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Conservation laws III: relaxation limit

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Abstract. In this paper, we apply the invariant region theory [1] and the compensated compactness method [2] to study the singular limits of stiff relaxation and dominant diffusion for the Cauchy problem of a system of quadratic flux and the Le Roux system, and obtain the convergence of the solutions to the equilibrium states of these systems.

Keywords and phrases. Relaxation limit, Dominant diffusion, Entropy-entropy flux, Equilibrium states.

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Resumen. En este artículo aplicamos la teoría de la región invariante [1] y el método de la compactificación compensada [2] para estudiar los límites singulares de la relajación rígida y difusión dominante para el problema de Cauchy de un sistema de flujo cuadrático y el sistema Roux obteniendo la convergencia de las soluciones para el estado de equilibrio de esos sistemas.

1. Introduction

We are concerned with singular limits of stiff relaxation and dominant diffusion for the Cauchy problem of two special quasilinear conservation laws with relaxation and diffusion: one is related to a system of quadratic flux

$$\begin{cases} u_t + \frac{3}{2}(u^2+v^2)_x + \frac{u-h(v)}{\tau} = \varepsilon u_{xx}, \\ v_t + (uv)_x = \varepsilon v_{xx} \end{cases} \tag{1.1}$$

with bounded measurable initial data

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)); \tag{1.2}$$

the other one is related to the Le Roux system:

$$\begin{cases} u_t + (u^2+v)_x + \frac{u-h(v)}{\tau} = \varepsilon u_{xx}, \\ v_t + (uv)_x = \varepsilon v_{xx} \end{cases} \tag{1.3}$$

with bounded measurable initial data

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x) + \varepsilon,\ v_0(x) \geq 0. \tag{1.4}$$

In this paper, we only consider the case of stiff relaxation and dominant diffusion, that is, $\tau = o(\varepsilon)$ as $\varepsilon \to 0$ (see [3]). We will show that the solutions of the Cauchy problem (1.1)-(1.2), (1.3)-(1.4) are uniformly bounded in L^∞ by the invariant region theory, and the relaxation limits are always stable and no oscillation arises.

2. The relaxation system of quadratic flux

The relaxation system of quadratic flux is described by

$$\begin{cases} u_t + \frac{1}{2}(u^2+v^2)_x + \frac{u-h(v)}{\tau} = \varepsilon u_{xx}, \\ v_t + (uv)_x = \varepsilon v_{xx}. \end{cases} \tag{2.1}$$

By simple calculations, the two eigenvalues of system (2.1) are

$$\lambda_1 = 2u - s^{\frac{1}{2}}, \quad \lambda_2 = 2u + s^{\frac{1}{2}}$$

and the two Riemann invariants are

$$w(u,v) = u + s^{\frac{1}{2}}, \quad z(u,v) = u - s^{\frac{1}{2}}.$$

Hereafter $s = u^2 + v^2$.

In this section, we use the compensated compactness method and the invariant region theory to study the Cauchy problem (1.1)-(1.2) and get the following theorem.

Theorem 2.1. *Let $\tau = o(\varepsilon)$ as $\varepsilon \to 0$, $h(v) \in C^2(R)$ and meas$\{v : g''(v) = 0\} = 0$, where $g(v) = vh(v)$. Suppose that there exists a region*

$$\Sigma_1 = \{(u,v) : w(u,v) \leq N, \quad z(u,v) \geq -L\}$$

for some $N,\ L > 0$, such that the curve $u = h(v)$ and the initial data $(u_0(x), v_0(x))$ are inside the region Σ_1, and $u = h(v)$ passes the two intersections (v_1, u_1) and (v_2, u_2) of the curves $w = N$ and $z = -L$ (see Figure 1). Then, the solutions

