x, v)$ – are the first integrals of Vlasov equation (1.2). Braasch in his own thesis [16] extended RSS results to the relativistic VM system.

Collisionless kinetic models (classical and relativistic VM systems). In this area existence theorems (and global stability) of renormalized solutions on bounded domains (when trace is defined on the boundary) were proved by Mischler [44, 45]. Abdallah and Dolbeault [5] also developed the entropic methods on bounded domains for qualitative study of behavior of global solutions of the VP system. Regularity theorems of weak solutions on the basis of scalar conservation laws and averaging lemmas were proved by Jabin, Perthame [38]. Jabin [39] also obtained local existence theorems of weak solutions of the VP system on bounded domains. For modeling of ionic beams Ambroso, Fleury, Lucquin-Desreux, Raviart [8] proposed some new kinetic models with a source. Existence theorems of global solutions of the Vlasov-Einstein system in the case of hyperbolic symmetry were proved by Andreasson, Rein, Rendall [9].

Quantum models: Vigner-Poisson (VP) and Schrödinger-Poisson (SP) systems. The paper of Abdallah, Degond, Markowich [3] considered the Child-Langmuir regime for stationary Schrödinger equation. The authors developed a semi-classical analysis for quantum kinetic equations passing from limit $h \to 0$ to classical Vlasov equation with special boundary "transition" conditions from quantum zone to classical. New results were obtained for Boltzmann-Poisson, Euler-Poisson, Vigner-Poisson-Fokker-Plank systems (like existence and uniqueness of the solutions, hydrodynamic limits, solutions with a minimum energy and dispersion properties).

Mixed quantum–classical kinetic systems. In the paper of Abdallah [4] the Vlasov-Schrödinger (VS) and Boltzmann-Schrödinger systems for one-dimensional stationary case are considered. Nonstationary problems for VS system with boundary "transition" conditions from classical zone (Vlasov equation) to quantum (Schrödinger equation) are studied in the paper by Abdallah, Degond, Gamba [4].

We study the special classes of stationary and nonstationary solutions of VM system. Being constructed, such solutions lead us to systems of nonlocal semilinear elliptic equations with boundary conditions. Applying the lower-upper solution method, existence theorems for solutions of the semilinear nonlocal elliptic boundary value problem under corresponding restrictions upon a distribution function are obtained. It was shown that under certain conditions upon electromagnetic field, the boundary conditions and specular reflection condition for distribution function are satisfied.

2. Stationary solutions of Vlasov-Maxwell system

In this section we consider the system

$$V \cdot \frac{\partial}{\partial r} f_i(r,V) + \frac{q_i}{m_i}\left(E + \frac{1}{c} V \times B\right) \cdot \frac{\partial}{\partial V} f_i(r,V) = 0, \tag{2.1}$$

$$\mathrm{rot} E = 0, \tag{2.2}$$

$$\mathrm{div} B = 0, \tag{2.3}$$

$$\mathrm{div} E = 4\pi \sum_{k=1}^{N} q_k \int_{\Omega_1} f_k(r,V) \mathrm{d}V, \tag{2.4}$$

$$\mathrm{rot} B = \frac{4\pi}{c} \sum_{k=1}^{N} q_k \int_{\Omega_1} V f_k(r,V) \mathrm{d}V. \tag{2.5}$$

Here $f_i(r,V)$ – distribution function of the particles of i-th kind; $r \overset{\triangle}{=} (x,y,z) \in \in \Omega_2$, $V \overset{\triangle}{=} (V_x, V_y, V_z) \in \Omega_1 \subset \mathbb{R}^3$ – coordinate and velocity of particle respectively; E, B – electric field strength and magnetic induction; m_i, q_i – mass and charge of particle of i-th kind.

We shall seek stationary distributions of the form

$$f_i(r,V) = f_i\left(-\alpha_i|V|^2 + \varphi_i, V \cdot d_i + \psi_i\right) \overset{\triangle}{=} \hat{f}_i(R,G) \tag{2.6}$$

and corresponding self consistent electromagnetic fields E and B satisfying (2.1)–(2.5). We assume that

i) $\hat{f}_i(R,G)$ – fixed differentiable functions of own arguments; $\alpha_i \in \mathbb{R}^+$, $d_i \in \mathbb{R}^3$ are free parameters, $|d_i| \neq 0$; $\varphi_i = c_{1i} + l_i\varphi$, $\psi_i = c_{2i} + k_i\psi$, where c_{1i}, c_{2i} – constant; for all φ_i, ψ_i the integrals

$$\int_{\mathbb{R}^3} f_i dV, \qquad \int_{\mathbb{R}^3} V f_i dV,$$

are convergent. Unknown functions $\varphi_i(r)$, $\psi_i(r)$ have to be defined in such a manner that system (2.1)–(2.5) will satisfies the relation $\big(E(r), d_i\big) = 0$, $i = 1, \dots, N$. The last condition is necessary for solvability of (2.1) in a class (2.6) for $\partial \hat{f}_i / \partial R|_{v=0} \neq 0$.

2.1. **Reduction of the problem (2.1)–(2.5) to the system of nonlinear elliptic equations.** We construct the system of equations to define the set of functions φ_i, ψ_i. Substituting (2.6) into (2.1) and equating to zero the coefficients at $\partial \hat{f}_i/\partial R$ and $\partial \hat{f}_i / \partial G$ we obtain

$$E(r) = \frac{m_i}{2\alpha_i q_i} \nabla \varphi_i, \tag{2.7}$$

$$B(r) \times d_i = -\frac{m_i c}{q_i} \nabla \psi_i, \tag{2.8}$$

$$\big(E(r), d_i\big) = 0. \tag{2.9}$$

Here φ_i, ψ_i – arbitrary functions satisfying conditions

$$(\nabla \varphi_i, d_i) = 0, \qquad i = 1, \dots, N, \tag{2.10}$$

$$(\nabla \psi_i, d_i) = 0. \tag{2.11}$$

Vector B is

$$B(r) = \frac{\lambda_i(r)}{d_i^2} d_i - [d_i \times \nabla \psi_i] \frac{m_i c}{q_i d_i^2}, \tag{2.12}$$

where $\lambda_i(r) = (B, d_i)$ – function which has to be defined. Having defined φ_i, ψ_i such that system (2.2)–(2.5) is satisfied, we can find unknown functions f_i, E, B by formulae (2.6), (2.7), (2.12).

Unknown vectors $\nabla \varphi_i, \nabla \psi_i$ are linearly dependent by virtue of (2.7), (2.8). Then we shall seek φ_i, ψ_i in the form

$$\varphi_i = c_{1i} + l_i \varphi, \qquad \psi_i = c_{2i} + k_i \psi, \tag{2.13}$$

where c_{1i}, c_{2i} – constants. Because of (2.7), (2.8) parameters l_i, k_i are linked by the following relations

$$l_i = \frac{m_1}{\alpha_1 q_1} \frac{\alpha_i q_i}{m_i}, \qquad i = 1, \dots, N, \tag{2.14}$$

$$k_i \frac{q_1}{m_1} d_1 = \frac{q_i}{m_i} d_i. \tag{2.15}$$

From (2.4) with (2.7) we obtain the system

$$\triangle \varphi_i = \frac{8\pi \alpha_i q_i}{m_i} \sum_{k=1}^{N} q_k \int_{\Omega_1} f_k(r, V) \mathrm{d}V. \tag{2.16}$$

Since $\operatorname{div}[d_i \times \nabla \psi_i] = 0$, then substituting (2.12) into (2.3) we have

$$\big(\nabla \lambda_i(r), d_i\big) = 0. \tag{2.17}$$

Taking into account (2.12), from (2.5) we obtain the system of linear algebraic equations for $\nabla\lambda_i$

$$\nabla\lambda_i \times d_i = \frac{m_i c}{q_i} d_i \triangle\psi_i + \frac{4\pi}{c} d_i^2 \sum_{k=1}^{N} q_k \int_{\Omega_1} V f_k \mathrm{d}V. \tag{2.18}$$

To solve (2.18) it is necessary and sufficient due to Fredholm's theorem (see [55]) that ψ_i satisfies the equation

$$\triangle\psi_i = -\frac{4\pi q_i}{m_i c^2} \sum_{k=1}^{N} q_k \int_{\Omega_1} (V, d_i) f_k \mathrm{d}V. \tag{2.19}$$

Furthermore, vector

$$C_i(r) d_i + d_i \times J(r) \tag{2.20}$$

is a general solution of (2.18) with

$$J \triangleq \frac{4\pi}{c} \sum_{k=1}^{N} q_k \int_{\Omega_1} V f_k \mathrm{d}V,$$

C_i – arbitrary function. Taking into account (2.13)–(2.15), it is easy to show that functions φ, ψ satisfy the system

$$\triangle\varphi = \frac{8\pi\alpha q}{m} \sum_{k=1}^{N} q_k \int_{\Omega_1} f_k \mathrm{d}V, \tag{2.21}$$

$$\triangle\psi = -\frac{4\pi q}{mc^2} \sum_{k=1}^{N} q_k \int_{\Omega_1} (V, d) f_k \mathrm{d}V, \tag{2.22}$$

with $\alpha \triangleq a_1$, $q \triangleq q_1$, $m \triangleq m_1$, $d \triangleq d_1$.

Lemma 2.1. *Vector $d_i \times J(r)$ is a potential and a unique solution of (2.18) satisfying condition (2.17).*

Proof. Since ψ satisfies (2.22), then (2.20) is a general solution of (2.18). Due to (2.17) we can set $C_i \equiv 0$. Therefore $d_i \times J$ – unique solution of (2.17), (2.18). We show that $d_i \times J$ – potential. In fact

$$\mathrm{rot}[d_i \times J] = -(d_i, \nabla)J + d(\nabla, J),$$

where

$$(\nabla, J) \equiv 0, \qquad (d_i, \nabla)J = (d_i, \nabla)\mathrm{rot}B = \mathrm{rot}(d_i, \nabla)B.$$

Due to (2.12)

$$\begin{aligned}(d_i, \nabla)B &= (d_i, \nabla)\left\{\frac{\lambda_i}{d_i^2} d_i - [d_i \times \nabla\psi_i]\frac{m_i c}{q_i d_i^2}\right\} \\ &= \frac{d_i}{d_i^2}(\nabla\lambda_i, d_i) - \frac{m_i c}{q_i d_i^2} \times [d_i \times \nabla(d_i, \nabla\psi_i)],\end{aligned}$$

$(\nabla\lambda_i, d_i) = 0$, $(\nabla\psi_i, d_i) = 0$.

Hence $\mathrm{rot}[d_i \times J] \equiv 0$, $d_i \times J = \nabla\lambda_i$, and Lemma is proved. ☑

Corollary 2.1.

$$\nabla\lambda_i(r) = [d_i \times J(r)]. \tag{2.23}$$

Lemma 2.2. *Let* $b(x) = (b_1(x), b_2(x), b_3(x))$, $x \in \mathbb{R}^3$,

$$\frac{\partial b_i}{\partial x_j} = \frac{\partial b_j}{\partial x_i}, \qquad i, j = 1, 2, 3. \tag{2.24}$$

Then $b(x) = \nabla\lambda(x)$, *where*

$$\lambda(x) = \int_0^1 (b(\tau x), x)\mathrm{d}\tau + \mathrm{const}. \tag{2.25}$$

The proof is developed by straight calculation.

