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On corepresentations of equipped posets and their differentiation

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Abstract. Corepresentations of equipped posets over the pair (F, G) are introduced and studied, where $F \subset G$ is a quadratic field extension. The reduction algorithms $\widehat{\mathrm{VII}}$ and Completion for corepresentations (being in some intuitive sense dual to the known algorithms VII and Completion for representations) are built and investigated, with some applications. The generalized short versions of Differentiations VII and $\widehat{\mathrm{VII}}$ for representations and corepresentations of equipped posets with additional relations are described.

Keywords and phrases: equipped poset, representation, corepresentation, matrix problem of mixed type, differentiation functor.

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Resumen. Se introducen y estudian corepresentaciones de posets equipados sobre la pareja (F,G), donde $F \subset G$ es una extensión cuadrática de campos. Los algoritmos de reducción $\widehat{VII}$ y Completación para corepresentaciones (que son en algún sentido intuitivo dual a los ya conocidos algoritmos VII y Completación para representaciones) se construyen e investigan, con algunas aplicaciones. Las versiones cortas generalizadas de las Diferenciaciones VII y $\widehat{VII}$ para representaciones y corepresentaciones de posets equipados con relaciones adicionales se describen.

1. Introduction

In modern representation theory, the diagrammatic methods and matrix problems (i.e. classification problems of linear algebra) play an important role. The study of representations of finite dimensional algebras, quivers, vectroids, orders, posets (inclusively those with additional relations or structures) leads in many cases to matrix problems (see for instance [2, 3, 7, 10]).

In particular, representations of equipped posets over the pair of fields of real and complex numbers $(\mathbb{R}, \mathbb{C})$ (introduced and studied in [9, 10, 11]) are reduced to certain matrix problems of mixed type over this pair. The equipped posets of finite representation type were described earlier (in a more general context) in fact in [6], while those of one-parameter type, of tame type and of finite growth in [9], [10], [11] respectively. It was elaborated in [10] a system of differentiation functors (reduction algorithms) for solving these problems, among them the Differentiation VII algorithm and the operation of Completion.

It becomes clear now that, the restriction to the case $(\mathbb{R}, \mathbb{C})$ is not essential and the representation theory of equipped posets can be extended (true, not quite automatically) to the case of an arbitrary quadratic field extension $F \subset G$. At the same time, some other matrix problems of mixed type over the pair (F, G) naturally appear.

In the present paper, we introduce and investigate corepresentations of equipped posets over the pair (F, G) which are in some sense dual to the mentioned above representations. Their classification leads to the dual matrix problem of mixed type over (F, G). In spite of this, it is not yet known any formal construction establishing a direct relationship between representations and corepresentations.

We define in Section 2 the category of corepresentations $\operatorname{corep} \mathcal{P}$ of an equipped poset $\mathcal{P}$ and develop for it in Sections 3–5 the reduction functor machinery sufficient at least for the finite and one-parameter cases. More precisely, the dual variants of Differentiation VII and Completion for corepresentations are described and (following the scheme of [12] for ordinary posets) the short generalized versions of the algorithms VII and $\widehat{\text{VII}}$ for representations and corepresentations of equipped posets with additional relations of lattice type are constructed.

Possible applications are observed in Section 7.

2. Main definitions and notations

Equipped posets and their representations over the pair of fields of real and complex numbers $(\mathbb{R}, \mathbb{C})$ were introduced and studied in [9, 10, 11]. Now we see that the chosen there definition of an equipped poset can be reformulated in the following equivalent form.

A poset $(\mathcal{P}, \leq)$ is called *equipped* if all the order relations between its points $x \leq y$ are separated into *strong* $(x \trianglelefteq y)$ and *weak* $(x \preceq y)$ in such a way that

$$x \leq y \trianglelefteq z \quad or \quad x \trianglelefteq y \leq z \quad implies \quad x \trianglelefteq z\,, \tag{2.1}$$

i.e. a composition of a strong relation with any other relation is strong*.

Clearly, in general both binary relations $\trianglelefteq$ and $\preceq$ are not order relations: they are antisymmetric but not reflexive and $\preceq$ is not transitive (meanwhile $\trianglelefteq$ is transitive).

*It is interesting that this definition coincides with that one for a *biordered* set, given in [1] in another context (our relation $x \trianglelefteq y$ corresponds to $x \triangleleft y$ in [1]).

A point $x \in \mathcal{P}$ will be called *strong* (*weak*) if $x \trianglelefteq x$ (resp. $x \preceq x$), with the notation in diagrams $\circ$ (resp. $\otimes$) (remark that in [9, 10, 11] strong and weak points were called *single* and *double* respectively). If there are no weak points, the equipment is *trivial* and the poset $\mathcal{P}$ is *ordinary.*

We write $x \prec y$ if $x \preceq y$ and $x \neq y$ (similarly, $x \triangleleft y$ means $x \trianglelefteq y$ and $x \neq y$). We call an abstract relation xRy between two points *strict* if $x \neq y$.

Remark 2.1. It follows from the definition (2.1) that, for any weak relation $x \prec y$, both the points x, y are weak and moreover it holds $x \prec t \prec y$ for any possible intermediate point $x < t < y$. This condition defines completely $\mathcal{P}$ (that was the original definition of an equipped poset used in [9, 10, 11]).

By a sum $X_1 + \cdots + X_n$ we denote a disjoint union of subsets $X_1, \ldots, X_n \subset \mathcal{P}$ (notice that elements belonging to different subsets X_i can be comparable).

For a point $x \in \mathcal{P}$, set

$$x^{\vee} = \{y \,:\, x \leq y\}, \quad x^{\triangledown} = \{y \,:\, x \trianglelefteq y\}, \quad x^{\curlyvee} = \{y \,:\, x \preceq y\},$$

and dually define subsets $x_{\wedge}, x_{\vartriangle}, x_{\curlywedge}$. Remark that $x^{\vee}$ and $x^{\triangledown}$ ($x_{\wedge}$ and $x_{\vartriangle}$) are upper (lower) cones in $\mathcal{P}$, while $x^{\curlyvee}$ and $x_{\curlywedge}$ in general are not cones (see the example below). Obviously $x^{\vee} = x^{\triangledown} + x^{\curlyvee}$ and the dual formula holds . Also it holds $x^{\curlyvee} = x_{\curlywedge} = \varnothing$ for a strong point x.

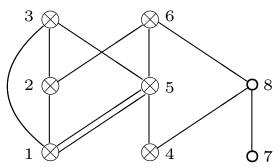
For a subset $X \subset \mathcal{P}$, set

$$X^{\vee} = \bigcup_{x \in X} x^{\vee}, \quad X^{\triangledown} = \bigcup_{x \in X} x^{\triangledown}, \quad X^{\curlyvee} = \bigcup_{x \in X} x^{\curlyvee}$$

and symmetrically (also by the union) define the corresponding sets $X_{\wedge}$, $X_{\vartriangle}$, $X_{\curlywedge}$. Sometimes we identify a one-point subset $\{x\} \subset \mathcal{P}$ with the point itself $\{x\} = x$.

Graphically each equipped poset is presented by its Hasse diagram (with strong and weak points) completed by additional lines symbolizing those strong strict relations between weak points which are not consequences of other relations.

Example 2.2. If an equipped poset $\mathcal{P}$ is given by the diagram



then (among strict relations) the relations $1 \triangleleft \{3,5,6\}$, $4 \triangleleft 8 \triangleleft 6$ and $7 \triangleleft 8$ are the only strong ones, hence all those in the rest are weak. In particular $3_{\vartriangle} = \{1\}$, $3_{\curlywedge} = \{2,3,4,5\}$, $3^{\triangledown} = \varnothing$, $3^{\curlyvee} = \{3\}$, $1^{\triangledown} = \{3,5,6\}$, $1^{\curlyvee} = \{1,2\}$, $1_{\vartriangle} = \varnothing$, $1_{\curlywedge} = \{1\}$, $8^{\triangledown} = \{6,8\}$, $8_{\vartriangle} = \{4,7,8\}$, $8_{\curlywedge} = 8^{\curlyvee} = \varnothing$, $7^{\triangledown} = \{6,7,8\}$, $7_{\vartriangle} = \{7\}$.

Let $F \subset G$ be an arbitrary quadratic field extension and $G = F(u)$ for some fixed element u. Then each element $x \in G$ is presented uniquely in the form $\alpha + \beta u$ with $\alpha, \beta \in F$ and (analogously to the case $(F, G) = (\mathbb{R}, \mathbb{C})$) the coefficients α and β are called the *real* and *imaginary* parts of x.

Each equipped poset $\mathcal{P}$ naturally defines a matrix problem of mixed type over the pair (F, G). Consider a rectangular matrix M separated into vertical stripes M_x, $x \in \mathcal{P}$, with M_x being over F (over G) if the point x is strong (weak):

$$M = \begin{array}{|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{x} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{y} & \multicolumn{1}{c}{} \\ \hline \dots & M_x & \dots & M_y & \dots \\ \hline \end{array} \tag{2.2}$$

Such partitioned matrices M are called *matrix representations* of $\mathcal{P}$ over (F, G). Their *admissible transformations* are as follows:

(a) F-elementary row transformations of the whole matrix M;

(b) F-elementary (G-elementary) column transformations of a stripe M_x if the point x is strong (weak);

(c) In the case of a weak relation $x \prec y$, additions of columns of the stripe M_x to the columns of the stripe M_y with coefficients in G;

(d) In the case of a strong relation $x \lhd y$, independent additions both real and imaginary parts of columns of the stripe M_x to real and imaginary parts (in any combinations) of columns of the stripe M_y with coefficients in F (assuming that, for y strong, there are no additions to the zero imaginary part of M_y).

Two representations are said to be *equivalent* or *isomorphic* if they can be turned into each other with help of the admissible transformations. The corresponding *matrix problem* of mixed type over the pair (F, G) consists of classifying the indecomposable in the natural sense matrices M, up to equivalence.

One can give another natural definition of representations, in terms of vector spaces over F and G. Identifying the direct sum U_0^2 of two copies of an F-space U_0 with a G-space $U_0^2 = U_0 \oplus uU_0$, we notice that, for each G-subspace $X \subset U_0^2$, its real and imaginary parts coincide $\operatorname{Re} X = \operatorname{Im} X$. Hence X is contained in its F-*hull* $F(X) = (\operatorname{Re} X)^2$, which is a G-subspace in U_0^2.