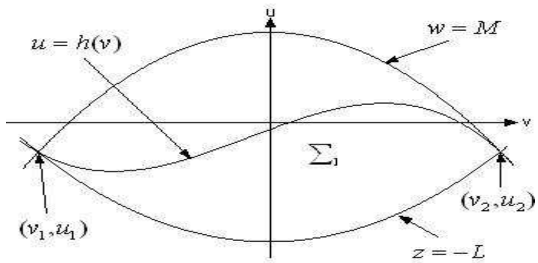

FIGURE 1

$(u^\varepsilon, v^\varepsilon) = (u^{\varepsilon,\tau(\varepsilon)},\ v^{\varepsilon,\tau(\varepsilon)})$ *of the Cauchy problem (1.1)-(1.2) satisfying*

$$|u^\varepsilon(x,t)| \le M, |v^\varepsilon(x,t)| \le M, \quad (x,t) \in R \times R^+,$$

where M *is independent of* ε*. Moreover, there exists a subsequence (still labeled)* $(u^\varepsilon, v^\varepsilon)$ *converging strongly to the functions* (u,v) *as* $\varepsilon \to 0$*, which are the equilibrium states uniquely determined by* $(E_1) - (E_2)$*:*

(E_1) $u(x,t) = h(v(x,t))$*, for almost all* $(x,t) \in R \times R^+$*;*

(E_2) $v(x,t)$ *is the* L^∞ *entropy solution of the Cauchy problem*

$$v_t + (vh(v))_x = 0, \quad v(x,0) = v_0(x).$$

Proof. By simple calculations, we have

$$w_u = 1 + \frac{u}{\sqrt{s}},\ w_v = \frac{v}{\sqrt{s}},\ w_{uu} = \frac{v^2}{s^{\frac{3}{2}}},\ w_{uv} = -\frac{uv}{s^{\frac{3}{2}}},\ w_{vv} = \frac{u^2}{s^{\frac{3}{2}}};$$

$$z_u = 1 - \frac{u}{\sqrt{s}},\ z_v = -\frac{v}{\sqrt{s}},\ z_{uu} = -\frac{v^2}{s^{\frac{3}{2}}},\ z_{uv} = \frac{uv}{s^{\frac{3}{2}}},\ z_{vv} = -\frac{u^2}{s^{\frac{3}{2}}}.$$

Then

$$w_{uu} \ge 0,\ w_{vv} \ge 0,\ w_{uu}w_{vv} - w_{uv}^2 \ge 0;$$
$$z_{uu} \le 0, \quad z_{vv} \le 0, \quad z_{uu}z_{vv} - z_{uv}^2 \ge 0,$$

which implies that both $w(u,v)$ and $-z(u,v)$ are convex. Thus

$$w_{uu}u_x^2 + 2w_{uv}u_xv_x + w_{vv}v_x^2 \ge 0, \quad z_{uu}u_x^2 + 2z_{uv}u_xv_x + z_{vv}v_x^2 \le 0.$$

Multiplying the first equation in system (1.1) by w_u and the second by w_v, then adding the result, we obtain

$$w_t + \lambda_2 w_x + w_u \frac{u - h(v)}{\tau} = \varepsilon w_{xx} - \varepsilon(w_{uu}u_x^2 + 2w_{uv}u_xv_x + w_{vv}v_x^2).$$

and hence

$$w_t + \lambda_2 w_x + w_u \frac{u - h(v)}{\tau} \le \varepsilon w_{xx};$$

similarly,

$$z_t + \lambda_1 z_x + z_u \frac{u - h(v)}{\tau} \ge \varepsilon z_{xx}.$$

If the curve $u = h(v)$ passes the two intersections points$(v_1, u_1),\ (v_2, u_2)$ of the curves $w = N,\ z = -L$ and is above the curve $z = -L$ and below the curve $w = N$ as $v_1 \le v \le v_2$, then it is easy to check that on the intersection of $\partial\Sigma_2$ and the curve $w(u,v) = N$, $w_u \frac{u-h(v)}{\tau} \ge 0$; on the intersection of $\partial\Sigma_2$ and the curve $z(u,v) = -L$, $z_u \frac{u-h(v)}{\tau} \le 0$. This shows that the region $\Sigma_1 = \{(u,v) : w(u,v) \le N, \ \ z(u,v) \ge -L\}$ is an invariant region by the Theorem 4.4 of [1]. Thus we get the estimates

$$|u^\varepsilon(x,t)| \le M, \quad |v^\varepsilon(x,t)| \le M$$

for a suitable positive constant M, which is independent of ε. Hence there exists a subsequence (still labeled) $(u^\varepsilon, v^\varepsilon)$ such that

$$w^\star - \lim(u^\varepsilon(x,t),\ v^\varepsilon(x,t)) = (u(x,t), v(x,t)),$$

where $w^\star-\lim$ denotes the weak-star limit.