Corollary 2.2.

$$\frac{d_i}{d_i^2}\lambda_i = \frac{d}{d^2}\left(\beta + \int_0^1 \big(d \times J(\tau x), x\big)\mathrm{d}\tau\right), \qquad i = 1, \dots, N, \qquad \beta - \mathrm{const}. \tag{2.26}$$

Result follows from Lemma 2.2, Corollary 2.1 and (2.15).

We are looking for the solutions (2.21), (2.22) satisfying orthogonality conditions (2.10), (2.11). Assuming $d_{1i} \neq 0$, $i = 1, 2, 3$ we shall seek solutions in the form $\varphi = \varphi(\xi, \eta)$, $\psi = \psi(\xi, \eta)$

$$\xi = \left(\frac{y}{d_{12}} - \frac{z}{d_{13}}\right) + \frac{d_{11}^2}{d_{11}^2 + d_{12}^2}\left(\frac{x}{d_{11}} - \frac{y}{d_{12}}\right),$$

$$\eta = \frac{|d_1| d_{11} d_{12}}{d_{13}(d_{11}^2 + d_{12}^2)}\left(\frac{x}{d_{11}} - \frac{y}{d_{12}}\right), \qquad d_1 \stackrel{\triangle}{=} (d_{11}, d_{12}, d_{13}). \tag{2.27}$$

Moreover the problem is reduced to the study of nonlinear (semilinear) elliptic equations

$$\triangle\varphi = \mu \sum_{k=1}^N q_k \int_{\Omega_1} f_k \mathrm{d}V, \tag{2.28}$$

$$\triangle\psi = \nu \sum_{k=1}^N q_k \int_{\Omega_1} (V, d) f_k \mathrm{d}V, \tag{2.29}$$

where

$$d \stackrel{\triangle}{=} d_1, \qquad \triangle\cdot = \frac{\partial^2 \cdot}{\partial \xi^2} + \frac{\partial^2 \cdot}{\partial \eta^2};$$

$$\mu = \frac{8\pi\alpha q}{m w(d)}; \qquad \nu = -\frac{4\pi q}{m c^2 w(d)}; \qquad w(d) = \frac{d^2}{d_{13}(d_{11}^2 + d_{12}^2)}.$$

We notice that every solution (2.28), (2.29) due to (2.27) satisfies orthogonality conditions (2.10), (2.11). From the preceding assertion it follows

Theorem 2.1. *Let the distribution function have the form (2.6). Then the electromagnetic field $\{E, B\}$ is defined by formulas*

$$E(\hat{r}) = \frac{m}{2\alpha q}\nabla\varphi,$$
$$B(\hat{r}) = \frac{d}{d^2}\left\{\beta + \int_0^1 (d \times J(\tau\hat{r}), \hat{r})\mathrm{d}\tau\right\} - [d \times \nabla\psi(\hat{r})]\frac{mc}{qd^2}, \tag{2.30}$$

where $\hat{r} \stackrel{\triangle}{=} (\xi, \eta)$; β – const; functions $\varphi(\hat{r})$, $\psi(\hat{r})$ satisfy system (2.28), (2.29).

Let us introduce a scalar and vector potentials $U(r)$, $A(r)$,

$$E(r) = -\nabla U(r), \quad B(r) = \mathrm{rot}A. \tag{2.31}$$

Due to (2.7), (2.12) and (2.26) field $\{E, B\}$ is defined via potentials $\{U, A\}$ by formulae

$$U = -\frac{m}{2\alpha q}\varphi, \quad A = \frac{mc}{qd^2}\psi d + A_1(r), \tag{2.32}$$

where $(A_1, d) = 0$. Unknown potentials U, A can be defined in a subspace D of smooth enough functions on the set $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$ and moreover to satisfy conditions

$$(\nabla U, d) = 0, \qquad \big(\nabla(A, d), d\big) = 0 \tag{2.33}$$

and on the boundary

$$U|_{\partial\Omega_2} = u_0(r), \qquad (A, d)|_{\partial\Omega_2} = u_1(r). \tag{2.34}$$

Corollary 2.3. *Let the distribution function have the form (2.6). Then the VM system (2.1)–(2.5) with boundary conditions (2.34) has a solution*

$$f_i = f_i\left(-\alpha_i|V|^2 + c_{1i} + l_i\varphi^*(r), d_iV + c_{2i} + k_i\psi^*(r)\right),$$

where l_i, k_i satisfy (2.14) and (2.15),

$$E = \frac{m}{2\alpha q}\nabla\varphi^*(r),$$
$$B = \frac{d}{d^2}\left\{\beta + \int_0^1 \big(d \times J^*(\tau r), r\big)\mathrm{d}\tau\right\} - \big[d \times \nabla\psi^*(r)\big]\frac{mc}{qd^2}, \tag{2.35}$$
$$J^*(r) = \frac{4\pi}{c}\sum_{k=1}^N q_k \int_{\Omega_1} V f \mathrm{d}V.$$

Functions φ^*, ψ^* belong to D *and are defined from system (2.28), (2.29) with boundary conditions*

$$\varphi|_{\partial\Omega_2} = -\frac{2\alpha q}{m}u_0(r), \tag{2.36}$$

$$\psi|_{\partial\Omega_2} = \frac{q}{mc}u(r). \tag{2.37}$$

2.2. **Reduction of system (2.28)–(2.29) to a single equation.**

Lemma 2.3. *If*

$$f(V+d,r) = f(-V+d,r), \qquad d \in \mathbb{R}^3, \tag{2.38}$$

then the following equality holds

$$j = d \cdot \rho, \tag{2.39}$$

where $j = \int_{\Omega_1} V f \mathrm{d}V$ *or is the vector of current density;* $\rho = \int_{\Omega_1} f \mathrm{d}V$ *–or is the charge density.*

Proof. Making change of variables in integral $\int_{\Omega_1} V f \mathrm{d}V$ of the form $V_i = \xi_i + d_i$ $(i = 1, 2, 3)$, we obtain

$$\int V_i f(V,r) \mathrm{d}V = J_1 + J_2 + J_3,$$

where

$$J_1 = d_i \int_{\Omega_1} f(\xi + d, r) \mathrm{d}\xi,$$

$$J_2 + J_3 = \int_0^\infty \int_0^\infty \int_0^\infty \xi_i f(\xi_i + d, r) \mathrm{d}\xi + \int_{-\infty}^0 \int_{-\infty}^0 \int_{-\infty}^0 \xi_i f(\xi_i + d, r) \mathrm{d}\xi.$$

It is easy to show that $J_3 = -J_2$ and (2.39) follows. ☑

Taking into account Lemma 2.3, (2.28), (2.29) can be transformed to the form

$$\triangle\varphi = \mu \sum_{i=1}^{N} q_i A_i, \tag{2.40}$$

$$\triangle\psi = \frac{\nu d^2}{2\alpha} \sum_{i=1}^{N} \frac{k_i q_i}{l_i} A_i, \tag{2.41}$$

where $A_i = \int_{\Omega_1} f_i \mathrm{d}V,\ \ i = 1, \dots, N$.

Let $(\xi, \eta) \in \Omega$ where Ω is a bounded domain in $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$. We set a value of scalar potential on the boundary $\partial\Omega$:

$$\varphi(\xi,\eta)|_{\partial\Omega} = \mathbb{A}(\xi,\eta). \tag{2.42}$$

Lets consider two cases, when (2.40), (2.41) is reduced to one equation.

Case 1. $l_i = k_i$, $i = 1, \dots, N$.

Lemma 2.4. *If* $l_i = k_i$ *and* u^* *satisfies equation*

$$\triangle u = a(d,\alpha) \sum_{k=1}^{N} q_i A_i (\gamma_i + l_i u) \tag{2.43}$$

with

$$\gamma_i = c_{1i} + c_{2i}, \quad i = 1, \dots, N, \quad u = \varphi + \psi,$$
$$a(d,\alpha) = 2\pi q \left(4\alpha^2 c^2 - d^2\right) / \left(mc^2 \alpha w(d)\right),$$

then system (2.40), (2.41) has a solution

$$\varphi = \Theta(d,\alpha)u^* + \varphi_0, \qquad \psi = \big(1-\Theta(d,\alpha)\big)u^* - \varphi_0, \tag{2.44}$$

where

$$\Theta(d,\alpha) = 4\alpha^2c^2/(4\alpha^2c^2 - d^2), \qquad 4\alpha^2c^2 \neq d^2.$$

Knowing some solution u^* of the equation (2.43) being solved under the conditions of Lemma 2.4 and the value of potential on the boundary $\varphi|_{\partial\Omega} = \mathbb{A}(\xi,\eta)$, we find φ_0 by means of the solution of the linear problem

$$\begin{gathered} \triangle\varphi_0 = 0, \\ \varphi_0|_{\Omega} = \mathbb{A}(\xi,\eta) - \Theta u^*|_{\partial\Omega}. \end{gathered} \tag{2.45}$$

Hence, in the first case we transformed the problem finding the "solving" equation (2.43) and linear Dirichlet problem (2.45). We have the following result.

Theorem 2.2. *Let $k_i = l_i$, $i = 1,\dots,N$. Then the VM system (2.1)–(2.5) with boundary condition (2.42) has a solution*

$$f_i = f_i\left(-\alpha_i|V|^2 + Vd_i + \gamma_i + l_iu^*(\xi,\eta)\right),$$

$$E = \frac{m}{2\alpha q}\left(\Theta(d,\alpha)\nabla u^*(\xi,\eta) + \nabla\varphi_0\right), \tag{2.46}$$

$$B = \frac{d}{d^2}\left\{\beta + \int_0^1 \big(d\times J(\tau\hat{r}),\hat{r}\big)\mathrm{d}\tau\right\} - \left[d\times\big(\nabla(1-\Theta(d,\alpha))u^*(\xi,\eta) - \varphi_0\big)\right]\frac{mc}{qd^2}.$$

$u^*(\xi,\eta)$ – function satisfying "solving" equation (2.43); γ_i, β_i – const*; $\hat{r} \triangleq (\xi,\eta)$ and $\varphi_0(\xi,\eta)$ is a harmonic function defined from the linear problem (2.45).*

Case 2. $l_2 = \ldots = l_N \triangleq l$, $k_2 = \ldots = k_N \triangleq k$, $l \neq k$. We notice that for $N = 2$ cases 1 and 2 exhaust all possible connections between parameters l_i and k_i. We construct a solution φ,ψ of (2.40), (2.41) satisfying condition

$$\varphi + \psi = l\varphi + k\psi. \tag{2.47}$$

Let $f_i \triangleq f_i(-\alpha_i|V|^2 + Vd_i + \varphi_i + \psi_i)$ be functions such that the following condition holds.