A G-subspace $X \subset U_0^2$ is called *strong* if $F(X) = X$ or, equivalently, if $X = Y^2$ for some F-subspace Y (remark that we use the notation $X \subset Y$ for an arbitrary inclusion of sets, not necessarily proper).

A *representation* of an equipped poset $\mathcal{P}$ over the pair (F, G) is any collection of the form

$$U = (U_0, U_x \,:\, x \in \mathcal{P}) \tag{2.3}$$

where U_0 is a finite-dimensional F-space and U_x are G-subspaces in U_0^2 such that the following conditions are satisfied

$$\begin{aligned} x \le y &\Longrightarrow U_x \subset U_y\,, \\ x \trianglelefteq y &\Longrightarrow F(U_x) \subset U_y\,. \end{aligned} \tag{2.4}$$

In particular, if a point x is strong, then the corresponding subspace $U_x = F(U_x) = (\operatorname{Re} U_x)^2$ is strong and is determined completely by its real part $\operatorname{Re} U_x$.

Representations are the objects of the category $\operatorname{rep} \mathcal{P}$, morphisms $U \xrightarrow{\varphi} V$ of which are F-linear maps $\varphi : U_0 \to V_0$ such that $\varphi^2(U_x) \subset V_x$ for each $x \in \mathcal{P}$. Two representations U, V are *isomorphic* ($U \simeq V$) if and only if for some F-isomorphism $\varphi : U_0 \to V_0$ it holds $\varphi^2(U_x) = V_x$ for all x.†

The described problem (for the classical pair of fields $(\mathbb{R}, \mathbb{C})$) was investigated in [9, 10, 11], where in particular the criteria for an equipped poset to be respectively one-parameter, tame and of finite growth were obtained (the finite representation type criterion for an arbitrary pair (F, G) follows from the earlier result of [6] on schurian vector space categories).

The aim of the present article is to observe one more matrix problem of mixed type over the pair (F, G), which also is defined by an equipped poset $\mathcal{P}$ and is in some intuitive sense dual to the mentioned above. Moreover, as will be shown, one can manage with that problem by similar technical means.

Nevertheless, it is not yet known any formal construction reducing one of these problems to another one. Our intension is in particular to present some primary information and facts concerning the new one, on the base of the experience with the old one. To avoid confusions in terminology, we will attach the particle *co* to some terms concerning the new dual problem (saying coproblem, corepresentation, etc.).

Consider again a rectangular separated matrix M of the form (2.2) supposing now that *all* its vertical stripes M_x, $x \in \mathcal{P}$, are over G. This is by definition a *matrix corepresentation* of an equipped poset $\mathcal{P}$ over the pair (F, G) to which one can apply the following *admissible transformations* (compare with the transformations $(a) - (d)$ above):

- (a') G-elementary row transformations of the whole matrix M;
- (b') G-elementary (F-elementary) column transformations of a stripe M_x if the point x is strong (weak);

† The matrix problem for representations $(a) - (d)$ naturally appears when classifying the objects of the category $\operatorname{rep} \mathcal{P}$, up to isomorphism. For this, one should attach to a representation U its matrix realization $M = (M_x : x \in \mathcal{P})$ in the following way. If a point x is strong (weak), then the columns of the stripe M_x are formed by the coordinates (with respect to some base of U_0) of a system of generators of the F-space $\operatorname{Re} U_x$ (resp. G-space U_x) modulo its radical subspace $\operatorname{Re} \underline{U_x}$ (resp. $\underline{U_x}$) defined analogously to [11], Section 3. Changing the base and the systems of generators, you get the problem $(a) - (d)$.

(c') In the case of a weak relation $x \prec y$, additions of columns of the stripe M_x to the columns of the stripe M_y with coefficients in F;

(d') In the case of a strong relation $x \lhd y$, additions of columns of the stripe M_x to the columns of the stripe M_y with coefficients in G;

The *matrix coproblem* over the pair (F, G), defined by an equipped poset $\mathcal{P}$, consists in classifying all indecomposable corepresentations, up to equivalence with respect to the admissible transformations $(a')-(d')$. In this situation also is possible to give a nice invariant definition in terms of subspaces over F and G.

Assume now that U_0 is a G-space. Then for any F-subspace $X \subset U_0$ it is defined its *G-hull* $G(X) = GX$ being nothing else but the ordinary G-span of X, i.e. the minimal G-subspace in U_0 containing X. If $G(X) = X$, the F-subspace X itself is a G-space and is said to be *G-strong*.

A *corepresentation* of an equipped poset $\mathcal{P}$ over the pair (F, G) is a collection of the form

$$U = (U_0, U_x \; : \; x \in \mathcal{P}) \tag{2.5}$$

where U_0 is a finite-dimensional G-space containing F-subspaces U_x such that

$$\begin{aligned} x \leq y &\Longrightarrow U_x \subset U_y\,, \\ x \unlhd y &\Longrightarrow G(U_x) \subset U_y\,. \end{aligned} \tag{2.6}$$

Notice that to a strong point x a strong subspace $U_x = G(U_x)$ corresponds.

Corepresentations are the objects of the category $\operatorname{corep} \mathcal{P}$, with morphisms $U \xrightarrow{\varphi} V$ being G-linear maps $\varphi : U_0 \to V_0$ such that $\varphi(U_x) \subset V_x$ for each $x \in \mathcal{P}$. It is clear that two corepresentations U, V are *isomorphic* if and only if for some G-isomorphism $\varphi : U_0 \to V_0$ it holds $\varphi(U_x) = V_x$ for all x.

Remark 2.3. The classification of indecomposable objects of the category $\operatorname{corep} \mathcal{P}$, up to isomorphism, corresponds precisely to the described above matrix coproblem $(a') - (d')$ (if to exclude from considerations formal indecomposable "empty" matrices having zero rows and one column). Namely, if M is a matrix corepresentation, one may attach to n rows of M a base $e_1, \ldots, e_n$ of some n-dimensional G-space U_0 and identify each column $(\lambda_1, \ldots, \lambda_n)^T$ of M with the element $u = \lambda_1 e_1 + \cdots + \lambda_n e_n \in U_0$. Denoting then by $F[X]$ (resp. $G[X]$) the F-span (G-span) in U_0 of any column set $X \subset M$, put

$$U_x = \sum_{y \preceq x} F[M_y] + \sum_{y \unlhd x} G[M_y]$$

and obtain immediately a collection (2.5) satisfying the conditions (2.6). It is clear that each vertical stripe M_x represents (by its columns) a system of generators of the space U_x modulo its *radical subspace*

$$\underline{U_x} = \sum_{y \prec x} F[M_y] + \sum_{y \lhd x} G[M_y],$$

hence the transformations $(a') - (d')$ of M reflect both base changing in U_0 and generator changing in subspaces U_x.

3. **Further notations and preliminaries**

The *dimension* of a matrix corepresentation M is a vector $d = \underline{\dim} M = (d_0, d_x : x \in \mathcal{P})$ with d_0 (resp. d_x) being the number of rows in M (of columns in M_x). Meanwhile the *dimension* of U is a vector $d = \underline{\dim} U = (d_0, d_x : x \in \mathcal{P})$ with $d_0 = \dim_G U_0$ and $d_x = \dim_F U_x/\underline{U_x}$ ($d_x = \dim_G U_x/\underline{U_x}$) for a weak (strong) point x.

Obviously $\underline{\dim} U \leq \underline{\dim} M$ (the equality holds if and only if the columns of each stripe M_x are linearly independent modulo the radical columns). A corepresentation U will be called *trivial* if $\dim_G U_0 = 1$.

A *sincere* vector has no zero coordinates by definitions. A representation or corepresentation is *sincere* if its dimension vector is sincere. Every equipped poset having at least one sincere indecomposable representation (corepresentation) is called *sincere* with respect to representations (corepresentations).

A subset of $\mathcal{P}$ is a *chain* (*anti-chain*) if all its points are pairwise comparable (incomparable). The *length* of a chain is the number of its points. A chain of the form $a_1 \prec a_2 \prec \cdots \prec a_n$ is called *weak*, if additionally $a_1 \prec a_n$ then it is *completely weak*.

An arbitrary subset $X \subset \mathcal{P}$ is said to be *completely weak* if all its points and possible relations between them are weak.

For a subset $X \subset \mathcal{P}$ and a matrix representation or corepresentation M, set $M_X = \bigcup_{x \in X} M_x$.

Denote by $\min X$ ($\max X$) the set of all minimal (maximal) points of a subset $X \subset \mathcal{P}$.

Let (X, Y) be any pair of subsets of $\mathcal{P}$ such that X is completely weak, Y is arbitrary and $X \cap Y^{\vee} = \varnothing$. We use in the sequel trivial indecomposable corepresentations $\widehat{T}(X, Y)$ of the form

$$\widehat{T}(X, Y) = (G, U_t \,:\, t \in \mathcal{P})$$

where

$$U_t = \begin{cases} G, & \text{if } t \in X^{\triangledown} \cup Y^{\vee}; \\ F, & \text{if } t \in X^{\vee} \setminus (X^{\triangledown} \cup Y^{\vee}); \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $\widehat{T}(X, Y) = \widehat{T}(\min X, \min Y)$, thus in principle one can deal with objects $\widehat{T}(X, Y)$ supposing X, Y to be antichains.

Setting

$$\widehat{T}(X, \varnothing) = \widehat{T}(X), \qquad \widehat{T}(\varnothing, Y) = \widehat{P}(Y),$$

we have in particular

$$\widehat{T}(\varnothing, \varnothing) = \widehat{T}(\varnothing) = \widehat{P}(\varnothing) = (G, 0, \ldots, 0).$$

In the case $X = \{x\}$ or $Y = \{y\}$, we write simply $\widehat{T}(x, Y)$ or $\widehat{T}(X, y)$, so for instance, the objects $\widehat{T}(x)$, $\widehat{T}(x, y)$ with $x \prec y$ and $\widehat{P}(y)$ are partial cases of $\widehat{T}(X, Y)$ and have the following matrix forms

$$\widehat{T}(x) = \begin{array}{c} x \\ \boxed{1} \end{array}, \qquad \widehat{T}(x,y) = \begin{array}{c} x \prec y \\ \begin{array}{|c|c|} \hline 1 & u \\ \hline \end{array} \end{array}, \qquad \widehat{P}(y) = \begin{array}{c} y \\ \boxed{1 \quad u} \end{array}$$

(clearly, the element u can be deleted from the matrix $\widehat{P}(y)$ if the point y is strong). In Section 5, some other objects of type $\widehat{T}(x, Y)$ will be considered (see Theorem 5.7).