We now prove that $\varepsilon(u_x^\varepsilon)^2,\ \varepsilon(u_x^\varepsilon)^2$ and $\frac{(u^\varepsilon-h(v^\varepsilon))^2}{\tau}$ are bounded in L^1_{loc}. For simplicity, we will drop the superscript ε.

Since (u,v) is bounded, we can choose a large constant C_1 such that the function $p(u,v)=\frac{u^2}{2}-h(v)u+\frac{C_1v^2}{2}$ satisfies

$$p_{uu}u_x^2+2p_{uv}u_xv_x+p_{vv}v_x^2\geq C_2(u_x^2+v_x^2) \tag{2.2}$$

for some constant $C_2>0$.

Multiplying system (1.1) by (p_u,p_v), we have from (2.2) that

$$\begin{aligned}p(u,v)_t+p_u(u,v)(\frac{3}{2}u^2+\frac{1}{2}v^2)_x+p_v(u,v)(uv)_x+\frac{(u-h(v))^2}{\tau}\leq\\ \varepsilon[p_{xx}(u,v)-C_2(u_x^2+v_x^2)].\end{aligned} \tag{2.3}$$

Direct calculations show that

$$\begin{aligned}&p_u(u,v)(\tfrac{3}{2}u^2+\tfrac{1}{2}v^2)_x\\ &\quad=\tfrac{3}{2}(p_u(u,v)(u^2-h^2(v)))_x+p_u(h(v),v)(\tfrac{3}{2}h^2(v)+\tfrac{1}{2}v^2)_x\\ &\qquad-\tfrac{3}{2}p_{ux}(u,v)(u^2-h^2(v))+(p_u(u,v)-p_u(h(v),v))(\tfrac{3}{2}h^2(v)+\tfrac{1}{2}v^2)_x\\ &\quad=(\tfrac{3}{2}p_u(u,v)(u^2-h^2(v))+\textstyle\int^v p_u(h(s),s)(3h(s)h'(s)+s)ds)_x\\ &\qquad-\tfrac{3}{2}(p_{uu}u_x+p_{uv}v_x)(u+h(v))(u-h(v))\\ &\qquad\quad+p_{uu}(\beta_1,v)(u-h(v))(3h(v)h'(v)+v)v_x\\ &\quad=T_1+T_2+T_3,\end{aligned}$$

and

$$\begin{aligned}&p_v(u,v)(uv)_x\\ &\quad=(p_v(u,v)v(u-h(v))+\textstyle\int^v p_v(h(s),s)(h(s)+h'(s)s)ds)_x\\ &\qquad-(p_{uv}u_x+p_{vv}v_x)v(u-h(v))+p_{uv}(\beta_2,v)(u-h(v))(h(v)+h'(v)v)v_x\\ &\quad=\tilde{T}_1+\tilde{T}_2+\tilde{T}_3,\end{aligned}$$

where $p_v(h(v),v)=p_v(u,v)|_{u=h(v)}$ and β_1,β_2 take values between u and $h(v)$. Using the elementary inequality $\delta a^2+\frac{b^2}{4\delta}\geq|ab|$ $(\delta>0)$ and noticing that (u,v) is bounded, we have

$$|T_2|\leq C\sqrt{\tau}(|u_x|+|v_x|)\frac{|u-h(v)|}{\sqrt{\tau}}\leq\delta\frac{(u-h(v))^2}{\tau}+C_1(\delta)\tau(u_x^2+v_x^2). \tag{2.4}$$

Similarly,

$$|\tilde{T}_2|\leq\delta\frac{(u-h(v))^2}{\tau}+C_2(\delta)\tau(u_x^2+v_x^2), \tag{2.5}$$

and

$$|T_3|\leq\delta\frac{(u-h(v))^2}{\tau}+C_3(\delta)\tau v_x^2,\quad|\tilde{T}_3|\leq\delta\frac{(u-h(v))^2}{\tau}+C_4(\delta)\tau v_x^2. \tag{2.6}$$

Let $q(u,v)_x=T_1+\tilde{T}_1,\ \delta=\frac{1}{8}$. It follows from (2.3)-(2.6) that

$$p(u,v)_t+q(u,v)_x+\frac{1}{2}\frac{(u-h(v))^2}{\tau}+(\varepsilon C_2-\tau C_3)(u_x^2+v_x^2)\leq\varepsilon p_{xx}(u,v) \tag{2.7}$$

for a positive constant C_3 depending on the bounds of second derivatives of $p(u,v)$.