(A). There are constants γ_i, $i = 1,\dots,N$ such that

$$\Theta q A_1(\gamma_1 + u) + \tau\sum_{i=2}^{N} q_iA_i(\gamma_i + u) = 0$$

for

$$\Theta = 4\alpha^2c^2(1-l) + d^2(k-1), \qquad \tau = 4\alpha^2c^2(1-l) + d^2(k-1)\frac{k}{l}.$$

We remark that the corresponding distribution function satisfies the condition of Lemma 2.3.

Lemma 2.5. *Let $l_2 = l_3 = \ldots = l_N \stackrel{\triangle}{=} l$, $k_2 = k_3 = \ldots = k_N \stackrel{\triangle}{=} k$, $l \neq k$. We assume that condition* **(A)** *holds. Then (2.40), (2.41) has a solution*

$$\varphi = \frac{k-1}{k-l} u^*, \qquad \psi = \frac{1-l}{k-l} u^*,$$

where u^* satisfies equation

$$\triangle u = \epsilon \frac{h}{a(\alpha, l) + \epsilon b(d, k, l)} A_1(\gamma_1 + u), \tag{2.48}$$

$$\epsilon = \frac{1}{c^2}, \quad h = \frac{d^2(k-l)^2 8\pi\alpha q^2}{mw(d)}, \quad a = 4\alpha^2(1-l)l, \quad b = d^2(k-1)k.$$

Proof. By changing $\varphi = lu$, $\psi = ku$ system is reduced to (2.48) due to **(A)**. Since

$$\varphi = \frac{k-1}{k-l} u, \quad \psi = \frac{1-l}{k-l} u,$$

then Lemma is proved. ☑

From Lemma 2.5 we obtain

Theorem 2.3. *Let $\alpha_2 q_2/m_2 = \ldots = \alpha_N q_N/m_N$, $k_2 = \ldots = k_N \stackrel{\triangle}{=} k$ such that $k \notin \left\{\frac{\alpha_N q_N}{m_N}, 1\right\}$ and condition* **(A)** *holds. Then the VM system (2.1)–(2.5) with boundary condition (2.42) on the scalar potential φ has a solution*

$$f_i = f_i\left(-\alpha_i |V|^2 + V d_i + \gamma_i + u^*\right),$$

$$E = \frac{m(k-1)}{2\alpha q(k-l)} \nabla u^*,$$

$$B = \frac{d}{d^2}\left\{\beta + \int_0^1 \big(d \times J(\tau\hat{r}), \hat{r}\big) \mathrm{d}\tau\right\} - [d \times \nabla u^*] \frac{cm(1-l)}{qd^2(k-l)}.$$

Here u^* satisfies (2.48) with condition

$$u^*|_{\partial\Omega} = \frac{k-1}{k-l} \frac{m}{2\alpha q} \mathbb{A}(\xi, \eta), \tag{2.49}$$

β, γ_i – constants, $\hat{r} \stackrel{\triangle}{=} (\xi, \eta)$.

The problem (2.48), (2.49) at $\epsilon \to 0$ has a solution $u^* = u_0 + O(\epsilon)$ where u_0 is a harmonic function satisfying condition (2.49). Existence of other solutions of (2.48), (2.49) can be shown using a parameter continuation method, results of branching theory (see [57]).

3. Existence of solutions of boundary value problem (2.40)–(2.42)

We present the form of the distribution function. Let

$$f_i = \exp\left(-\alpha_i|V|^2 + Vd_i + \gamma_i + l_i\varphi + k_i\psi\right). \tag{3.1}$$

Distributions of the form (3.1) have meaning in applications. Substituting (3.1) into (2.40) and (2.41) and taking into account (2.13)–(2.15) and (2.39), we get the system

$$\begin{aligned}\triangle\varphi &= \mu\sum_{k=1}^{N} q_i\left(\frac{\pi}{a_i}\right)^{3/2}\exp\left(\gamma_i + \frac{d_i^2}{4\alpha_i}\right)\exp(l_i\varphi + k_i\psi),\\ \triangle\psi &= \frac{d^2\nu}{2\alpha}\sum_{i=1}^{N} q_i\left(\frac{\pi}{\alpha_i}\right)^{3/2}\exp\left(\gamma_i + \frac{d_i^2}{4\alpha_i}\right)\exp(l_i\varphi + k_i\psi)\frac{k_i}{l_i}.\end{aligned} \tag{3.2}$$

Introducing the normalization condition

$$\int_\Omega\int_{\mathbb{R}^3} f_i \mathrm{d}V\mathrm{d}x = 1, \tag{3.3}$$

$$i = 1,\dots,N;\quad \Omega\subseteq\mathbb{R}^2;\quad x \stackrel{\triangle}{=} (\xi,\eta),$$

we transform (3.2) into

$$\begin{aligned}\triangle\varphi &= \mu\sum_{i=1}^{N} q_i\exp(l_i\varphi + k_i\psi)\left(\int_\Omega \exp(l_i\varphi + k_i\psi)\mathrm{d}x\right)^{-1},\\ \triangle\psi &= \frac{d^2\nu}{2\alpha}\sum_{i=1}^{N} q_i\frac{k_i}{l_i}\exp(l_i\varphi + k_i\psi)\left(\int_\Omega \exp(l_i\varphi + k_i\psi)\mathrm{d}x\right)^{-1}.\end{aligned} \tag{3.4}$$

Consider the general case, when it is not possible to transform (3.4) into one equation. Without loss of generality we can consider that $l_2 \neq k_2$; $q \stackrel{\triangle}{=} q_1$. Let $q_1 < 0$, $q_i > 0$, $i = 2,\dots,N$. Introduce new variables

$$u_1 = \varphi + \psi,\quad u_i = -(l_i\varphi + k_i\psi),\quad i = 2,\dots,N. \tag{3.5}$$

Using (3.5) taking into account of boundary conditions (2.36)–(2.37), we obtain the system

$$-\triangle u_i = \sum_{j=1}^{N} C_{ij}A_j,\quad i = 1,\dots,N, \tag{3.6}$$

where

$$A_1 = e^{u_1}\left(\int_\Omega e^{u_1}dx\right)^{-1},\quad A_j = e^{-u_j}\left(\int_\Omega e^{-u_j}\mathrm{d}x\right)^{-1},\quad j = 2,\dots,N,$$

$$C_{ij} = \frac{8\pi}{w(d_1)} \cdot \frac{a_i}{m_i}|q_i|q_j\left(1 - \frac{1}{2d_1^2c^2}Z_iZ_j\right), \quad Z_i = \frac{(d_1, d_i)}{\alpha_i},$$

$$u_i = u_{0i}, \quad x \in \partial\Omega, \quad i = 1, \dots, N. \tag{3.7}$$

It is easy to check that (3.4) and (3.6) are equivalent in the sense that solutions of (3.6) completely define solutions of (3.4). In fact φ, ψ are defined via u_1, u_2, because l_2, k_2 and u_i are linearly dependent for $i = 3, \dots, N$. Here we assume that $u_{0i} \in C^{2+\alpha}$, $\partial\Omega \in C^{2+\alpha}$, $\alpha \in (0, 1)$.

We give auxiliary results.

Lemma 3.1. *Let*

$$\sum_{j=1}^{N} C_{ij} > 0, \qquad \left(\sum_{j=1}^{N} C_{ij} < 0\right).$$

Then

$$F_i(u) = \sum_{j=1}^{N} C_{ij}A_j(u) \geq 0, \qquad u_i \geq \min_{\partial\Omega} u_{0i},$$

$$\left(F_i(u) = \sum_{j=1}^{N} C_{ij}A_j(u) \leq 0, \qquad u_i \leq \max_{\partial\Omega} u_{0i}\right).$$

Proof. It is easy to see that $\int_\Omega F_i(u)\mathrm{d}x = \sum_{j=1}^{N} C_{ij} > 0$. Moreover the set $\Omega^+ = \{x \in \Omega : F_i(u(x)) > 0\}$ is nonempty. We denote with $\Omega^- = \{x \in \Omega : F_i\big(u(x)\big) < 0\}$, and we show that $\Omega^- = \emptyset$. Hence, on one hand, $F_i\big(u(x)\big) = 0$ where $x \in \partial\Omega$, and on the other hand,

$$-\triangle u_i(x) = F_i\big(u(x)\big) < 0, \qquad x \in \Omega.$$

Thus, u_i is bounded in Ω and it attains its maximum on $\partial\Omega = \bar{\Omega}\backslash\Omega$, i.e. $\max_{x\in\bar{\Omega}} u(x) = u(x_0)$, $x_0 \in \partial\Omega$. However, since the function $F_i(u)$ decreases for fixed $\left(\int_\Omega e^{-u_j}\mathrm{d}x\right)^{-1}$, then we obtain $F_i\big(u(x)\big) > F_i\big(u(x_0)\big) = 0$, so $x \in \bar{\Omega}$ contradicts definition of the set Ω^-. Analogously case $\sum_{j=1}^{N} C_{ij} < 0$ is considered as well (see [41]). The lemma is now proved. ☑

Lemma 3.2 (Gogny, Lions [29]). *Let*

$$\max_{\Omega}(u - v)(x) = (u - v)(x_0) > 0.$$

Then

$$e^{u(x_0)}\left(\int_\Omega e^{u(x)}\mathrm{d}x\right)^{-1} > e^{v(x_0)}\left(\int_\Omega e^{v(x)}\mathrm{d}x\right)^{-1},$$

$$e^{-u(x_0)}\left(\int_\Omega e^{-u(x)}\mathrm{d}x\right)^{-1} < e^{-v(x_0)}\left(\int_\Omega e^{-v(x)}\mathrm{d}x\right)^{-1}.$$

We define the vector-function $v(x), w(x) \in C^2(\Omega)^N \cap C^1(\bar{\Omega})^N$ as *a lower* and *upper solutions* of (3.6), (3.7) in the following sense

$$-\triangle v_i \leq \sum_{j=2}^{N} C_{ij} \frac{e^{-w_j}}{\int_\Omega e^{-v_j} \mathrm{d}x} + C_{i1} \frac{e^{w_1}}{\int_\Omega e^{v_1} \mathrm{d}x} \leq F_i(v), \qquad x \in \Omega,$$

$$(3.8)$$

$$-\triangle w_i \geq \sum_{j=2}^{N} C_{ij} \frac{e^{-v_j}}{\int_\Omega e^{-w_j} \mathrm{d}x} + C_{i1} \frac{e^{v_1}}{\int_\Omega e^{w_1} \mathrm{d}x} \geq F_i(w), \qquad x \in \Omega,$$

$$v_i \leq u_{0i}, \qquad w_i \geq u_{0i}, \qquad x \in \partial\Omega \tag{3.9}$$

with $v = (v_1, \dots, v_N)'$, $w = (w_1, \dots, w_N)'$.