The following simple fact holds.

Lemma 3.1. *The corepresentations $\widehat{P}(\varnothing), \widehat{P}(x), \widehat{T}(x)$ and $\widehat{T}(x, y)$ are all possible (up to isomorphism) indecomposable corepresentations of an arbitrary weak chain.*

Sketch of the proof. Use induction on n. The case $n = 1$ is in fact trivial. If $n \geq 2$, set $X = \{x_2, \dots, x_n\}$ and consider a matrix corepresentation M of a weak chain $x_1 \prec \cdots \prec x_n$. First reduce the stripe M_{x_1} to the natural canonical form, with direct summands $\widehat{P}(x_1), \widehat{T}(x_1)$ and some zero-rows. Since each direct summand $\widehat{P}(x_1)$ annuls (by admissible column additions $M_{x_1} \longrightarrow M_X$) the same row in M_X and is in fact a direct summand of the whole matrix M, one can assume M_{x_1} containing (besides zero-rows) only direct summands $\widehat{T}(x_1)$. Thus M takes the form

$$M = \begin{array}{ccccc} x_1 & x_2 & \dots & x_n & \\ \begin{array}{|c|c|c|c|} \hline I & * & * & * \\ \hline 0 & * & * & * \\ \hline \end{array} & & & & \begin{array}{c} K \\ L \end{array} \end{array} \tag{3.1}$$

Now you can reduce, by induction step, the stripe $M_X \cap L$ to the canonical form with direct summands mentioned in Lemma and then finally reduce (using admissible column additions $M_{x_1} \longrightarrow M_X$ and row additions $L \xrightarrow{G} K$) the stripe $M_X \cap K$ getting the desired result (some more proof details for the case $(\mathbb{R}, \mathbb{C})$ are given in [8]). ☑

We recall (see [10, 12]) that, for given subspaces A, B, X, Y of some vector space V over a field, the pair (X, Y) is called (A, B)-*cleaving* if $V = X \oplus Y$ and $A = X + (A \cap B)$, $B = Y \cap (A + B)$.

Denote by U^m (resp. φ^m) the direct sum of m copies of a representation, corepresentation or a space U (of a morphism or linear map φ).

If X is a set and U a vector space, then U^X means the direct some of $|X|$ copies of U numbered by the elements of X.

In the sequel, $K\{e_1,\dots,e_n\}$ is a notation for the vector space over a field K generated by the given vectors $e_1,\dots,e_n$.

By $[U]$ we denote the isomorphism class of an object U. For a collection of object $\mathcal{X}$, set $[\mathcal{X}] = \{\,[U] \,:\, U \in \mathcal{X}\,\}$. Let $\operatorname{Ind}\mathcal{P}$ (resp. $\widehat{\operatorname{Ind}}\,\mathcal{P}$) be the set of all isomorphism classes of indecomposable objects in $\operatorname{rep}\mathcal{P}$ ($\operatorname{corep}\mathcal{P}$).

Sometimes (if no confusions) a one-point set $\{a\}$ is denoted simply by a.

4. **Differentiation $\widehat{\mathrm{VII}}$**

The combinatorial action of Differentiation $\widehat{\mathrm{VII}}$ coincides with that one of Differentiation VII described in [10]. Namely, a pair of incomparable points (a,b) of an equipped poset $\mathcal{P}$ is called VII-*suitable* or $\widehat{\mathrm{VII}}$-*suitable* if a is weak, b is strong and

$$\mathcal{P} = a^{\triangledown} + b_{\vartriangle} + \{a \prec c_1 \prec \cdots \prec c_n\}$$

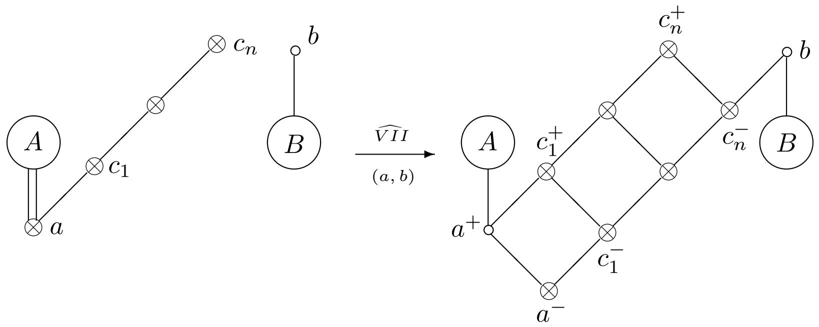
where $\{a \prec c_1 \prec \cdots \prec c_n\}$ is a completely weak chain incomparable with the point b. Putting $a = c_0$, we assume $n \geq 0$. Denote $A = a^{\triangledown}$, $B = b_{\vartriangle} \setminus b$ and $C = \{c_1 \prec \cdots \prec c_n\}$.

The *derived* poset $\mathcal{P}' = \mathcal{P}'_{(a,b)}$ of $\mathcal{P}$ with respect to such a pair (a,b) has the form

$$\mathcal{P}'_{(a,b)} = (\mathcal{P} \setminus (a + C)) + \{a^- < a^+\} + C^+ + C^-$$

where the point a^- is weak, the point a^+ is strong, $C^- = \{c_1^- \prec \cdots \prec c_n^-\}$ and $C^+ = \{c_1^+ \prec \cdots \prec c_n^+\}$ are completely weak chains, $c_i^- \prec c_i^+$ for all $i = 1,\dots,n$; $a^- \prec c_1^-$; $a^+ < c_1^+$; $c_n^- < b$ and the following natural conditions are satisfied:

- (a) Each of the points a^-, a^+ (c_i^-, c_i^+) inherits all previous order relations of the original point $a\,(c_i)$ with the points of the subset $\mathcal{P} \setminus (a + C)$.
- (b) The order relation in $\mathcal{P}'_{(a,b)}$ is induced by the initial order relation in the subset $\mathcal{P} \setminus (a + C)$ and by the listed above relations.



The *differentiation functor* $\widehat{D}_{(a,b)} : \operatorname{corep}\mathcal{P} \longrightarrow \operatorname{corep}\mathcal{P}'$ (denoted also by $'$) of the algorithm $\widehat{\mathrm{VII}}$ assigns to each co-representation U of $\mathcal{P}$ the *derived* one U'

of $\mathcal{P}'$ by the rule

$$\begin{aligned} U'_0 &= U_0 \\ U'_{c_i^-} &= U_{c_i} \cap U_b \quad \text{for } i = 0,1,\dots,n \\ U'_{c_i^+} &= U_{c_i} + G(U_a) \quad \text{for } i = 0,1,\dots,n \\ U'_x &= U_x \quad \text{for the remaining points } x \in \mathcal{P}'. \end{aligned} \tag{4.1}$$

And, for a morphism $U \xrightarrow{\varphi} V$ of the category $\operatorname{corep}\mathcal{P}$ (considered as a linear map $\varphi : U_0 \longrightarrow V_0$), set $\varphi' = \varphi$. One checks trivially that the functor is well defined.

The objects $\widehat{P}(a)$, $\widehat{T}(a)$ and $\widehat{T}(a, c_i)$, $i = 1,\dots,n$, play an important role in the description of properties of the algorithm $\widehat{\text{VII}}$. Their derivative all coincide $\widehat{P}(a)' = \widehat{T}(a)' = \widehat{T}(a, c_i)' = \widehat{P}(a^+)$, thus we have to consider the *reduced* objects of the category $\operatorname{corep}\mathcal{P}$ ($\operatorname{corep}\mathcal{P}'$) as those not containing direct summands $\widehat{P}(a)$, $\widehat{T}(a)$ and $\widehat{T}(a, c_i)$, $i = 1,\dots,n$, (resp. $\widehat{P}(a^+)$).

We point out that, as a rule, the derived object U' contains trivial direct summands $\widehat{P}(a^+)$, even if U is indecomposable. That's why the *reduced derived* object $U^{\downarrow}$ (which is unique up to isomorphism) is defined for any object $U \in \operatorname{corep}\mathcal{P}'$ as the largest direct summand of U' not containing trivial summands $\widehat{P}(a^+)$, i.e. by setting $U' = U^{\downarrow} \oplus \widehat{P}^m(a^+)$, with $m = \dim_G G(U_a)/(G(U_a) \cap U_b) = \dim_G (G(U_a) + U_b)/U_b$. Evidently $(U_1 \oplus U_2)^{\downarrow} \simeq U_1^{\downarrow} \oplus U_2^{\downarrow}$.

An equivalent definition of $U^{\downarrow}$ is as follows: take any $(G(U_a), U_b)$-cleaving pair of subspaces (E_0, W_0) of the G-space U_0 and set $U^{\downarrow} = W = (W_0; W_x \mid x \in \mathcal{P}')$ where $W_x = U'_x \cap W_0$ for each $x \in \mathcal{P}'$.

It is clear that, the reduced derived object $U^{\downarrow}$ can be viewed as a corepresentation not only of $\mathcal{P}'_{(a,b)}$ but also of the *completed derived* equipped poset $\overline{\mathcal{P}}'_{(a,b)}$ obtained from $\mathcal{P}'_{(a,b)}$ by adding one additional relation $a^+ < b$ (since due to the definition $W_{a^+} \subset W_b$).

The *integration* procedure for the algorithm $\widehat{\text{VII}}$ (which is in some sense inverse to the differentiation) is described in the following way. For a given corepresentation W of the completed derived poset $\overline{\mathcal{P}}'_{(a,b)}$, present each F-space $W_{c_i^+}$ $(i = 1,\dots,n)$ in the form $W_{c_i^+} = \underline{W}_{c_i^+} \oplus S_i \oplus H_i$, where S_i, H_i are some complements such that $S_i \subset W_b$ and $H_i \cap W_b = 0$. Choose in each S_i some F-base $s_{i1},\dots,s_{im_i}$. Analogously present the G-space W_{a^+} in the form $W_{a^+} = \underline{W}_{a^+} \oplus T_0$ where $T_0 = G\{t_{01},\dots,t_{0m_0}\}$ is some complement for the G-subspace $\underline{W}_{a^+}$.