Since $\tau = o(\varepsilon)$ as $\varepsilon \to 0$, $2\tau C_3 \le \varepsilon C_2$ if ε is sufficiently small. Let $K \subset R \times R^+$ be an arbitrary compact set. Choose $\phi \in C_0^\infty(R \times R^+)$ such that $\phi_K = 1, 0 \le \phi \le 1$ and write $S = \text{supp}\phi$. Then, multiplying (2.7) by ϕ and integrating by parts over $R \times R^+$, we get

$$\begin{aligned}&\int_0^\infty \int_{-\infty}^\infty \frac{C_2}{2}\varepsilon(u_x^2+v_x^2)\phi + \frac{(u-h(v))^2}{\tau}\phi dxdt\\ \le &\int_0^\infty \int_{-\infty}^\infty p\phi_t + q\phi_x + \varepsilon p\phi_{xx} dxdt \le M(\phi),\end{aligned}$$

that is, εu_x^2, εu_x^2 and $\frac{(u-h(v))^2}{\tau}$ are bounded in L^1_{loc}.

Next, we verify the compactness of $\eta(v)_t + q(v)_x$ in H^{-1} for any entropy-entropy flux pair $(\eta(v), q(v))$ of the scalar equation

$$v_t + (h(v)v)_x = 0. \tag{2.8}$$

We rewrite the second equation in (1.1) as follows:

$$v_t + (h(v)v)_x = \varepsilon v_{xx} + ((h(v)v) - (uv))_x. \tag{2.9}$$

Let $(\eta(v), q(v))$ be any entropy-entropy flux pair of (2.8). Then, multiplying (2.9) by $\eta'(v)$, we have

$$\begin{aligned}\eta(v)_t + q(v)_x &= -\eta'(v)((uv) - (h(v)v))_x + \varepsilon\eta'(v)v_{xx}\\ &= -(\eta'(v)v(u-h(v)))_x + \varepsilon\eta(v)_{xx}\\ &\quad + v\eta''(v)(u-h(v))v_x - \eta''(v)v_x^2.\end{aligned} \tag{2.10}$$

In view of the boundedness of $\varepsilon(u_x^\varepsilon)^2$, $\varepsilon(u_x^\varepsilon)^2$ and $\frac{(u^\varepsilon - h(v^\varepsilon))^2}{\tau}$ in L^1_{loc}, we obtain

$$\int_\Omega |v\eta''(v)(u-h(v)v_x| dxdt \le c\left(\int_\Omega \frac{(u-h(v))^2}{\tau} dxdt\right)^{\frac12} \left(\int_\Omega \tau v_x^2 dxdt\right)^{\frac12} \to 0,$$

$$\begin{aligned}\left|\int_\Omega (\eta'(v)v(u-h(v)))_x \Phi dxdt\right| &= \left|\int_\Omega \eta'(v)v(u-h(v))\Phi_x dxdt\right|\\ &\le c\left(\int_\Omega \tau\Phi_x^2 dxdt\right)^{\frac12} \left(\int_\Omega \frac{(u-h(v))^2}{\tau} dxdt\right)^{\frac12} \to 0;\end{aligned}$$

and

$$\begin{aligned}\left|\int_\Omega (\varepsilon\eta(v)_{xx})\Phi dxdt\right| &= \left|\int_\Omega (\varepsilon^{\frac12}\eta'(v)v_x)(\varepsilon^{\frac12}\Phi_x) dxdt\right|\\ &\le c\left(\int_\Omega \varepsilon v_x^2)^{\frac12}(\int_\Omega \varepsilon\Phi_x^2\right)^{\frac12} \to 0\end{aligned}$$

as $\varepsilon \to 0$, where $\Omega \subset R \times R^+$ is any bounded open set and $\Phi \in H_0^1(R \times R^+)$. Moreover, since $\varepsilon\eta'' v_x^2$ is bounded in L^1_{loc}, the right-hand side of (2.10) is compact in $W^{-1,q}$ for a constant $q \in (1,2)$. Noticing that the left-hand side of (2.10) is bounded in $W^{-1,\infty}$, we have from the Murat lemma [5] that $\eta(v^\varepsilon)_t + q(v^\varepsilon)_x$ is compact in H^{-1}_{loc} with respect to the viscosity solution v^ε.