It is easy to show that $A_j(u)$ is invariant under a translation on the constant vector, therefore (3.9) can be changed to

$$v_i \leq 0, \qquad w_i \geq 0, \qquad x \in \partial\Omega. \tag{3.10}$$

Theorem 3.1. *Let there exists a lower $v_i(x) \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and an upper $w_i(x) \in C^2(\Omega) \cap C^1(\bar{\Omega})$ solution satisfying inequalities (3.8), (3.10), such that $v_i(x) \leq w_i(x)$ in $\bar{\Omega}$. Let $u_{0i} \in C^{2+\alpha}(\partial\Omega)$. Then (3.6), (3.7) has a unique classical solution $u_i(x) \in C^{2+\alpha}(\bar{\Omega})$ and moreover $v_i(x) \leq u_i(x) \leq w_i(x)$ in $\bar{\Omega}$, $i = 1, \dots, N$.*

Proof. Let $z_i(x) \in C(\bar{\Omega})$ be given functions, $v_i \leq z_i \leq w_i$. We define the operator $T : C(\bar{\Omega})^N \to C(\bar{\Omega})^N$ by formulae $u = Tz$, $z(x) \in \in C(\bar{\Omega})^n$, where $u = (u_1, \dots, u_N)'$ is a unique solution of the problem

$$-\triangle u_i = \sum_{j=1}^{N} C_{ij} A_j(p(z)) + q(z_i) \stackrel{\triangle}{=} \hat{F}_i(z), \quad u_i = u_{0i}, \quad x \in \partial\Omega, \tag{3.11}$$

where $p(z) = \max\{v, \min\{z, w\}\}$,

$$q(z_i) = \begin{cases} \frac{w_i - z_i}{1 + z_i^2}, & z_i \geq w_i, \\ 0, & v_i \leq z_i \leq w_i, \\ \frac{v_i - z_i}{1 + z_i^2}, & v_i \leq z_i. \end{cases}$$

It is evident that the function $\hat{F}(z)$ is continuous and bounded. Then due to smoothness of $\partial\Omega$ and boundary conditions, (3.11) is only solvable in $C^{1+\alpha}(\bar{\Omega})^N$, i.e. $u(x) \in C^{1+\alpha}(\bar{\Omega})^N$. Here we used Theorem 8.34 from [24]. Due to compactness of embedding $C^{1+\alpha}(\bar{\Omega}) \subset C(\bar{\Omega})$ and continuity of $\hat{F}(z)$, it follows that the operator T is a completely continuous (compact) operator. Then by Schauder theorem (see [66]), the operator T has a fixed point $u = Tu$ with $u \in C(\bar{\Omega})^N$. On the other hand, since $u \in C^{1+\alpha}(\bar{\Omega})^N$, then $\hat{F}(u) \in C^{\alpha}(\bar{\Omega})^N$ and from classical theory it follows that $u \in C^{2+\alpha}(\bar{\Omega})^N$.

Now we will show that $v_i \le u_i \le w_i$. We suppose that there exists a number $k \in \{1, \dots, N\}$ and a point $x_0 \in \bar{\Omega}$ such that

$$(v_k - u_k)(x_0) = \max_{\bar{\Omega}}(v_k - u_k) = \epsilon > 0.$$

Evidently, x_0, due to (3.9), can not belong to the boundary $\partial\Omega$. Then, due to the maximum principle, we have a contradiction

$$0 \le -\triangle(v_k - u_k)(x_0) \le C_{k1}\frac{e^{w_1(x_0)}}{\int_\Omega e^{v_1}\mathrm{d}x} + \sum_{j=2}^{N} C_{kj}\frac{e^{-w_j(x_0)}}{\int_\Omega e^{-v_j}\mathrm{d}x}$$

$$- C_{k1}\frac{e^{p(u_1)(x_0)}}{\int_\Omega e^{p(u_1)}\mathrm{d}x} - \sum_{j=2}^{N} C_{kj}\frac{e^{-p(u_j)(x_0)}}{\int_\Omega e^{-p(u_j)}\mathrm{d}x} + \frac{(u_k - v_k)(x_0)}{1 + u_k^2(x_0)} < 0.$$

Thus, $v_i \le u_i$. Analogously, the proof of the inequality $u_i \le w_i$ follows.

We assume that there exists a number $l \in \{1, 2, \dots, N\}$ and a point $y_0 \in \bar{\Omega}$ such that there are two solutions u^1, u^2 of (3.6), (3.7), $u_i^1 \equiv u_i^2$, $i \ne l$, $u_l^1(y_0) > u_l^2(y_0)$. Using Lemma 3.2 a contradiction arises again to contradiction: $0 \le -\triangle\left(u_l^1 - u_l^2\right)(y_0) < 0$, which proves uniqueness. The Theorem is now proved. ☑

We construct an upper and lower solutions of (3.6), (3.7). Let $\sum_{j=1}^{N} C_{ij} > 0$, $i = 1, \dots, N$. Then from Lemma 3.1 it follows $u_i \ge 0$. First, we construct an upper solution of the form: $v_i \equiv 0$,

$$-\triangle w_i = \sum_{j=2}^{N} \frac{C_{ij}}{\int_\Omega e^{-w_j}\mathrm{d}x} - \frac{|C_{i1}|}{\int_\Omega e^{w_1}\mathrm{d}x}, \tag{3.12}$$

$$w_i|_{\partial\Omega} = \max_{i,\partial\Omega} u_{0i} \equiv w_0 \tag{3.13}$$

with $x = (\xi, \eta) \in \Omega \subset \mathbb{R}^2$. From (3.8) it follows that w_i must satisfy the inequalities

$$\sum_{j=2}^{N} C_{ij}e^{-w_j} - |C_{i1}|e^{w_1} \ge 0, \qquad i = 1, \dots, N. \tag{3.14}$$

Consider the auxiliary problem

$$-\triangle g = 1, \qquad g|_{\partial\Omega} = w_0.$$

We assume that Ω(domain) is contained in a strip $0 < x_1 < r$ and we introduce the function $q(x) = w_0 + e^r - e^{x_1}$. It is easy to show that $\triangle(q - g) = = -e^{x_1} + 1 < 0$ on Ω, $q - g = e^r - e^{x_1} \ge 0$ on $\partial\Omega$. Therefore, according to maximum principle (see [24]) $q - g \ge 0$, if $x \in \bar{\Omega}$ and

$$w_0 \le g(x) \le w_0 + e^r - 1 \overset{\triangle}{=} M. \tag{3.15}$$

We denote with $z_i = \text{const} \geq 0$ in (3.12). Then from (3.12) and (3.15) we obtain $w_i \leq Mz_i$, $w_i = z_i g(x)$ and (3.12), (3.13) is equivalent to the following finite-dimensional algebraic system

$$z_i = \sum_{j=2}^{N} \frac{C_{ij}}{\int_\Omega e^{-z_j g} \mathrm{d}x} - \frac{|C_{i1}|}{\int_\Omega e^{z_1 g} \mathrm{d}x} \triangleq L_i(z).$$

Let us introduce the norm $|z| = \max_{1 \leq i \leq N} |z_i|$. Then, due to (3.15) we obtain the following chain of inequalities

$$\begin{aligned} |L(z)| &\leq \max_{1 \leq i \leq n} \left| \sum_{j=2}^{N} \frac{C_{ij}}{\int_\Omega e^{-z_j g} \mathrm{d}x} - \frac{|C_{i1}|}{\int_\Omega e^{z_1 g} \mathrm{d}x} \right| \leq \\ &\leq \frac{1}{|\Omega|} \max_{1 \leq i \leq N} \left\{ \sum_{j=2}^{N} C_{ij} e^{Mz_j} - |C_{i1}| e^{-Mz_1} \right\} \leq \\ &\leq \frac{1}{|\Omega|} \max_{1 \leq i \leq N} \left\{ \sum_{j=2}^{N} C_{ij} e^{M|z|} - |C_{i1}| e^{-M|z|} \right\}, \end{aligned} \tag{3.16}$$

where $|\Omega| = \operatorname{mes} \Omega$, $\Omega \subset \mathbb{R}^2$.

Lemma 3.3. *Let $\sum_{j=1}^{N} C_{ij} > 0$. We introduce notations*

$$\sum_{j=2}^{N} C_{ij} \triangleq a_i, \quad |C_{i1}| = b_i, \quad \min_{1 \leq i \leq N} \frac{a_i}{b_i} = \alpha^2 > 1.$$

Let the inequalities

$$\alpha a_i - \frac{1}{\alpha} b_i \leq \frac{|\Omega|}{M} \ln \alpha, \quad i = 1, \dots, N \tag{3.17}$$

hold. Then the equation $Lz = z$ has a solution with $z_i \leq \frac{1}{M} \ln \alpha$ and functions $v_i \equiv 0$, $w_i = z_i g(x)$ are a lower and upper solutions of the problems (3.6), (3.7).

Proof. Let $|z| = R$. From (3.14) it follows

$$a_i e^{-MR} - b_i e^{MR} \geq 0$$

with $R \leq \frac{1}{M} \ln \alpha$. Substituting a maximum value $R = \frac{1}{M} \ln \alpha$ into (3.16), it is easy to check that (3.17) gives an estimation $|L(z)| \leq |z|$ and the existence of the fixed point $Lz = z$ follows from Brayer theorem (see [66]). ☑

Let now $\sum_{j=1}^{N} C_{ij} \leq 0$, $i = 1, \dots, N$. Analogously, the latter suggest the following result.

Lemma 3.4. *Let $\sum_{j=1}^{N} C_{ij} < 0$, $\beta^2 = \min_{1 \leq i \leq N} (b_i / a_i) > 1$ and suppose that inequalities*

$$\frac{b_i}{\beta} - \beta a_i \leq \frac{|\Omega|}{M} \ln \beta, \qquad i = 1, \dots, N \tag{3.18}$$

hold. Then, the functions $v_i = -z_i g(x)$, $w_i \equiv 0$ *are a lower and an upper solutions of (3.6), (3.7) with* $z_i = -L_i(-z)$.

It follows from Theorem 3.1 and from smoothness of the function $F_i(u)$ under the fixed functional coefficients $\left(\int_\Omega e^{-u_j} \mathrm{d}x\right)^{-1}$, that there exists a constant $M(v,w) > 0$ such that $\frac{\partial}{\partial u_j} F_i \geq -M$ with $i,j = 1,\ldots,N$. Moreover the mapping $G : C(\bar{\Omega})^N \to C(\bar{\Omega})^N$ defined by formulae $G_i u = F_i + M u_i$ will be monotonic increasing in u_i because of the monotonicity of the coefficients. Let $T_1 : z = T_1 z$,be an operator with

$$-\triangle z_i + M z_i = G_i u > 0, \qquad z_i|_{\partial\Omega} = u_{0i}. \tag{3.19}$$

Due to the maximum principle $z_i > 0$ $(u_{0i} > 0)$, thus operator T_1 is positive and monotonic. Moreover, T_1 is completely continuous and it can be proved in the same way as we did for operator T. It is evident, that $v \leq T_1 v$ and $T_1 w \leq w$. We notice that a cone of nonnegative functions is normal in $C(\bar{\Omega})$. Therefore due to uniqueness (Theorem 3.1), we can apply the classical theory of monotonic operators (see. [40]) for problem (3.19) and we obtain the following result.

Theorem 3.2. *Operator* T_1 *has a unique fixed point* $u = T_1 u$, $v_i \leq u_i \leq w_i$, *where for any* $y_0 : v_i \leq y_{0i} \leq w_i$, *successive approximations* $y_{n+1} = T_1 y_n$ *are uniformly convergent to* u.