Taking now a new G-space E_0 with a base $\{e_{ij} : i = 0, \dots, n; j = 1, \dots, m_i\}$, attach to W its *primitive* object $W^{\uparrow} = U = (U_0, U_x : x \in \mathcal{P})$ where

$$\begin{aligned} &U_0 = W_0 \oplus E_0 \ , \\ &U_x = W_x \oplus E_0^{A\cap\{x\}} \quad \text{for } \ x \neq a, c_i \ , \\ &U_a = W_{a^-} + F\{s_{0j} + ue_{0j} \ : \ j = 1, \dots, m_0\} + \\ &\qquad\qquad + F\{e_{ij} \ : \ \ i = 0, \dots, n; j = 1, \dots, m_i\} \ , \\ &U_{c_i} = U_{c_{i-1}} + W_{c_i^-} + H_i + F\{s_{ij} + ue_{ij} \ : \ \ j = 1, \dots, m_i\} \quad (i \geq 1) \ , \end{aligned} \tag{4.2}$$

and $U_{c_0} = U_a$. The primitive object $W^{\uparrow}$ depends no, up to isomorphism, on the choice of subspaces T_i, H_i and their bases, moreover $(W_1 \oplus W_2)^{\uparrow} \simeq W_1^{\uparrow} \oplus W_2^{\uparrow}$. There hold also equalities $\dim_G S_0 = \dim_G (W_{a^+}/G(W_{a^-}))$ and $\dim_F S_i = \dim_F (W_{c_i^+} \cap W_b)/(W_{c_i^-} + W_{c_{i-1}^+} \cap W_b)$ for $i = 1, \dots, n$.

The main result on Differentiation $\widehat{\text{VII}}$ is as follows.

Theorem 4.1. *In the case of Differentiation $\widehat{VII}$, the operations $\uparrow$ and $\downarrow$ induce mutually inverse bijections*

$$\widehat{\operatorname{Ind}}\ \mathcal{P} \setminus [\widehat{P}(a), \widehat{T}(a), \widehat{T}(a, c_i), i = 1, \dots, n] \leftrightarrows \widehat{\operatorname{Ind}}\ \overline{\mathcal{P}}'_{(a,b)} = \widehat{\operatorname{Ind}}\ \mathcal{P}'_{(a,b)} \setminus [\widehat{P}(a^+)] \, .$$

Proof. For given corepresentations U of $\mathcal{P}$ and W of $\overline{\mathcal{P}}'$, one has to prove that $[U^{\downarrow}]^{\uparrow} \simeq U$ and $[W^{\uparrow}]^{\downarrow} \simeq W$. The second isomorphism is verified without difficulties by a standard routine procedure using the formulas (4.1) and (4.2), this is left to the reader as an exercise.

To prove the first one, consider the matrix M of a reduced corepresentation U of $\mathcal{P}$ chosen in such a way that the columns of each vertical stripe M_x, $x \in \mathcal{P}$, generate U_x. Applying to M suitable G-elementary row transformations, place at its bottom linearly independent rows corresponding to some base of the G-subspace U_b obtaining all zeroes above them in the block M_{b+B} (our convention for matrix pictures is that empty blocks denote zero-blocks, but blocks marked by $*$ are arbitrary, and I denotes the identity block of arbitrary order):

$$\begin{array}{c|cccccccccccccc|c}
 & & a & & & & A & & c_1 & & & c_n & & B & b & \\
\hline
 & & uI & I & & & & & & & & & & & & \\
E_0 & & & & I & & & & uI & & & & & & & \\
 & & & & & I & & & & & & uI & & & & \\
\hline
W_0 & & & & & & * & & & X_1 & & & X_n & & & Q \\
 & * & * & & & & * & * & S_1 & Y_1 & * & S_n & Y_n & * & * & U_b \\
\hline
 & a^- & a^+ & & & & & c_1^- & \multicolumn{2}{c}{\underbrace{\qquad\quad}_{c_1^+}} & c_n^- & \multicolumn{2}{c}{\underbrace{\qquad\quad}_{c_n^+}} & & &
\end{array} \tag{4.3}$$

Further, select linearly independent over G rows in M_a above the horizontal stripe U_b denoting the new horizontal stripe by E_0 and obtaining (by suitable

G-elementary transformations of rows) all zeroes in the intermediate horizontal stripe $Q \cap M_a$. Applying then to the block $Q \cap (M_{c_1} \cup \cdots \cup M_{c_n})$ suitable admissible column transformations, we can leave there only those cells $X_1, \ldots, X_n$ the columns of which are F-linearly independent (all together).

Consider each matrix M_{c_i} $(i = 1, \ldots, n)$ as a union of two vertical stripes $M_{c_i} = M'_{c_i} \cup M''_{c_i}$ where M''_{c_i} is formed by the columns containing the block X_i and M'_{c_i} consists of the rest of the columns. Reduce to the canonical form the block $E_0 \cap (M_a \cup M'_{c_1} \cdots \cup M'_{c_n})$ considering it as a corepresentation of the completely weak chain $a \prec c_1 \prec \cdots \prec c_n$ and applying Lemma 3.1 (select the matrix forms $\widehat{T}(a) = \boxed{u \quad 1}$ and $\widehat{T}(a, c_i) = \boxed{1 \mid u}$, $i = 1, \ldots, n$). Omit the direct summands $\widehat{P}(a)$ which obviously are split as direct summands of the whole M. Make (by row additions) zeroes below the identity blocks I in M_a and get the block M_a as shown in (4.3).

Then (using row additions $Q \xrightarrow{G} E_0$ and column addition $M_a \xrightarrow{F} M_{\{c_1, \ldots, c_n\}}$) annul all the blocks M''_{c_i} (this is possible because the matrix $X_1 \cup \cdots \cup X_n$ can be viewed as a corepresentation of a completely weak chain $c_1 \prec \cdots \prec c_n$ and hence presented as a direct sum of the mentioned in Lemma 3.1 trivial blocks). Remark that the shown matrix blocks S_i and $X_i \cup Y_i$ correspond to the subspaces S_i and H_i respectively, considered above in the integration procedure.

Annul finally (by column additions $M_a \xrightarrow{G} M_A$) the block $E_0 \cap A$ and obtain the shown matrix form (4.3). An immediate comparison with the formulas (4.1) and (4.2) confirms evidently that in the horizontal stripe $W_0 = Q \cup U_b$ we have just the reduced derived corepresentation $U^{\downarrow}$ and certainly the isomorphism $(U^{\downarrow})^{\uparrow} \simeq U$ holds. The proof is complete. ☑

Remark 4.2. Analogously to the case of Differentiation VII (see [10], Remark 3.5), one can expect that the differentiation functor $\widehat{\text{VII}}$ induces an equivalence of the quotient categories

$$\operatorname{corep} \mathcal{P}/\langle \widehat{P}(a), \widehat{T}(a), \widehat{T}(a, c_i),\ i = 1, \ldots, n\rangle \xrightarrow{\sim} \operatorname{corep} \mathcal{P}'/\langle \widehat{P}(a^+)\rangle$$

where the brackets $\langle \ldots \rangle$ denote the ideals of all morphisms passed through finite direct sums of the shown objects. The proof may be carried out by the same scheme as in [12] (Section 7) using short generalized steps of the algorithm $\widehat{\text{VII}}$, as explained below.

5. Short generalized versions of Differentiations VII and $\widehat{\text{VII}}$

To deal with differentiation algorithms more effectively, one has to reduce (if possible) long differentiation steps to shorter ones, probably passing to a more wide (but suitable) class of matrix problems.

Such a possibility exists in the case of the algorithms VII and $\widehat{\mathrm{VII}}$ and can be realized analogously to [12] via passing to a class of representations or corepresentations of equipped posets with additional relations. We outline here briefly the main scheme (a more detailed exposition will be placed elsewhere).

First we decompose easily the algorithms VII and $\widehat{\mathrm{VII}}$ in two intermediate steps, one of which will be decomposed then more.

Preliminary decomposition. Let (a,b) be a VII-suitable pair of points of an equipped poset $\mathcal{P}$ as defined above, i.e. $\mathcal{P} = a^{\triangledown} + b_{\vartriangle} + (a + C)$ where $C = \{c_1 \prec \cdots \prec c_n\}$ is a completely weak chain ($n \geq 0$, $c_0 = a$) incomparable with b (set $A = a^{\triangledown}$, $B = b_{\vartriangle} \setminus b$).

We recall that the combinatorial action of the algorithms VII and $\widehat{\mathrm{VII}}$ coincide and are going to present it as a combination of two steps.

The *long step* or briefly *l-step* or *Differentiation VII*$_l$ consists in transition $\mathcal{P} \mapsto \dot{\mathcal{P}}_{(a,b)}$ from the ordinary equipped poset $\mathcal{P}$ to an equipped poset with relation of the form

$$\dot{\mathcal{P}}_{(a,b)} = (\mathcal{P}^{\,l}_{(a,b)} \mid \Sigma_{(a,b)}),$$

where $\mathcal{P}^{\,l}_{(a,b)}$ is a new ordinary equipped poset differing slightly from $\mathcal{P}'_{(a,b)}$, namely

$$\mathcal{P}^{\,l}_{(a,b)} = (\mathcal{P} \setminus C) + C^{+} + C^{-}$$

where $C^{-} = \{c_1^{-} \prec \cdots \prec c_n^{-}\}$ and $C^{+} = \{c_1^{+} \prec \cdots \prec c_n^{+}\}$ are completely weak chains, $c_i^{-} \prec c_i^{+}$ for $i = 1, \ldots, n$; $a \lhd c_1^{+}$; $c_n^{-} < b$ and the standard conditions hold:

- (a_l) Each of the points c_i^{-}, c_i^{+}, ($i = 0, 1, \ldots, n$) inherits all previous order relations of the original point c_i with the points of the subset $\mathcal{P} \setminus (a + C)$.
- (b_l) The order relation in $\mathcal{P}^{\,l}_{(a,b)}$ is induced by the initial order relation in the subset $\mathcal{P} \setminus C$ and by the listed above relations.

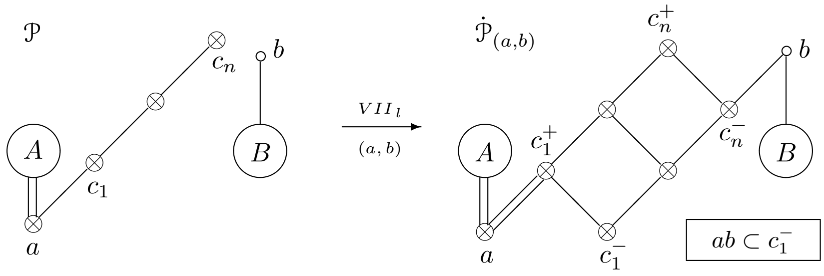
As for the set of relations $\Sigma_{(a,b)}$, it consists of one relation only

$$\Sigma_{(a,b)} = \{\, ab \subset c_1^{-} \,\}$$

which means conditionally that the categories $\operatorname{rep} \dot{\mathcal{P}}_{(a,b)}$ and $\operatorname{corep} \dot{\mathcal{P}}_{(a,b)}$ are by definition the full subcategories of the categories $\operatorname{rep} \mathcal{P}^{\,l}_{(a,b)}$ and $\operatorname{corep} \mathcal{P}^{\,l}_{(a,b)}$ respectively formed by all those objects V which satisfy the relations

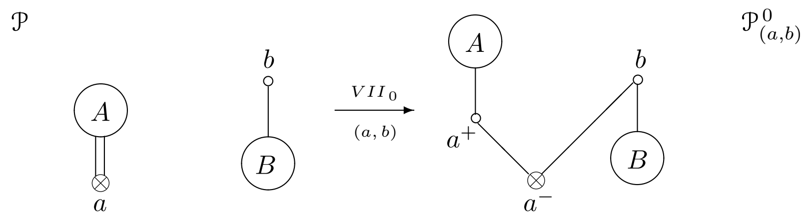
$$V_a \cap V_b \subset V_{c_1^{-}}\ .$$

The diagram of the long step is as follows:



Notice that if $n = 0$ and therefore $C = \varnothing$, then $\Sigma_{(a,b)} = \varnothing$ and actually $\dot{\mathcal{P}}_{(a,b)} = \mathcal{P}^l_{(a,b)} = \mathcal{P}$.

The additional *0-step* or *Differentiation VII*$_0$ is a transition from $\dot{\mathcal{P}}_{(a,b)}$ to $\mathcal{P}'_{(a,b)}$ where $\mathcal{P}'_{(a,b)}$ is the defined in Section 4 complete (a,b)-derived poset. In other words, Differentiation VII$_0$ is nothing else but a particular case of Differentiation VII applied in the situation $C = \varnothing$:



So, the obtained combinatorial decomposition $\mathcal{P} \longmapsto \dot{\mathcal{P}}_{(a,b)} \longmapsto \mathcal{P}'_{(a,b)}$ (compare with the shown in Section 4 diagram of the algorithm $\widehat{\text{VII}}$) corresponds to the functor decomposition for representations and corepresentations

$$D_{(a,b)} = D^0_{(a,b)} D^l_{(a,b)} \qquad \textit{and} \qquad \widehat{D}_{(a,b)} = \widehat{D}^0_{(a,b)} \widehat{D}^l_{(a,b)}$$

with the functors being defined in the following unified way.

For a representation or corepresentation U of $\mathcal{P}$ and a point $x \in \mathcal{P}$, let $\widetilde{U_x}$ be the *hull* of the space U_x in the following sense combining two possibilities

$$\widetilde{U_x} = \begin{cases} F(U_x), & \text{if U is a representation;} \\ G(U_x), & \text{if U is a corepresentation.} \end{cases}$$

Then both the differentiation functors $D_{(a,b)}$ and $\widehat{D}_{(a,b)}$ of the algorithms VII and $\widehat{\text{VII}}$ (described in [10] and Section 4 above respectively) are given by the same formulas

$$\begin{aligned} U'_0 &= U_0 \\ U'_{c_i^-} &= U_{c_i} \cap U_b \quad \text{for } i = 0, 1, \ldots, n \\ U'_{c_i^+} &= U_{c_i} + \widetilde{U_a} \quad \text{for } i = 0, 1, \ldots, n \\ U'_x &= U_x \quad \text{for the remaining points } x \in \mathcal{P}' \\ \varphi' &= \varphi \quad \text{for a linear map-morphism } \varphi : U_0 \longrightarrow V_0. \end{aligned} \tag{5.1}$$

To get from (5.1) the l-step differentiation functors $D^{\,l}_{(a,b)}$ and $\widehat{D}^{\,l}_{(a,b)}$, you have simply to exclude the case $i = 0$. Meanwhile to get the 0-step functors $D^0_{(a,b)}$ and $\widehat{D}^0_{(a,b)}$, you have on the contrary to assume $n = 0$.

Denote by $U^{(l)}$ (resp. $U^{(0)}$) the derivative of some object (representation or corepresentation) U with respect to the l-step (0-step) of Differentiation. Then it holds evidently for representations

$$\begin{aligned} &P(a)^{(l)} = P(a), \quad T(a)^{(l)} = T(a, c_i)^{(l)} = T(a), \\ &P(a)^{(0)} = P(a^+), \quad T(a)^{(0)} = P^2(a^+) \end{aligned} \tag{5.2}$$

and for corepresentations

$$\begin{aligned} &\widehat{T}(a)^{(l)} = \widehat{T}(a, c_i)^{(l)} = \widehat{T}(a), \quad \widehat{P}(a)^{(l)} = \widehat{P}(a), \\ &\widehat{T}(a)^{(0)} = \widehat{P}(a)^{(0)} = \widehat{P}(a^+) \end{aligned} \tag{5.3}$$

where $P(x), T(x)$ and $T(x,y)$ are representations in the matrix form

$$P(x) = \begin{array}{c} x \\ \boxed{1} \end{array}\,, \qquad T(x) = \begin{array}{c} x \\ \left|\begin{array}{c} 1 \\ u \end{array}\right| \end{array}\,, \qquad T(x,y) = \begin{array}{c} x \prec y \\ \left|\begin{array}{c|c} 1 & 0 \\ u & 1 \end{array}\right| \end{array}$$

(recall that $\widehat{P}(x), \widehat{T}(x)$, $\widehat{T}(x,y)$ have been already defined in Section 3).

Taking the equalities (5.2) and (5.3) into account, analogously to Differentiations VII and $\widehat{\text{VII}}$, one can define naturally the *reduced* derived object $U^{\downarrow}$ of an object U for the algorithms VII_l (resp. $\widehat{\text{VII}}_l$) as a maximal direct summand of $U^{(l)}$ not containing summands $T(a)$ (resp. $\widehat{T}(a)$). Also analogously to the algorithms VII and $\widehat{\text{VII}}$ one should define the *primitive* object $W^{\uparrow}$ for each object W of the derived category (free of direct summands $T(a)$ (resp. $\widehat{T}(a)$)) and to deduce then the following main property of the l-step algorithm (compare with Theorem 4.1).

Theorem 5.1. *In the case of Differentiations VII_l and $\widehat{VII}_l$, the operations* $\uparrow$ *and* $\downarrow$ *induce mutually inverse bijections*

$$\begin{aligned} &\text{Ind}\ \mathcal{P} \setminus [T(a), T(a, c_i), i = 1, \ldots, n] \leftrightarrows \text{Ind}\ \dot{\mathcal{P}}_{(a,b)} \setminus [T(a)]\,, \\ &\widehat{\text{Ind}}\ \mathcal{P} \setminus [\widehat{T}(a), \widehat{T}(a, c_i), i = 1, \ldots, n] \leftrightarrows \widehat{\text{Ind}}\ \dot{\mathcal{P}}_{(a,b)} \setminus [\widehat{T}(a)]\,. \end{aligned}$$

We left the details of the proof (which is very similar to the proofs of Theorem 3.5 in [10] and Theorem 4.1 above[‡]) as an exercise for the interested reader.

Since the 0-step algorithms VII_0 and $\widehat{\mathrm{VII}}_0$ are special cases of VII and $\widehat{\mathrm{VII}}$, we obtain for them immediately from Theorem 3.5 in [10] and Theorem 4.1 above the following corollary.

Corollary 5.2. *In the case of Differentiations* VII_0 *and* $\widehat{VII}_0$*, the operations* $\uparrow$ *and* $\downarrow$ *induce mutually inverse bijections*

$$\mathrm{Ind}\ \mathcal{P} \setminus [P(a), T(a)] \leftrightarrows \mathrm{Ind}\ \mathcal{P}^0_{(a,b)} \setminus [P(a^+)]\,,$$

$$\widehat{\mathrm{Ind}}\ \mathcal{P} \setminus [\widehat{P}(a), \widehat{T}(a)] \leftrightarrows \widehat{\mathrm{Ind}}\ \mathcal{P}^0_{(a,b)} \setminus [\widehat{P}(a^+)]\,.$$

Remark 5.3. In accordance with the previous statements, one should expect that the differentiation functors VII_l, $\widehat{\mathrm{VII}}_l$, VII_0, $\widehat{\mathrm{VII}}_0$ induce respectively equivalences of the quotient categories

(i) $\mathrm{rep}\,\mathcal{P}/\langle T(a), T(a,c_i), i=1,\dots,n\rangle \xrightarrow{\sim} \mathrm{rep}\,\dot{\mathcal{P}}_{(a,b)}/\langle T(a)\rangle\,,$

($\widehat{i}$) $\mathrm{corep}\,\mathcal{P}/\langle \widehat{T}(a), \widehat{T}(a,c_i), i=1,\dots,n\rangle \xrightarrow{\sim} \mathrm{corep}\,\dot{\mathcal{P}}_{(a,b)}/\langle \widehat{T}(a)\rangle\,,$

(ii) $\mathrm{rep}\,\mathcal{P}/\langle P(a), T(a)\rangle \xrightarrow{\sim} \mathrm{rep}\,\mathcal{P}^0_{(a,b)}/\langle P(a^+)\rangle \qquad (C=\varnothing),$

($\widehat{ii}$) $\mathrm{corep}\,\mathcal{P}/\langle \widehat{P}(a), \widehat{T}(a)\rangle \xrightarrow{\sim} \mathrm{corep}\,\mathcal{P}^0_{(a,b)}/\langle \widehat{P}(a^+)\rangle \qquad (C=\varnothing).$

Our next goal is to decompose more essentially the l-step.

Main decomposition. First we define the combinatorial action of the short generalized algorithm VII_s with respect to a triple of points.