Corollary 3.1. *We define successive approximations in the following way*

$$u_i^0 = 0,$$
$$-\triangle u_i^{n+1} + M u_i^{n+1} = F_i(u^n) + M u_i^n,$$
$$u_i^{n+1}|_{\partial\Omega = u_{0i}}, \quad i = 1,2, \quad n = 0,1,\ldots;$$
$$u_k^n = \frac{q_k \alpha_k}{m_k(z_2 - z_1)} \left[\frac{m_1}{|q_1|\alpha_1}(z_2 - z_k) u_1^n + \frac{m_2}{q_2\alpha_2}(z_k - z_1) u_2^n \right], \quad k = 3,\ldots,n.$$

Then $\{u_i^n\}$, $i = 1,\ldots,n$ *are monotonic and uniformly convergent to the solution of (3.6), (3.7).*

Remark 3.1. In the case $n = 1$, boundary value problem (2.55), (2.56) was considered in Gogny, Lions [29], Krzywicki, Nadzieja [41].

4. Existence of solution of nonlocal boundary value problem (2.40), (2.41), (2.36), (2.37)

Consider plasma on domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\partial\Omega \in C^1$ consisting of N kinds of charged particles. It is assumed that particles interact among themselves only by means of their own charges $q_1,\ldots,q_N \in \mathbb{R}\backslash\{0\}$. Every particle of i-th kind is described by the distribution function $f_i = f_i(x,v,t) \geq 0$, where $t \geq 0$ – time, $x \in \Omega$ – position and $v \in \mathbb{R}^3$ – velocity. Plasma motion is described by the classical VM system (2.1)–(2.5) with boundary conditions (2.34). We impose the reflection condition (1.17) for distribution functions.

In this section we studied stationary solutions $(f_1,\dots,f_N,E,B)$ of the VM system of special form

$$f_i = \hat{f}_i\left(-\alpha_i v^2 + c_{1i} + l_i\varphi(x), vd_i + c_{2i} + k_i\psi(x)\right), \tag{4.1}$$

$$E(x) = \frac{m}{2\alpha q}\nabla\varphi, \tag{4.2}$$

$$B(x) = -\frac{cm}{qd^2}(d\times\nabla\psi), \tag{4.3}$$

where functions $\hat{f}_i : \mathbb{R}^2 \to [0,\infty[$ and parameters $d \in \mathbb{R}^3\backslash\{0\}$, $\alpha_i > 0$, c_{1i}, c_{2i}, l_i, k_i (see formulae of connection (2.13)–(2.15)) are given, and functions φ,ψ have to be defined. In Section 3 by the lower-upper solutions method, the existence theorem of classical solutions of boundary value problem (2.40)–(2.42) is proved for the distribution function $\hat{f}_i = \exp(\varphi+\psi)$. In the proof of the existence theorem 3.1, we essentially applied the monotonic property of the right parts of (3.6). In the general case of distribution function (4.1) system (2.40), (2.41) does not have good monotonic properties and therefore we can not apply techniques of lower and upper solutions for nonlinear elliptic systems in a cone developed by Amann [6]. Therefore we show existence of solutions of the boundary value problem (2.40), (2.41), (2.36), (2.37) by the method of lower-upper solutions without monotonic conditions. We notice that the approach (4.1)–(4.3) is connected with papers of P. Degond [20] and J. Batt, K. Fabian [15]. In these papers they introduce integrals $\mathbb{E}$, $F(x,v)$ and $P(x,v)$ of the Vlasov equation and solutions of the VM system for the distribution function ($i = 1$ – particles of single kind) of the form $\hat{f}(\mathbb{E})$, $\hat{f}(\mathbb{E},F)$ or $\hat{f}(\mathbb{E},F,P)$ are considered. The case of the distribution function of $\hat{f}(\mathbb{E},P)$ and particles of various kinds (species $i = 1,\dots,N$) in these papers are not considered.

Thus we consider the boundary value problem (2.40), (2.41), (2.36), (2.37). Let $q < 0$ (electrons), $q_i > 0$ (positive ions), $i = = 2,\dots,N$. Then (2.40), (2.41) takes the form

$$\triangle\varphi = \frac{8\pi\alpha q}{m\omega(d)}\left(qA_1 - \sum_{i=2}^{N}|q_i|A_i\right) = h^1, \tag{4.4}$$

$$-\triangle\psi = \frac{4\pi q}{mc^2 w(d)}\frac{d^2}{2\alpha}\left(qA_1 - \sum_{i=2}^{N}\frac{k_i}{l_i}|q_i|A_i\right) = h^2, \tag{4.5}$$

where $A_i = \int_\Omega f_i \mathrm{d}v$, $i = 1,\dots,N$, and f_i is ansatz (4.1).

Remark 4.1. In case $k_i = l_i$ system (4.4), (4.5) is transformed to one equation and we may use Theorem 3.1.

Theorem (McKenna-Walter [43]). *Let $\Omega \subset \mathbb{R}^N$ – bounded domain with boundary $\partial\Omega \in C^{2,\mu}$ for some $\mu \in]0,1[$. Let $h : \bar{\Omega}\times\mathbb{R}^N \to \mathbb{R}^N$ satisfy the following smoothness conditions: $\forall r > 0$ there exists $C_r > 0$ such that $\forall x, x_1, x_2 \in \bar{\Omega}$ and $\forall y, y_1, y_2 \le r$:*

I. *Inequalities*

$$|h(x_1, y) - h(x_2, y)| \le C_r |x_1 - x_2|^\mu,$$

$$|h(x, y_1) - h(x, y_2)| \le C_r |y_1 - y_2| \text{ hold };$$

II. *There exists an ordered pair (v, w) of lower v and upper w solutions, i.e. $v, w \in C^2(\Omega)^N \bigcap C^1(\bar{\Omega})^N$, $v \le w$ in $\bar{\Omega}$, $v \le 0 \le w$ on $\partial\Omega$,*

$$\forall x \in \Omega : \ \forall z \in \mathbb{R}^N, \ v(x) \le z \le w(x), \ z_k = v_k(x) : \triangle v_k(x) \ge h_k(x, z)$$

and

$$\forall x \in \Omega : \ \forall z \in \mathbb{R}^N, \ v(x) \le z \le w(x) : \ z_k = w_k : \triangle w_k(x) \le h_k(x, z)$$

for all $k \in \{1, \dots, N\}$ (Here the vector inequality $v(x) \le z \le w(x)$ means a component wise comparison).

Then there is a solution $u \in C^{2,\mu}(\bar{\Omega})^N$ of the problem

$$\triangle u = h\big(\cdot, u(\cdot)\big) \ \text{ in } \ \Omega$$

$$u = 0 \qquad\qquad \partial\Omega$$

such that $v \le u \le w$ in $\bar{\Omega}$.

Because the right parts in (4.4), (4.5) are nonlocal, then we give sufficient conditions on functions $\hat{f}_i$ in order to make possible applying McKenna-Walter theorem.

Lemma 4.1. *Let $\alpha > 0$ and $\hat{f} : \mathbb{R}^2 \to [0, \infty[$ satisfy the following conditions:*

1. $\hat{f} \in C^1\big(\mathbb{R}^2\big)$;

2. *$\hat{f}$ and $\hat{f}'$ are bounded and there exists $R_0 \in \mathbb{R}$ such that* $\operatorname{supp}\left(\hat{f}\right) \subset [R_0, \infty[\times\mathbb{R}$.

Then the function $h_{\alpha,\hat{f}} : \mathbb{R}^2 \to \mathbb{R}^2$, given via

$$h_{\alpha,\hat{f}}(u) = \frac{4\pi q}{m w(d)} \int_{\mathbb{R}^3} \begin{pmatrix} 2\alpha q \\ -\frac{1}{c^2}\frac{k_i}{l_i} \end{pmatrix} \hat{f}\left(-\alpha v^2 + u_1, vd + u_2\right) dv$$

is continuously differentiable and there are R, C_1, C_2 such that

$$\begin{pmatrix} 0 \\ -C_2(u_1 + R)_+^2 \end{pmatrix} \le h_{\alpha,\hat{f}}(u) \le \begin{pmatrix} C_1(u_1 + R)_+^{3/2} \\ C_2(u_1 + R)_+^2 \end{pmatrix}$$

for any function $u \in \mathbb{R}^2$.

Proof. Transforming into spherical coordinates

$$v_1 = \rho \sin\Theta \cos\varphi, \quad v_2 = \rho \sin\Theta \sin\varphi, \quad v_3 = \rho\cos\Theta,$$

we obtain

$$
\begin{aligned}
h^1_{\alpha,\hat f}(u) &= \frac{8\pi\alpha q^2}{mc^2 w(d)} \int_{\mathbb{R}^3} \hat f\left(-\alpha v^2 + u_1, vd + u_2\right) dv \\
&= P(q,\alpha,d,m)\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} \hat f\left(-\alpha\varphi^2 + u_1, \varphi k(\rho,\Theta) + u_2\right)\sin(\Theta)\varphi^2 d\rho d\Theta d\varphi \\
&= \frac{P(q,\alpha,d,m)}{\alpha^2}\int_{-\infty}^{u_1}\int_0^{\pi}\int_0^{2\pi} \hat f\left(s, \alpha^{-1}k(\rho,\Theta)\sqrt{(u_1-s)} + u_2\right) \\
&\qquad\qquad\qquad\qquad \times \sin(\Theta)(u_1-s)\sqrt{(u_1-s)} d\rho d\Theta ds \\
&= \frac{P(q,\alpha,d,m)}{\alpha^2}\int_{-\infty}^{u_1} K_1(s, u_1-s, u_2)(u_1-s)\sqrt{(u_1-s)} ds,
\end{aligned}
$$

where

$$k(\rho,\Theta) = d_1\cos(\rho)\sin(\Theta) + d_2\sin(\rho)\sin(\Theta) + d_3\cos(\Theta)$$

and

$$K_1(s,t,\varphi) = \int_0^{\pi}\int_0^{2\pi} \hat f(s, \alpha^{-1}k(\rho,\Theta)\sqrt{t} + \varphi)\sin(\Theta) d\rho d\Theta.$$

Similar expressions are satisfied for $h^2_{\alpha,\hat f}$ and $K_2(s,t,\varphi)$. Due to condition (**2**) kernels K_1, K_2 are bounded, and applying Lebesgue dominated convergence theorem, it is easy to prove that $h_{\alpha,\hat f} \in C^1(\mathbb{R}^2)^2$. ☑

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^2$ – two-dimensional domain with boundary $\partial\Omega \in \in C^{2,\mu}$, $\mu \in]0,1[$. Let $\hat f_1, \dots, \hat f_N : \mathbb{R}^2 \to [0,\infty[$ satisfy conditions* (**1**), (**2**) *of Lemma 4.1. Then the problem (2.40), (2.41), (2.36), (2.37) has a smooth solution $\varphi \in C^2(\bar\Omega)$, $\psi \in C^2(\bar\Omega)$. Moreover, the distribution function $f_N \in C^1(\bar\Omega \times \mathbb{R}^3)$ generates the classical stationary solution $(f_1, \dots, f_N, E, B)$ of the VM system of the form (4.1)–(4.3) in Ω.*

Proof. Consider the system (4.4), (4.5). The right parts in it may change sign depending on relations

A1. $qA_1 - \sum_{i=2}^N |q_i|A_i = G(q,A) > 0 \Rightarrow qA_1 > \sum_{i=2}^N |q_i|A_i > \sum_{i=2}^N T_-|q_i|A_i$.

A2. $qA_1 - \sum_{i=2}^N |q_i|A_i = G_1(q,A) < 0 \Rightarrow qA_1 < \sum_{i=2}^N |q_i|A_i < \sum_{i=2}^N T^+|q_i|A_i$.

Here

$$T_- = \min\left\{\frac{k_i}{l_i}\right\} = \min\left\{\frac{(d_i,d)\alpha}{d_i^2\alpha_i}\right\},$$

$$T^+ = \max\left\{\frac{k_i}{l_i}\right\} = \max\left\{\frac{(d_i,d)\alpha}{d_i^2\alpha_i}\right\}.$$

It follows from Lemma 4.1 and conditions (**A1**), (**A2**) that right parts h^1, h^2 of (4.4), (4.5) satisfy smoothness conditions of McKenna-Walter theorem and that there is an $R > 0$ and matrix $(2 \times N)$ with positive components such that

$$\begin{pmatrix} -\sum_{G_1<0} c_{1i}|G_1|(l_i u_1 + R)_+^2 \\ \\ -\sum_{G>0} c_{2i}|G|(l_i u_1 + R)_+^2 \end{pmatrix} \le h(u) \le \begin{pmatrix} \sum_{G>0} c_{1i}|G|(l_i u_1 + R)_+^{3/2} \\ \\ \sum_{G_1<0} c_{2i}|G_1|(l_i u_1 + R)_+^2 \end{pmatrix}$$

for all $u \in \mathbb{R}^2$. Now we continue with the construction of lower-upper solutions (v, w) of (4.4), (4.5), (3.6), (3.7). Let us introduce the following notations

$$l^+ = \min\{|l_i| \mid \ l_i > 0\}, \quad l^- = \min\{|l_i| \mid l_i < 0\} \ \text{ and } \ l = \min(l^+, l^-).$$

We define a lower and an upper solution in Ω

$$v = \begin{pmatrix} -\epsilon l^+ \\ \\ \triangle^{-1} \sum_{i=1}^N c_{2i}|G|\left(1 + \frac{|l_i|}{l}\right)^2 R^2 \end{pmatrix}$$

and

$$w = \begin{pmatrix} \epsilon l^- \\ \\ -\triangle^{-1} \sum_{i=1}^N c_{2i}|G|\left(1 + \frac{|l_i|}{l}\right)^2 R^2 \end{pmatrix}$$

and on the boundary

$$v_i \le u_{0i}, \qquad w_i \ge u_{0i}^1, \qquad\qquad x \in \partial\Omega$$

with $v = (v_1, v_2)', w = (w_1, w_2)'$. Assuming that the right parts $h^1(\cdot), h^2(\cdot)$ of (4.4), (4.5) are invariant under a constant vector translation, we can change the last conditions to the following ones

$$v_i \le 0, \qquad w_i \ge 0, \qquad\qquad x \in \partial\Omega.$$

Moreover operator $\triangle^{-1}$ is defined with respect to zero boundary conditions and $v \le 0 \le w$ in $\bar{\Omega}$.

Due to the previously given estimation for h_f and conditions $(A1), (A2)$, we obtain

$$\triangle v_1 = 0 \ge h_f^1(v_1, z_2), \qquad z_2 \in \mathbb{R},$$

$$\triangle w_1 = 0 \le h_f^1(w_1, z_2), \qquad z_2 \in \mathbb{R},$$

$$\triangle v_2 \ge \sum_{i=1}^N c_{2i}|G|(l_i z_1 + R)_+^2 \ge h_f^2(z_1, v_2), \qquad z_1 \in [v_1, w_1]$$

and

$$\triangle w_2 \le -\sum_{i=1}^N c_{2i}|G|(l_i z_1 + R)_+^2 \le h_f^2(z_1, w_2), \qquad z_1 \in [v_1, w_1].$$

Thus existence of solutions $U \in C^{2,\mu}(\bar{\Omega})$, $U = (\varphi, \psi)'$ of (2.54), (2.55) (respectively (2.40, (2.41)) (2.36), (2.37) follows from McKenna-Walter theorem and it proves Theorem 4.1. ☑

Remark 4.2. Existence of stationary solutions for the relativistic VM system has been proved in the dissertation of P.Braasch [16] using RSS [52] ansatz.

5. Nonstationary solutions of the Vlasov-Maxwell system

5.1. **Reduction of the VM system to nonlinear wave equation.** Let us consider the nonstationary VM system (1.2)–(1.6) for an N-component distribution function with the additional condition

$$\sum_{i=1}^{N} \frac{q_i^2}{m_i} \int_{\mathbb{R}^3} \left\{E + \frac{1}{c}[V \times B]\right\} \cdot \nabla_V f_i \mathrm{d}V = 0. \tag{5.1}$$

We shall look for distribution functions of the form

$$f_i = f_i\left(-\alpha_i |V|^2 + V d_i + F_i(r,t)\right), \qquad d_i \in \mathbb{R}^3, \qquad \alpha_i \in [0,\infty) \tag{5.2}$$

and the corresponding fields $E(r,t), B(r,t)$ satisfying equations (1.2)–(1.6), (5.1). If functions $F_i(r,t)$, vectors d_i and vector-functions E, B are connected among themselves by relations

$$\frac{\partial F_i}{\partial t} + \frac{q_i}{m_i}(E, d_i) = 0, \tag{5.3}$$

$$\nabla F_i - \frac{2\alpha_i q_i}{m_i} E + \frac{q_i}{m_i c}[B \times d_i] = 0, \qquad i = 1, \ldots, N, \tag{5.4}$$

then functions (5.2) satisfy (1.2) and we have the following equations

$$\frac{\partial F_i}{\partial t} + \frac{1}{2\alpha_i}(\nabla F_i, d_i) = 0, \tag{5.5}$$

$$\frac{\partial f_i}{\partial t} + \frac{1}{2\alpha_i}(\nabla f_i, d_i) = 0. \tag{5.6}$$

Introducing auxiliary vectors $K_i = \big(K_{ix}(r,t), K_{iy}(r,t), K_{iz}(r,t)\big)$, we transform (5.4) to the system

$$\nabla F_i - \frac{2\alpha_i q_i}{m_i} E = K_i, \tag{5.7}$$

$$\frac{q_i}{m_i c}[B \times d_i] = -K_i. \tag{5.8}$$

We notice that equation (5.8) is solvable with respect to vector B, iff

$$(K_i, d_i) = 0. \tag{5.9}$$

We define functions $F_i(r,t)$ and vectors $K_i(r,t)$ as

$$F_i = \lambda_i + l_i U(r,t), \tag{5.10}$$

$$K_i = k_i K(r,t), \tag{5.11}$$

where λ_i, k_i, l_i – constants, $l_1 = k_1 = 1$. Then from (5.7) and (5.8) follows that

$$E(r,t) = \frac{m_i}{2\alpha_i q_i}(l_i \nabla U - k_i K), \tag{5.12}$$

$$B(r,t) = \frac{\gamma}{d_i^2} d_i + \big[K \times d_i\big] \frac{k_i m_i c}{q_i d_i^2}, \tag{5.13}$$

where $\gamma_i(r,t) = (B, d_i)$ are still arbitrary functions. Let

$$l_i = k_i = \frac{m_1}{\alpha_1 q_1} \frac{\alpha_i q_i}{m_i}, \tag{5.14}$$

$$\alpha_i d_1 = \alpha_1 d_i, \qquad \alpha_i \gamma_1 = \alpha_1 \gamma_i, \qquad i = 1, \ldots, N. \tag{5.15}$$

Then

$$E(r,t) = \frac{m}{2\alpha q}(\nabla U - K), \tag{5.16}$$

$$B(r,t) = \frac{\gamma}{d^2} d + [K \times d] \frac{mc}{qd^2}, \tag{5.17}$$

where the following notations are introduced

$$m \triangleq m_1, \qquad \alpha \triangleq \alpha_1, \qquad d \triangleq d_1, \qquad \gamma \triangleq \gamma_1.$$

Moreover $K \perp d$. Due to (5.3), (5.9) the function $U(r,t)$ satisfies the linear equation

$$2\alpha \frac{\partial U}{\partial t} + (\nabla U, d) = 0. \tag{5.18}$$

Having defined U, K such that the Maxwell equations (1.2)–(1.5) are satisfied for the distribution function

$$f_i = f_i\left(-\alpha_i |V|^2 + V d_i + \lambda_i + l_i U(r,t)\right), \tag{5.19}$$

we can find unknown functions f_i, E, B using (5.16), (5.17) and (5.19).

Lemma 5.1. *Densities of charge ρ and current j defined by formulae*

$$\rho(r,t) = 4\pi \int_{R^3} \sum_{i=1}^{N} q_i f_i \mathrm{d}V, \qquad j(r,t) = 4\pi \int_{R^3} \sum_{i=1}^{N} q_i V f_i \mathrm{d}V,$$

are connected among themselves by the following relation

$$j = \frac{1}{2\alpha} d\rho + \mathrm{rot} Q(r) + \nabla \varphi^0(r), \qquad \triangle \varphi^0(r) = 0. \tag{5.20}$$

The equality (5.20) follows directly from the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \times j = 0 \tag{5.21}$$

and

$$\frac{\partial \rho}{\partial t} + \frac{1}{2\alpha}(d, \nabla \rho) = 0, \tag{5.22}$$

which is a corollary of (5.6).