A triple of points (a,b,c) of an equipped poset $\mathcal{P}$ will be called *VII_s-suitable* if the points a,c are weak, b is strong incomparable with a,c and

$$\mathcal{P} = a^{\triangledown} + b_{\vartriangle} + \{a \prec X \prec c \prec Y\}$$

where $\{a \prec X \prec c \prec Y\}$ is a completely weak set containing arbitrary subsets X, Y (probably empty).

The *derived* or (a,b,c)-*derived* equipped poset *with relations* $\mathcal{P}'_{(a,b,c)}$ of the poset $\mathcal{P}$ is a pair

$$\mathcal{P}'_{(a,b,c)} = (\mathcal{P}^s_{(a,b,c)} \mid \Sigma_{(a,b,c)})$$

where

$$\mathcal{P}^s_{(a,b,c)} = (\mathcal{P} \setminus c) + \{c^-, c^+\}$$

is an equipped poset such that the pairs $c^- \prec c^+$, $X \prec c^+$ and $c^- \prec Y$ are completely weak, $a \lhd c^+$, $c^- < b$ and the partial order in $\mathcal{P}^s_{(a,b,c)}$ is induced both by these relations and by the initial order in $\mathcal{P} \setminus c$ (it is assumed that each of the points c^-, c^+ inherits the order relations of the point c with the points of the subset $a^{\triangledown} + b_{\vartriangle}$).

[‡] The only difference with the complete matrix Differentiations VII and $\widehat{\mathrm{VII}}$ is that one needs no more to separate the block M_a into the parts corresponding to the points $a^{\pm}$.

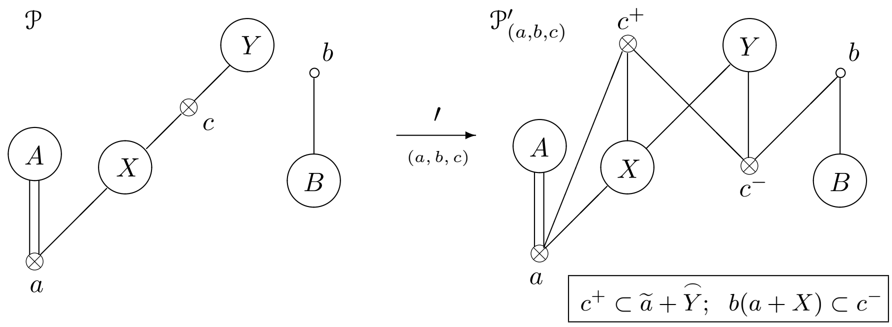
Further, $\Sigma_{(a,b,c)}$ is a set of two formal relations

$$\Sigma_{(a,b,c)} = \{c^+ \subset \widetilde{a} + \widehat{Y};\ bX \subset c^-\},$$

and by definition the categories $\operatorname{rep} \mathcal{P}'_{(a,b,c)}$ and $\operatorname{corep} \mathcal{P}'_{(a,b,c)}$ are the full subcategories of the categories $\operatorname{rep} \mathcal{P}^s_{(a,b,c)}$ and $\operatorname{corep} \mathcal{P}^s_{(a,b,c)}$ respectively formed by those objects W which satisfy the relations

$$W_{c^+} \subset \widetilde{W}_a + \widehat{W}_Y \qquad \textit{and} \qquad W_b \cap W_X \subset W_{c^-}$$

where $\widehat{W}_Y = \bigcap_{y\in Y} W_y$ and $W_X = \sum_{x\in X} W_x$ (with the commonly accepted convention that $\widehat{W}_\varnothing = W_0$ and $W_\varnothing = 0$).



Then both the *differentiation functors* $D_{(a,b,c)} : \operatorname{rep} \mathcal{P} \longrightarrow \operatorname{rep} \mathcal{P}'_{(a,b,c)}$ and $\widehat{D}_{(a,b,c)} : \operatorname{corep} \mathcal{P} \longrightarrow \operatorname{corep} \mathcal{P}'_{(a,b,c)}$ (also denoted briefly by $'$) are given by the same formulas

$$\begin{aligned} U'_0 &= U_0 \\ U'_{c^+} &= U_c + \widetilde{U_a}; \qquad U'_{c^-} = U_c \cap U_b \\ U'_x &= U_x \quad \text{for the remaining points } x \in \mathcal{P}'_{(a,b,c)} \\ \varphi' &= \varphi \quad \text{for a linear map-morphism } \varphi : U_0 \longrightarrow V_0. \end{aligned} \tag{5.4}$$

Remark 5.4. Certainly, the action of the functor is naturally extended to those situations when the initial poset $\mathcal{P}$ itself is an equipped poset with relations. In such cases some more relations have to be added to $\Sigma_{(a,b,c)}$.

Proposition 5.5. *Let $\mathcal{P} = a^\triangledown + b_\vartriangle + \{a \prec c_1 \prec \cdots \prec c_n\}$ be an equipped poset with VII-suitable pair of points (a, b). Then the long differentiation functors $D^l_{(a,b)}$ and $\widehat{D}^l_{(a,b)}$ of the algorithms VII_l and $\widehat{VII}_l$ are presented as compositions*

$$D^l_{(a,b)} = D_{(a,b,c_1)} D_{(a,b,c_2)} \dots D_{(a,b,c_n)}\,,$$
$$\widehat{D}^l_{(a,b)} = \widehat{D}_{(a,b,c_1)} \widehat{D}_{(a,b,c_2)} \dots \widehat{D}_{(a,b,c_n)}$$

which do not depend on the order of the factors. As a consequence, it holds

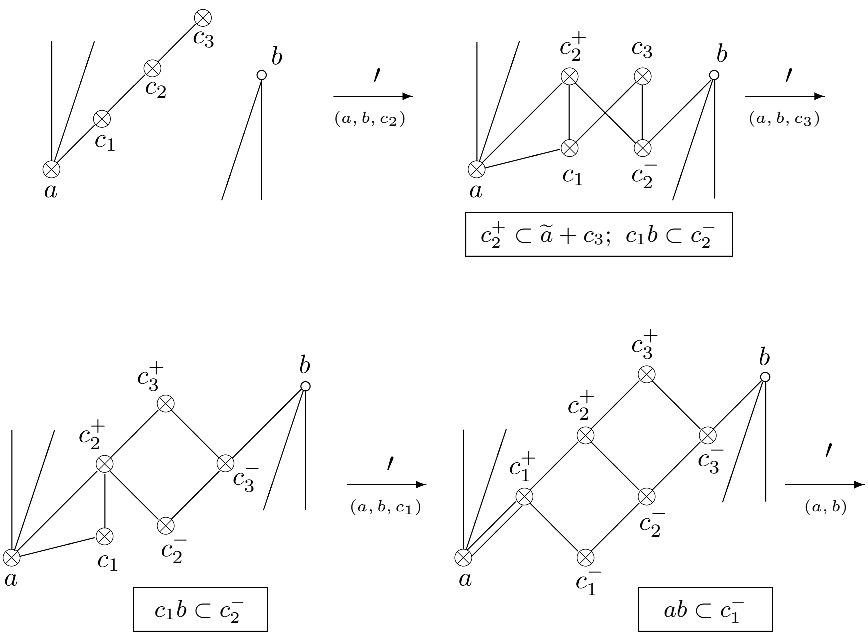
$$D_{(a,b)} = D^0_{(a,b)} D^l_{(a,b)} = D^0_{(a,b)} D_{(a,b,c_1)} D_{(a,b,c_2)} \dots D_{(a,b,c_n)} \,,$$

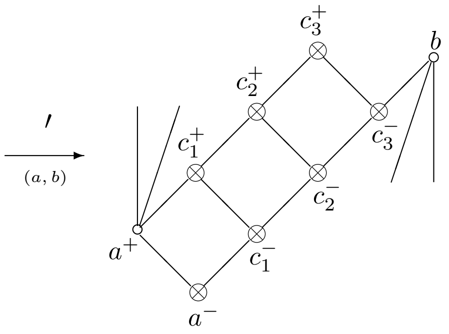
$$\widehat{D}_{(a,b)} = \widehat{D}^0_{(a,b)} \widehat{D}^l_{(a,b)} = \widehat{D}^0_{(a,b)} \widehat{D}_{(a,b,c_1)} \widehat{D}_{(a,b,c_2)} \dots \widehat{D}_{(a,b,c_n)} \,.$$

Proof. Apply the functors $D_{(a,b,c_i)}$ or $\widehat{D}_{(a,b,c_i)}$ $(i = 1, \dots, n)$ in arbitrary order and check in each step the relations. The rule of their forming will become clear (in particular, take into account the mentioned convention $\widehat{W}_\varnothing = W_0$ and $W_\varnothing = 0$). After n steps, the equipped poset $\mathcal{P}^l_{(a,b)}$ appears and the only one relation $ab \subset c_1^-$ remains, i.e. you get the set with relation $\dot{\mathcal{P}}_{(a,b)}$ which obviously is transformed by the subsequent 0-step to $\mathcal{P}'_{(a,b)}$. These transformations are conformed with the shown functor decompositions.

☑

Example 5.6.





Consider now a corepresentation $\widehat{T}(a,Y)$ of the poset $\mathcal{P}$ defined as in Section 3. One can introduce analogously a representation $T(a,Y)$ having the matrix form $T(a,Y) = \begin{array}{cccc} x & y_1 & \dots & y_r \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{\dots} & \multicolumn{1}{c|}{0} \\ \multicolumn{1}{|c|}{u} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{\dots} & \multicolumn{1}{c|}{1} \\ \hline \end{array}$ where $\{y_1,\dots,y_r\} = \min Y$. Since for the short Differentiations VII_s and $\widehat{\mathrm{VII}}_s$ it holds obviously

$$\begin{aligned} &T(a)' = T(a), \quad T(a,Y)' = T(a,c)' = T(a,Y), \quad P(a)' = P(a), \\ &\widehat{T}(a)' = \widehat{T}(a), \quad \widehat{T}(a,Y)' = \widehat{T}(a,c)' = \widehat{T}(a,Y), \quad \widehat{P}(a)' = \widehat{P}(a)\,, \end{aligned} \tag{5.5}$$

we define the *reduced* objects of the category $\operatorname{rep}\mathcal{P}$ ($\operatorname{rep}\mathcal{P}'_{(a,b,c)}$) as those not containing direct summands $T(a,Y)$ and $T(a,c)$ (resp. $T(a,Y)$). Symmetrically, the *reduced* objects of the category $\operatorname{corep}\mathcal{P}$ ($\operatorname{corep}\mathcal{P}'_{(a,b,c)}$) are those without direct summands $\widehat{T}(a,Y)$ and $\widehat{T}(a,c)$ (resp. $\widehat{T}(a,Y)$).

The standard operations of integration $\uparrow$ and reduced differentiation $\downarrow$ can be defined analogously to the algorithms VII and $\widehat{\mathrm{VII}}$. Again by the matrix considerations similar to those used in the proofs of Theorem 3.5 in [10] and Theorem 4.1 above, one can establish the following main property of the short algorithm.

Theorem 5.7. *Let (a,b,c) be a VII_s-suitable triple of points of an equipped poset $\mathcal{P}$ and $\mathcal{P}'_{(a,b,c)} = (\mathcal{P}^s_{(a,b,c)}, \Sigma_{(a,b,c)})$ the corresponding derived equipped poset with relations. Then the short generalized differentiation functors $D_{(a,b,c)}$ and $\widehat{D}_{(a,b,c)}$ defined by (5.4) induce bijections between indecomposables*

$$\operatorname{Ind}\mathcal{P} \setminus [T(a,Y), T(a,c))] \rightleftarrows \operatorname{Ind}\mathcal{P}'_{(a,b,c)} \setminus [T(a,Y)]\,,$$

$$\widehat{\operatorname{Ind}}\,\mathcal{P} \setminus [\widehat{T}(a,Y), \widehat{T}(a,c))] \rightleftarrows \widehat{\operatorname{Ind}}\,\mathcal{P}'_{(a,b,c)} \setminus [\widehat{T}(a,Y)]\,,$$

realized by the corresponding operations $\downarrow$ and $\uparrow$.

Remark 5.8. In the short generalized case, one should expect that the differentiation functors VII_s and $\widehat{\mathrm{VII}}_s$ induce equivalences of the quotient categories

(a) $\operatorname{rep} \mathcal{P}/\langle T(a,Y), T(a,c)\rangle \xrightarrow{\sim} \operatorname{rep} \mathcal{P}'_{(a,b,c)}/\langle T(a,Y)\rangle$,

(b) $\operatorname{corep} \mathcal{P}/\langle \widehat{T}(a,Y), \widehat{T}(a,c)\rangle \xrightarrow{\sim} \operatorname{corep} \mathcal{P}'_{(a,b,c)}/\langle \widehat{T}(a,Y)\rangle$.

The proof may be carried out analogously to [12] (Section 7). In particular, in the case $Y = 0$ one should expect the equivalences of the quotient categories

(a') $\operatorname{rep} \mathcal{P}/\langle T(a), T(a,c)\rangle \xrightarrow{\sim} \operatorname{rep} \mathcal{P}'_{(a,b,c)}/\langle T(a)\rangle$,

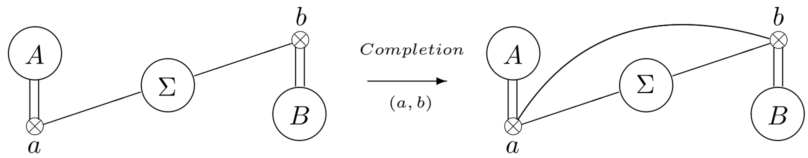
(b') $\operatorname{corep} \mathcal{P}/\langle \widehat{T}(a), \widehat{T}(a,c)\rangle \xrightarrow{\sim} \operatorname{corep} \mathcal{P}'_{(a,b,c)}/\langle \widehat{T}(a)\rangle$.

It is well known that, in the poset representation theory, the main reduction of representations (real differentiation) usually is combined with the subsequent easier operation of *completion* consisting in fact in deleting one simple object from the representation category. A full categorical description of the completion for ordinary posets was given in [12]. For representations of equipped posets, the completion was observed briefly in [10] (Section 4) only in the language of objects, without paying attention to morphisms. Below we give similar concise description (also in terms of objects) of the completion for corepresentations. Its more deep categorical properties concerning morphisms can be established analogously to [12] (Section 5).

6. **Completion for corepresentations**

Recall from [10] that a pair of weakly comparable points $a \prec b$ of an equipped poset $\mathcal{P}$ is called *special* if $\mathcal{P} = a^{\nabla} + b_{\vartriangle} + \{a \prec \Sigma \prec b\}$ where Σ is some subset of $\mathcal{P}$ (possibly empty). Note that automatically the set $\{a \prec \Sigma \prec b\}$ is completely weak. Set $A = a^{\nabla} \setminus a$, $B = b^{\nabla} \setminus b$.

The *completion* of $\mathcal{P}$ with respect to this pair (a,b) consists in *strengthening* the relation $a \prec b$, i.e. in converting it into a strong one $a \lhd b$. The resulting poset is denoted by $\overline{\mathcal{P}} = \overline{\mathcal{P}}_{(a,b)}$.



Obviously, the category $\operatorname{corep} \overline{\mathcal{P}}$ is a full subcategory of the category $\operatorname{corep} \mathcal{P}$ and moreover the following is true (compare with Lemma 4.1 in [10]).

Lemma 6.1. *The category* $\operatorname{corep} \overline{\mathcal{P}}_{(a,b)}$ *is a full subcategory of the category* $\operatorname{corep} \mathcal{P}$ *formed by the objects without trivial direct summands* $\widehat{T}(a)$. *In particular*

$$\widehat{\operatorname{Ind}}\ \overline{\mathcal{P}}_{(a,b)} = \widehat{\operatorname{Ind}}\ \mathcal{P} \setminus [\widehat{T}(a)].$$

Proof. Let $U \in \operatorname{corep} \mathcal{P} \setminus \operatorname{corep} \overline{\mathcal{P}}$, i.e. $\widetilde{U}_a \not\subset U_b$. Consider the matrix M of U and select in its lower part the horizontal stripe corresponding to the maximal

G-subspace U_b^G of the F-space U_b. You have zeroes above this stripe in the block M_B. Assume the matrix M to be chosen in such a way that the columns of the stripe M_b generate the whole space U_b. Reduce the upper parts of the matrices M_a, M_b (situated above the horizontal stripe U_b^G), as a representation of a completely weak chain $a \prec b$, to the canonical form in accordance with Lemma 3.1. You get there the blocks $\widehat{T}(a)$ and $\widehat{T}(b)$ only, otherwise the G-space U_b^G could be extended more.

$$
\begin{array}{r}
\\
M = \\
\\
\\
\end{array}
\begin{array}{cccccccc}
\multicolumn{2}{c}{a} & A & \Sigma & B & \multicolumn{2}{c}{b} & \\
\cline{1-7}
\multicolumn{1}{|c|}{I} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & L \\
\cline{1-7}
\multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{I} & \\
\cline{1-7}
\multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \\
\cline{1-7}
\multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} & U_b^G \\
\cline{1-7}
\end{array}
\tag{6.1}
$$

Since (by our assumption on the election of U_b) each column of the block M_Σ is an F-linear combination of columns of the stripes M_a and M_b, the block $M_\Sigma \cap L$ is over F and therefore can be annulled by column additions $M_a \xrightarrow{F} M_\Sigma$. Certainly, one can annul also the block $M_A \cap L$ (by column additions $M_a \xrightarrow{G} M_A$), as well as all the elements below the shown identity matrix I in M_a (by row transformations over G). Hence the matrix M takes the form (6.1) and evidently contains at least one trivial direct summand $\widehat{T}(a)$, i.e. U contains it. ☑

7. On some applications

In conclusion, a few words on possible applications. Recall from [10, 11] that the *evolvent* of an equipped poset $\mathcal{P}$ is an ordinary poset $E(\mathcal{P})$ obtained from $\mathcal{P}$ by doubling some its points and relations

$$E(\mathcal{P}) = \bigcup_{x \in \mathcal{P}} \{x', x''\}$$

where $x' = x'' = x$ for a strong point $x \in \mathcal{P}$ and $x' \neq x''$ is a pair of new incomparable strong points replacing each old weak point x, with the order relations defined by the rule

1) if $x \prec y$, then $x' < y'$ and $x'' < y''$;
2) if $x \lhd y$, then $x' < y'$; $x' < y''$; $x'' < y'$ and $x'' < y''$.

Many properties of the categories $\operatorname{rep} \mathcal{P}$ and $\operatorname{corep} \mathcal{P}$ can be expressed in terms of the evolvent $E(\mathcal{P})$. In particular, the following holds

(a) An equipped poset $\mathcal{P}$ is representation-finite (corepresentation-finite) if and only if the evolvent $E(\mathcal{P})$ is a representation-finite ordinary poset, i.e.

doesn't contain any of the Kleiner's critical subsets [4] (this criterion is a consequence of a more general result in [6] on representation-finite schurian vector space categories).

(b) In the case $(F, G) = (\mathbb{R}, \mathbb{C})$, an equipped poset $\mathcal{P}$ is representation one-parameter (of finite growth, tame) if and only if the evolvent $E(\mathcal{P})$ is an ordinary poset of the same type, i.e. also doesn't contain some well known special subsets (these criteria were proved in [9],[10] and [11] respectively).

The Differentiation VII and Completion reduction pair works effectively and is sufficient in the representation finite and one-parameter cases (the same is true for the pair $\widehat{\mathrm{VII}}$ and Completion for corepresentations). It allows not only to give logically transparent and relatively short proofs of the corresponding criteria but also classify all the indecomposables.

In particular, the criterion (a) was reproved rather briefly by differentiation: for representations when preparing the paper [9], and recently for corepresentations in [8]. One can find also in [9] and [8] the complete matrix lists of indecomposable representations and corepresentations respectively of non-trivially equipped posets of finite type. The combined list is shown in Appendix B below, completing the Kleiner's matrix list [5] of indecomposables of ordinary representation-finite posets to the equipped case. Though in principle this combined list can be also extracted from [6] (where it is contained in a less evident form), in practice our form of presentation is indispensable for certain matrix calculations.

In [9], there is given also a full matrix classification of indecomposable representations of one-parameter non-trivially equipped posets (over the pair $(\mathbb{R}, \mathbb{C})$), obtained on the base of using some algorithm which is reduced to Differentiation VII and Completion. Certainly, the same can be done for corepresentations, using Differentiation $\widehat{\mathrm{VII}}$ and Completion for corepresentations. We place in Appendix A below the series of indecomposable representations and corepresentations for the not ordinary critical one-parameter equipped posets, which may be found useful in various considerations.