Substituting (5.16), (5.17) into (1.3), (1.5) we obtain

$$\triangle U = \mathrm{div} K + \frac{8\pi\alpha q}{m} \sum_{i=1}^{N} q_i \int_{\mathbb{R}^3} f_i \mathrm{d}V, \tag{5.23}$$

$$(d, \nabla\gamma) + \frac{mc}{q}(d, \mathrm{rot} K) = 0. \tag{5.24}$$

Due to Lemma 5.1 and taking into account that $\mathrm{rot}Q(r) + \nabla\varphi^0 = 0$ (it can always be assured by calibrating)

$$\int_{\mathbb{R}^3} V f_i \mathrm{d}V = \frac{d}{2\alpha}\int_{\mathbb{R}^3} f_i \mathrm{d}V. \tag{5.25}$$

Thus after substitution (5.16), (5.17) into (1.2) we obtain the relation

$$\nabla\gamma \times d = \frac{md^2}{2\alpha cq}\frac{\partial}{\partial t}(\nabla U - K) + \frac{2\pi d^2}{\alpha c} d\sum_{i=1}^{N} q_i \int_{R^3} f_i \mathrm{d}V - \frac{mc}{q}\mathrm{rot}[K\times d]. \tag{5.26}$$

Having used Fredholm's alternative, we set the function $U(r,t)$, and from the condition that its solution $\nabla\gamma$ is a gradient of function $\gamma(r,t)$, we find $K(r,t)$ as a function of U. Thus from the solvability condition of (5.26) with respect to (3.18), we obtain

$$\frac{\partial^2 U}{\partial t^2} = \frac{2\pi q d^2}{\alpha m}\sum_{i=1}^{N} q_i \int_{\mathbb{R}^3} f_i \mathrm{d}V + c^2 \mathrm{div}K. \tag{5.27}$$

Due to (5.23), (5.27) is transformed into

$$\frac{\partial^2 U}{\partial t^2} = c^2 \triangle U + \frac{2\pi q}{\alpha m}\left(d^2 - 4\alpha^2 c^2\right)\sum_{i=1}^{N} q_i \int_{\mathbb{R}^3} f_i \mathrm{d}V. \tag{5.28}$$

Now we apply (5.28) for solvability of (5.26). If a function U satisfies (5.28), then (5.26) is satisfied and moreover

$$\nabla\gamma = \frac{\nu}{d^2}d + \left[d\times\left\{-\frac{mc}{q}\mathrm{rot}[K\times d] + \frac{md^2}{2\alpha cq}\frac{\partial}{\partial t}(\nabla U - K)\right\}\right]\frac{1}{d^2} \triangleq \mathbb{F}, \tag{5.29}$$

where $\nu(r,t) = (\nabla\gamma, d)$ is arbitrary. It follows from (5.29) that the vector field $\mathbb{F}(r,t)$ must be irrotational. Since U satisfies (5.18), we define K in a class of vectors satisfying condition

$$2\alpha\frac{\partial K}{\partial t} + (d\cdot\nabla)K = 0. \tag{5.30}$$

Then $d\times\mathrm{rot}[K\times d] = -2\alpha[d\times\partial K/\partial t]$ and (5.29) is transformed into

$$\nabla\gamma = \frac{\nu}{d^2}d + \left[d\times\left\{\left(4\alpha^2c^2 - d^2\right)\frac{\partial K}{\partial t} + d^2\frac{\partial}{\partial t}\nabla U\right\}\right]\frac{m}{2\alpha cqd^2}. \tag{5.31}$$

Up to an arbitrary function $b(U)$ and arbitrary vector-function $a(r)$, we can set

$$K(r,t) = \frac{d^2}{d^2 - 4\alpha^2c^2}\big(\nabla U + b(U)d + a(r)\big). \tag{5.32}$$

Then

$$\nabla\gamma = \frac{\nu}{d^2}. \tag{5.33}$$

If

$$b(U) = -\frac{1}{d^2}(\nabla U, d), \qquad a(r) = \nabla\varphi_0(r),$$

where $\nabla\varphi \perp d$, then (5.32) satisfies (5.30). The proof is developed by direct substitution (5.32) into (5.30) taking into account (5.18). Thus

$$K(r,t) = \frac{d^2}{d^2 - 4\alpha^2 c^2}\left\{\nabla U - \frac{1}{d^2}(\nabla U, d)d + \nabla\varphi_0(r)\right\}, \tag{5.34}$$

where $\nabla\varphi \perp d$ satisfies condition (5.30). Moreover it is evident that $K \perp d$. If

$$\triangle\varphi_0(r) = 0, \tag{5.35}$$

then for any $U(r,t)$ satisfying (5.28) the vector-function (5.34) satisfies (5.23) which can be shown by substituting (5.34) into (5.23). We show that in (5.33) $\nu \equiv 0$. In fact

$$\big(d, \mathrm{rot}(\nabla U, d)\big) = \big(d, \nabla(\nabla U, d) \times d\big) \equiv 0$$

for an arbitrary U, $(d, \mathrm{rot}K) = 0$ and due to (5.24) $d \perp \nabla\gamma$. Hence in (5.33), $\nu \equiv 0$, implying $\nabla\gamma = 0$, thus γ is constant.

It remains to show that functions (5.16), (5.17) where $U(r,t)$ satisfies (5.28) and $K(r,t)$ is expressed via U and φ_0 by formula (5.34), satisfy (1.4). ¿From substitution (5.16) and (5.17) in (1.4) we obtain the chain of equalities

$$\frac{m}{q}\left\{\frac{1}{d^2}\left[\frac{\partial K}{\partial t} \times d\right] - \frac{1}{2\alpha}\mathrm{rot}K\right\} =$$

$$\frac{m}{q(d^2 - 4\alpha^2c^2)}\left\{\frac{\partial}{\partial t}[\nabla U \times d] + \frac{1}{2\alpha}\mathrm{rot}((\nabla U, d)d)\right\} =$$

$$\frac{m}{q(d^2 - 4\alpha^2c^2)}\left[\nabla\left(\frac{\partial U}{\partial t} + \frac{1}{2\alpha}\mathrm{rot}(\nabla U, d)\right) \times d\right] = 0.$$

Remark 5.1. If (5.20) holds, then the function $\gamma \neq \mathrm{const}$, $\nabla\gamma = d \times \mathrm{rot}Q$.

Hence it follows:

Theorem 5.1. *Let $f_i(S)$ – the arbitrary differentiable functions, moreover*

$$\int_{\mathbb{R}^3} f_i(-|V|^2 + T)\mathrm{d}V < \infty, \qquad T \in (-\infty, +\infty), \qquad \alpha_i \in [0, \infty), \qquad d_i \in \mathbb{R}^3,$$

$$\alpha_i d = \alpha d_i, \qquad \alpha \stackrel{\triangle}{=} \alpha_1, \qquad d \stackrel{\triangle}{=} d_1,$$

then every solution $U(r,t)$ of the hyperbolic equation (5.28) with condition (5.18) corresponds to a solution of the system (1.1)–(1.5) of the form

$$f_i = f_i\left(-\alpha_i|V|^2 + Vd_i + \lambda_i + l_iU(r,t)\right), \tag{5.36}$$

$$B = \frac{\gamma}{d^2}d + \frac{mc}{q(d^2 - 4\alpha^2c^2)}[\nabla(U + \varphi_0(r)) \times d], \tag{5.37}$$

$$E = \frac{m}{2\alpha q(4\alpha^2c^2 - d^2)}\left\{\nabla\left(4\alpha^2c^2U + d^2\varphi_0(r) - (\nabla U, d)d\right)\right\}, \tag{5.38}$$

where $\varphi_0(r)$ – arbitrary function satisfying $\triangle\varphi_0 = 0$, $\nabla\varphi_0 \perp d$.

Corollary 5.1. *In the stationary case, (5.28) is transformed to the form*

$$\triangle U(r) = \frac{2\pi q}{\alpha m c^2}\left(4\alpha^2 c^2 - d^2\right)\sum_{i=1}^{N} q_i \int_{\mathbb{R}^3} f_i \mathrm{d}V \tag{5.39}$$

with condition

$$(\nabla U, d) = 0. \tag{5.40}$$

Remark 5.2. If

$$f_i = e^s, \qquad S = -\alpha_i |V|^2 + V d_i + \lambda_i + l_i U, \qquad l_i = \frac{\alpha_i m q_i}{\alpha m_i q},$$

then

$$\int_{\mathbb{R}^3} f_i \mathrm{d}V = \left(\frac{\pi}{\alpha_i}\right)^{3/2} \exp\left\{d_i^2/4\alpha_i + \lambda_i + l_i U\right\}.$$

In that case, "solving" equation (5.28) can be expressed as

$$\frac{\partial^2 U}{\partial t^2} = c^2 \triangle U + \frac{2\pi q}{\alpha m}\left(d^2 - 4\alpha^2 c^2\right)\pi^{3/2}\sum_{i=1}^{N} q_i (\alpha_i)^{-3/2} \exp\left\{d_i^2/4\alpha_i + \lambda_i + l_i U\right\}. \tag{5.41}$$

Due to paper [53], for case $N = 2$ (two-component system), equation (5.41) is transformed into

$$\frac{\partial^2 U}{\partial t^2} = c^2 \triangle U + \lambda b\left(e^U - e^{lU}\right), \qquad l \in \mathbb{R}^-, \qquad \lambda \in \mathbb{R}^+, \tag{5.42}$$

$$b = \frac{2\pi q^2}{\alpha m}\left(\frac{\pi}{\alpha}\right)^{3/2}\left(d^2 - 4\alpha^2 c^2\right) e^{d^2/4\alpha}.$$

Due to $l = -1$, (5.42) is a wave sinh-Gordon equation

$$\frac{\partial^2 U}{\partial t^2} = c^2 \triangle U + 2\lambda b \sinh U. \tag{5.43}$$

Remark 5.3. By the conditions of Theorem 5.1, a scalar Φ and a vector A potentials are defined by formulae

$$\Phi = \frac{m}{2\alpha q\left(d^2 - 4\alpha^2 c^2\right)}\left\{4\alpha^2 c^2 U(r,t) + d^2 \varphi_0\right\}, \tag{5.44}$$

$$A = \frac{mc}{q\left(d^2 - 4\alpha^2 c^2\right)} d\{U(r,t) + \varphi_0\} + \triangle\Theta(r), \tag{5.45}$$

where

$$\triangle\Theta(r) = \frac{\gamma}{d^2}(d_2 z, d_3 x, d_1 y)' + \nabla p(r), \qquad d \overset{\triangle}{=} (d_1, d_2, d_3)$$

and $p(r)$ is an arbitrary harmonic function. Since the function $U(r,t)$ satisfies (5.18), then potentials Φ, A are connected among themselves by Lorentz calibration.

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \operatorname{div} A = 0. \tag{5.46}$$

For analysis (5.43), we direct a constant vector $d \in \mathbb{R}^3$ along Z axis, i.e. we assume that $d \triangleq d_1(0,0,d_z)$. Moreover the solution $U(x,y,z,t)$ for(5.18) has the form

$$U = U\left(x, y, z - \frac{d}{2\alpha}t\right). \tag{5.47}$$

Solution (5.47) describes the wave spreading velocity running in the positive direction along Z axis with a constant velocity $d/2\alpha$, where $d/2\alpha < c$. With the substitution $\xi = z - (d/2\alpha)t$ we reduce (5.43) to

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\left(4\alpha^2 c^2 - d^2\right)}{4\alpha^2 c^2}\frac{\partial^2 U}{\partial \xi^2} = 2\lambda p \,\mathrm{sinh} U, \tag{5.48}$$

where

$$p \triangleq \frac{2\pi q^2}{\alpha m c^2}\left(\frac{\pi}{\alpha}\right)^{3/2}\left(4\alpha^2c^2 - d^2\right)\exp(d^2/4\alpha) > 0; \qquad \lambda \in \mathbb{R}^+.$$

Moreover introducing a new variable $\eta = \left(4\alpha^2c^2/\left(4\alpha^2c^2 - d^2\right)\right)^{1/2}\xi$, we transform (5.48) into

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial \eta^2} = 2\lambda p \,\mathrm{sinh} U, \qquad U \triangleq U(x,y,\eta). \tag{5.49}$$

Using formulae (2.69), it is easy to reconstruct some solutions of (5.49) using the Xirota method [67].