Among other fields of applications, one can mention the categorical description of differentiation algorithms. For instance, the results of Section 5 (with the source in [12], Section 7) give a nice opportunity to establish in a clear way the main categorical properties of the algorithms VII and $\widehat{\mathrm{VII}}$ (see in particular Remarks 4.2, 5.3, 5.8 and Proposition 5.5).

We hope, further developing of the corepresentation differentiation technic and establishing relationships between objects of the categories $\operatorname{rep} \mathcal{P}$ and $\operatorname{corep} \mathcal{P}$ will allow to solve the remaining open problems in the tame corepresentation situation.

Notations for Appendices. With an equipped poset $\mathcal{P}$ there are associated the *Tits quadratic form* $f = f_{\mathcal{P}}$ and *coform* $\widehat{f} = \widehat{f}_{\mathcal{P}}$ given on a vector $d = (d_0, d_x : x \in \mathcal{P})$ by

$$f(d) = d_0^2 + \sum_{x\in\mathcal{P}} l_x d_x^2 + \sum_{x<y} l_{xy} d_x d_y - d_0 \sum_{x\in\mathcal{P}} l_x d_x\,,$$

where $l_x = 1$ ($l_x = 2$) if x is strong (weak), $l_{xy} = l_x l_y$ ($l_{xy} = 2$) if $x \lhd y$ ($x \prec y$), and

$$\widehat{f}(d) = 2d_0^2 + \sum_{x\in\mathcal{P}} \widehat{l}_x d_x^2 + \sum_{x<y} \widehat{l}_{xy} d_x d_y - 2d_0 \sum_{x\in\mathcal{P}} d_x\,,$$

where $\widehat{l}_x = 2$ ($\widehat{l}_x = 1$) if x is strong (weak), $\widehat{l}_{xy} = 2$ ($\widehat{l}_{xy} = 1$) if $x \lhd y$ ($x \prec y$).

In Appendix B, for each of the listed indecomposable representation or corepresentation of dimension $d = (d_0, d_1, \dots, d_n)$, the values $f = f(d)$ and $\widehat{f} = \widehat{f}(d)$ are shown.

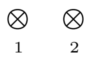
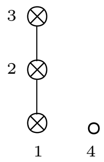
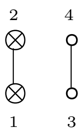
The second column N contains the numbers of indecomposable representations or corepresentations (which coincide) for each of the considered sincere non-trivially equipped posets of finite type $F_{13} - F_{18}$ (the notations for posets coincide with those in [11], Table 1).

The presented in Appendix A infinite series of matrix representations and corepresentations of the critical equipped posets $K_6, \dots, K_9$ cover almost all indecomposables of any shown dimension d when the square matrix blocks X, Y (of arbitrary order $n \geq 1$) run through the following families of matrices:

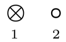
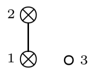
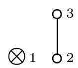
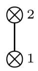
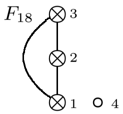
(a) X is an arbitrary indecomposable (under ordinary similarity) Frobenius Canonical Form block over F (in other terminology, Rational Canonical Form block). It holds $X \sim X' \;\Leftrightarrow\; M(X) \sim M(X')$ where $M(X)$ is the shown matrix depending on X. This property holds because the series containing X are reduced to the ordinary Kronecker pencil problem over F.

(b) Y also is an arbitrary indecomposable Frobenius Canonical Form block over F, but here for non-equivalent blocks Y and Y' it may happen $M(Y) \sim M(Y')$. Whether this really happens, depends on the pair (F, G). For instance, if $(F, G) = (\mathbb{R}, \mathbb{C})$, the series is reduced to the case $|\det Y| \leq 1$ and under this convention $Y \sim Y' \;\Leftrightarrow\; M(Y) \sim M(Y')$. The last property follows from the fact that the $(\mathbb{R}, \mathbb{C})$-series containing Y are reduced to those of the set K_6, for which the property was established in [9] using the result of [2].

Appendix A. Series of indecomposables of critical equipped posets

Critical posets	Representation series	Corepresentation series
K_6 	$\begin{array}{cc} 1 & 2 \\ \hline \multicolumn{1}{\|c\|}{I} & \multicolumn{1}{c\|}{I} \\ \hline \multicolumn{1}{\|c\|}{uI} & \multicolumn{1}{c\|}{uY} \\ \hline \end{array}$ $d = (2n; n, n)$	$\begin{array}{cc} 1 & 2 \\ \hline \multicolumn{1}{\|c\|}{I} & \multicolumn{1}{c\|}{uI + X} \\ \hline \end{array}$ $d = (n; n, n)$
K_7 	$\begin{array}{\|c\|c\|c\|c\|c\|} \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{4}} \\ \hline I & & & & \\ \hline uI & I & & & \\ \hline & uI & I & I & \\ \hline & Y & uI & & I \\ \hline \end{array}$ $d = (4n; n, n, n, 2n)$	$\begin{array}{\|c\|c\|c\|c\|} \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} \\ \hline I & X & I & I \\ \hline I & uI & & \\ \hline \end{array}$ $d = (2n; n, n, n, n)$
K_8 	$\begin{array}{\|c\|c\|c\|} \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} \\ \hline I & I & \\ \hline uI & X & I \\ \hline \end{array}$ $d = (2n; n, n, n)$	$\begin{array}{\|c\|c\|c\|c\|} \multicolumn{2}{c}{\overbrace{\qquad\quad}^{1}} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} \\ \hline I & uI & I & \\ \hline I & uY & & I \\ \hline \end{array}$ $d = (2n; 2n, n, n)$
K_9 	$\begin{array}{\|c\|c\|c\|c\|} \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} \\ \hline I & & & \\ \hline uI & uI & I & I \\ \hline & X & I & \\ \hline \end{array}$ $d = (3n; n, n, n, n)$	$\begin{array}{\|c\|c\|c\|c\|c\|c\|} \multicolumn{2}{c}{\overbrace{\qquad\quad}^{1}} & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{2}} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} \\ \hline & & I & uI & I & \\ \hline & I & uI & Y & & I \\ \hline I & uI & & & & \\ \hline \end{array}$ $d = (3n; 2n, 2n, n, n)$

Appendix B. Indecomposables of representation-finite equipped posets

Poset	N	Representations $f=1$	Representations $f=2$	Corepresentations $\widehat{f}=1$	Corepresentations $\widehat{f}=2$
F_{13} 	2	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline u\\ 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	$\begin{array}{\|cc\|}\hline 1 & u\\\hline\end{array}$
F_{14} 	3	$\begin{array}{\|c\|c\|}\hline 1 & 1\\\hline\end{array}$ $\begin{array}{\|c\|c\|}\hline u & 1\\ 1 & 1\\\hline\end{array}$	$\begin{array}{\|c\|cc\|}\hline u & 1 & 0\\ 1 & 0 & 1\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 1\\\hline\end{array}$	$\begin{array}{\|cc\|c\|}\hline 1 & u & 1\\\hline\end{array}$ $\begin{array}{\|cc\|c\|}\hline 1 & 0 & 1\\ 1 & u & 0\\\hline\end{array}$
F_{15} 	7	$\begin{array}{\|c\|c\|c\|}\hline u & 0 & 0\\ 1 & 1 & 1\\\hline\end{array}$ $\begin{array}{\|c\|c\|c\|}\hline u & 0 & 0\\ 1 & u & 0\\ 0 & 1 & 1\\\hline\end{array}$ $\begin{array}{\|c\|c\|cc\|}\hline u & 0 & 0 & 0\\ 1 & u & 1 & 0\\ 0 & 1 & 0 & 1\\\hline\end{array}$	$\begin{array}{\|c\|c\|cc\|}\hline u & 0 & 1 & 0\\ 1 & 1 & 0 & 1\\\hline\end{array}$ $\begin{array}{\|c\|c\|cc\|}\hline u & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & u & 1 & 0\\ 0 & 1 & 0 & 1\\\hline\end{array}$ $\begin{array}{\|c\|cc\|cc\|}\hline u & 0 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 0\\ 0 & u & 0 & 1 & 0\\ 0 & 1 & 1 & 0 & 1\\\hline\end{array}$ $\begin{array}{\|cc\|c\|cc\|}\hline u & 0 & 0 & 0 & 0\\ 0 & u & 0 & 0 & 0\\ 1 & 0 & u & 1 & 0\\ 0 & 1 & 1 & 0 & 1\\\hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 0 & u\\ 0 & 1 & 1\\\hline\end{array}$ $\begin{array}{\|c\|c\|c\|}\hline 1 & u & 1\\\hline\end{array}$ $\begin{array}{\|cc\|c\|c\|}\hline 1 & 0 & u & u\\ 0 & 1 & 0 & 1\\\hline\end{array}$ $\begin{array}{\|c\|cc\|c\|}\hline 0 & 1 & u & 1\\ 1 & 0 & 0 & 1\\\hline\end{array}$	$\begin{array}{\|cc\|cc\|c\|}\hline 1 & 0 & u & 0 & u\\ 0 & 1 & 0 & u & 1\\\hline\end{array}$ $\begin{array}{\|cc\|cc\|c\|}\hline 0 & 0 & 1 & u & 1\\ 1 & 0 & 0 & 0 & u\\ 0 & 1 & 0 & 0 & 1\\\hline\end{array}$ $\begin{array}{\|cc\|cc\|cc\|}\hline 0 & 0 & 1 & u & 1 & 0\\ 1 & 0 & 0 & 0 & 0 & u\\ 0 & 1 & 0 & 0 & 1 & 1\\\hline\end{array}$
F_{16} 	1	$\begin{array}{\|c\|c\|c\|}\hline u & 1 & 0\\ 1 & 0 & 1\\\hline\end{array}$			$\begin{array}{\|cc\|c\|c\|}\hline 1 & 0 & u & 1\\ 0 & 1 & 1 & 0\\\hline\end{array}$
F_{17} 	1		$\begin{array}{\|c\|c\|}\hline u & 0\\ 1 & 1\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & u\\\hline\end{array}$	
F_{18} 	1		$\begin{array}{\|c\|c\|c\|cc\|}\hline u & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 1 & 0\\ 0 & u & 0 & 0 & 1\\\hline\end{array}$	$\begin{array}{\|c\|c\|c\|c\|}\hline 1 & 0 & 0 & 1\\ 0 & 1 & u & u\\\hline\end{array}$	

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