5.2. **Existence of nonstationary solutions of the VM system on bounded domains.** Here we consider the classical solutions $(f_1,\ldots,f_N,E,B)$ of the VM system of the special form of(5.36)–(5.38), which we write in the following form

$$f_i(x,v,t) = \hat{f}_i\left(-\alpha_i v^2 + v d_i + l_i U(x,t)\right), \tag{5.50}$$

$$E(x,t) = \frac{m}{2\alpha q(4\alpha^2c^2 - d^2)}\left(4\alpha^2c^2\nabla U(x,t) + \partial_t U(x,t) d\right), \tag{5.51}$$

$$B(x,t) = -\frac{mc}{q(4\alpha^2c^2 - d^2)}\nabla U(x,t) \times d, \tag{5.52}$$

where the functions $\hat{f}_i : \mathbb{R} \to [0,\infty[$ and the vector $d \in \mathbb{R}^3\backslash\{0\}$ are given, and the function $U : [0,\infty[\times\bar{\Omega} \to \mathbb{R}$ has to be defined. Assuming that $\partial\Omega \in$ $\in C^1$, we add the VM system with the boundary conditions for the electromagnetic field

$$E(x,t) \times n_\Omega(x) = 0, \quad B(x,t) n_\Omega(x) = 0, \quad t \geq 0,\ x \in \partial\Omega, \tag{5.53}$$

and a specular reflection condition for the distribution function on the boundary

$$f_i(t,x,v) = f_i\big(t,x,v - 2\,(v n_\Omega(x))\, n_\Omega(x)\big), \quad t \geq 0,\ x \in \partial\Omega,\ v \in \mathbb{R}^3, \tag{5.54}$$

where n_Ω – normal unit vector to $\partial\Omega$.

To prove existence of classical solutions of (1.2)–(1.6), (5.50)–(5.54) we apply the method of lower-upper solutions developed for nonlinear elliptic systems. In contrast to the stationary problem, nonstationary is more complicated because we need to add the equation of first order (5.18) to the nonlinear wave equation (5.28). Hence the problem is not "strongly" elliptic and it demands a further development of the method of lower-upper solutions.

Lemma 5.2. *Let $\Omega \subset \mathbb{R}^N$ – bounded domain with boundary $\partial\Omega \in \in C^{2,\alpha}$, $\alpha \in]0,1[$. Let $u_0 \in C^{2,\alpha}(\bar{\Omega})$ and $h \in C^{0,1}_{loc}(\bar{\Omega} \times \mathbb{R})$ such that $h(x,\cdot)$ – monotonic increasing function for every $x \in \Omega$. Then the boundary value problem*

$$\triangle u = h(\cdot, u(\cdot)) \qquad in \qquad \Omega,$$
$$(5.55)$$
$$u = u_0 \qquad on \qquad \partial\Omega$$

has a unique solution $u \in C^{2,\alpha}(\bar{\Omega})$.

Proof. Due to monotonicity of h it is easy to check that there exist $p_1, p_2 \in C^{0,\alpha}(\Omega)$ such that $p_2(x) \le 0 \le p_1(x)$ and

$$h(x,s) \begin{cases} \le p_1(x) & \text{for} \quad s \le 0, \\ \ge p_2(x) & \text{for} \quad s \ge 0, \end{cases}$$

for all $x \in \bar{\Omega}$. Let $u_{01} = \min(u_0, 0)$ and $u_{02} = \max(u_0, 0)$. Let $u_k \in \in C^{2,\alpha}\left(\bar{\Omega}\right)$ – solution of the linear boundary value problem for $k \in (1,2)$

$$\begin{cases} \triangle u_k = p_k & \text{in} \quad \Omega, \\ u_k = u_{0k} & \text{on} \quad \partial\Omega. \end{cases}$$

Due to the maximum principle, $u_1 \le 0 \le u_2$ in $\bar{\Omega}$. ¿From the latter it follows that u_1 is a lower solution and u_2 is an upper solution for (5.55). Then from theorem of existence (see Pao [47, Theorem 7.1]) it follows that (5.55) has a unique solution $u \in C^{2,\alpha}(\bar{\Omega})$. ☑

Remark 5.4. Lemma (5.2) is a well-known statement and does not require additional comments. We remark only that the condition of monotonicity of the function $h(x,\cdot)$ for the VP system is applied first by Vedenyapin [58–60].

Introduce the following conditions to the function $\hat{f} : \mathbb{R} \to [0,\infty[$:

(f1) $\hat{f} \in C^1(\mathbb{R})$;

(f2) $\forall u \in \mathbb{R} : f \in L^1(u,\infty)$;

(f3) f is a measurable function and $f(s) \le C\exp(-s)$ for a.e. $s \in \mathbb{R}$;

(f4) f is decreasing, $f(0) = 0$ and $\exists \mu \ge 0 : \forall s \le 0 : f(s) \le C|s|^\mu$.

Lemma 5.3 (Braasch [16]). *Let $f : \mathbb{R} \to [0,\infty[$ be a given function and*

$$h_f(u) = c\int_{\mathbb{R}^3} f\left(v^2 + vd + u\right), \qquad u \in \mathbb{R}.$$

Then the following claims hold.

(1) *Assume conditions* (f2), (f3). *Then* $h_f : \mathbb{R} \to \mathbb{R}$ *is continuous and nonnegative,*

$$h_f(u) = \frac{c_1}{|d|} \int_1^\infty \int_{-|d|s^2}^{|d|s^2} s f(s+t+u) dt ds$$

for all $u \in \mathbb{R}$.

(2) *Assume condition* (f3). *Let* $\psi : \to [0,\infty[$ *be a measurable function and* $\psi \le f$ *(a.e.). Then* $h_\psi \le h_f$.

(3) *Assume* (f4) *and* $|d| < 1$. *Then the following conditions* (f2), (f3), h_f *– continuously differentiable and* $h_f(u) \le C\exp(-u)$ *for all* $u \in \mathbb{R}$ *are satisfied.*

(4) *Assume* (f4) *and* $|d| < 1$. *Then from* (f4) *it follows that* h_f *is a decreasing function and*

$$|h_f(u)| \le C|u|^\mu$$

for all $u \in \mathbb{R}$, *where* $C = C(\mu, |d|)$.

Lemma 5.4. *Let* $\Omega \subseteq \mathbb{R}^2$ *have a smooth boundary* $\partial\Omega \in C^1$. *Let* $\hat{f}_1, \ldots, \hat{f}_N : \mathbb{R} \to [0,\infty[$ *be functions satisfying conditions* (f1)–(f3) *and* $|d| < 1$. *Let* $h_f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ *be given by*

$$h_f(x,U) = -\frac{2\pi q}{\alpha m}\big(4\alpha^2 c^2 - d^2\big) \sum_{i=1}^N q_i \int_{\mathbb{R}^3} \hat{f}_i\big(-\alpha v^2 + v d_i + l_i U(x,t)\big) dv,$$

and we assume $U \in C^2(\bar{\Omega})$ *– solution of the boundary problem*

$$\begin{cases} LU \triangleq \frac{\partial^2 U}{\partial t^2} - c^2 \triangle U = h_f(\cdot, U) \quad in \quad \Omega, \\ \\ U = 0 \quad on \quad \partial\Omega. \end{cases} \tag{5.56}$$

We define

$$U(x,t) = \tilde{U}(x+td), \qquad t \ge 0, \qquad x \in \bar{\Omega},$$

$$K(x,t) \triangleq -\frac{d^2}{4\alpha^2 c^2 - d^2}\bigg(\nabla U(x,t) - |d|^{-2}\partial_t U(x,t) d\bigg), \qquad t \ge 0, \qquad x \in \bar{\Omega},$$

$K \in C^1\big([0,\infty[\times\Omega\big)^3$ *and* E, B *by means of (5.51), (5.52). Then* $(f_1, \ldots, f_N, E, B,)$ *is a classical solution of the VM system in* Ω *and it satisfies boundary conditions (5.53), (5.54).*

Proof. Due to Lemma 5.3, h_f is a continuously differentiable function. The function U satisfies equation (5.18). Therefore it follows from Theorem 5.1 that $f_1 \ldots, f_N$ is a solution of the Vlasov equation, and E, B is a solution of the Maxwell system. Since U vanishes on $\partial\Omega$, then from the definition of U and the translation invariance Ω in d we obtain that U and $\partial_t U$ vanish on $[0,\infty[\times\partial\Omega$. Hence $\nabla U \times n_\Omega = K \times n_\Omega = 0$ on $[0,\infty[\times\partial\Omega$. From the latter we obtain

$$E(x,t) \times n_\Omega(x) = (K(x,t) - \nabla U(x,t)) \times n_\Omega(x) = 0$$

and

$$B(x,t) \times n_\Omega(x) = |d|^{-2}\big(n_\Omega(x) \times K(x,t)\big)d = 0$$

at $t \geq 0$ and $x \in \partial\Omega$. Therefore the boundary conditions (5.53) are satisfied. ☑

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^3$. Let $f_1, \ldots, f_N : \mathbb{R} \to [0, \infty[$ be functions satisfying condition* (f1) *and (pointwise) smaller than corresponding functions $\psi_1, \ldots, \psi_N : \mathbb{R} \to [0, \infty[$ satisfying condition* (f4) *with $\mu > 0$. We suppose that $|d| < 1$ and that there exists a function $\tilde{U} \in C^2_C(\Omega)$ such that*

$$U(x,t) = \tilde{U}(x + td), \qquad t \geq 0, \qquad x \in \Omega.$$

Then (5.56) in Lemma 5.4 has a smooth solution and $f_1, \ldots, f_N$ generates the classical solution $(f_1, \ldots, f_N, E, B)$ of the VM system in Ω of the form (5.50)–(5.52).

Proof. Since the elliptic operator L in (5.56) has constant coefficients, then by a linear change of coordinates, it is possible to transform it to the Laplace operator $L = \triangle$. Introducing notations $F \overset{\triangle}{=} (f_1, \ldots, f_N)$ we write the right part h_F of (5.56) as

$$h_F(x,U) = -c_1(c_2 - d_2)\sum_{i=1}^{N} q_i h_{f_i}(l_i U(x)),$$

where functions $h_{f_1}, \ldots, h_{f_N}$ are defined in Lemma 5.4. From Lemmas 5.3 and 5.4 we obtain

$$h_F(x,U)\begin{cases} \geq -c_1(c_2 - d_2)\sum_{q_i>0} C_i|q_i|h_{\psi_i}\big(|l_i|\tilde{U}(x)\big) \overset{\triangle}{=} h_1(x,U), \\ \leq c_1(c_2 - d^2)\sum_{q_i<0} C_i|q_i|\big(-|l_i|\tilde{U}(x)\big) \overset{\triangle}{=} h_2(x,U), \end{cases}$$

where $h_{\psi_1} \ldots, h_{\psi_N} : \mathbb{R} \to \mathbb{R}$ are continuously differentiable, decreasing, non-negative functions. Moreover functions h_1, h_2 are continuously differentiable and increasing functions in U and $h_1 \leq 0 \leq h_2$. ☑

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