Finally, we use the compactness framework [4] about the scalar equation to show that v^ε converge to the weak solution v of (2.8) almost everywhere. Let $(\eta_1(\theta), q_1(\theta)) = (\theta - k, g(\theta) - g(k))$, $(\eta_2(\theta), q_2(\theta)) = (g(\theta) - g(k), \int_k^\theta (g'(s))^2 ds)$, where k is an arbitrary constant, $g(\theta) = \theta h(\theta)$. Then using the Tartar-Murat Lemma(see[5-7]), we have

$$\overline{\eta_1(v^\varepsilon) q_2(v^\varepsilon)} - \overline{\eta_2(v^\varepsilon) q_1(v^\varepsilon)} = \overline{\eta_1(v^\varepsilon)}\ \overline{q_2(v^\varepsilon)} - \overline{\eta_2(v^\varepsilon)}\ \overline{q_1(v^\varepsilon)}. \tag{2.11}$$

Here and below we use the notation $\overline{\eta(v^\varepsilon)} = w^\star - \lim \eta(v^\varepsilon)$, $\overline{q(v^\varepsilon)} = w^\star - \lim q(v^\varepsilon)$.

By simple calculations, we have from the equality (2.11)

$$\overline{(v^\varepsilon - v)\int_v^{v^\varepsilon} (g'(s))^2 ds - (g(v^\varepsilon) - g(v))^2} + (\overline{g(v^\varepsilon) - g(v)})^2 = 0. \tag{2.12}$$

Since both terms in the left-hand side of (2.12) are nonnegative, we get $g(v) = \overline{g(v^\varepsilon)}$ and

$$\overline{(v^\varepsilon - v)\int_v^{v^\varepsilon} (g'(s))^2 ds - (g(v^\varepsilon) - g(v))^2} = 0. \tag{2.13}$$

Thus for any bounded open set $\Omega \subset R \times R^+$, we have

$$\lim_{\varepsilon \to 0} \int_\Omega (v^\varepsilon - v)\int_v^{v^\varepsilon} (g'(s))^2 ds - (g(v^\varepsilon) - g(v))^2 dxdt = 0$$

and hence

$$\lim_{\varepsilon \to 0} \int_{\Omega(|v^\varepsilon - v| > \alpha)} (v^\varepsilon - v)\int_v^{v^\varepsilon} (g'(s))^2 ds - (g(v^\varepsilon) - g(v))^2 dxdt = 0.$$

Since

$$\frac{d}{d\theta}((\theta - v)\int_v^\theta (g'(s))^2 ds - (\theta) - g(v))^2 = \int_v^\theta (g'(\theta) - g'(s))^2 ds$$

and if $g''(v) \neq 0$, a.e., then

$$\int_{\Omega(v^\varepsilon - v > \alpha)} (v^\varepsilon - v)\int_v^{v^\varepsilon} (g'(s))^2 ds - (g(v^\varepsilon) - g(v))^2 dxdt \geq C_\alpha \text{meas}(\Omega(v^\varepsilon - v > \alpha))$$

and

$$\int_{\Omega(v^\varepsilon - v < -\alpha)} (v^\varepsilon - v)\int_v^{v^\varepsilon} (g'(s))^2 ds - (g(v^\varepsilon) - g(v))^2 dxdt$$
$$\geq C_\alpha \text{meas}(\Omega(v^\varepsilon - v < -\alpha))$$

for a suitable positive constant C_α, which is independent of ε. Therefore for any given $\alpha > 0$, we have

$$\lim_{\varepsilon \to 0} \text{meas}(\Omega(|v^\varepsilon - v| > \alpha)) = 0,$$

i.e., v^ε converge in measure to v which satisfies E_2. This implies the pointwise convergence of a subsequence (still denoted by) v^ε. Because $\frac{(u^\varepsilon - h(v^\varepsilon))^2}{\tau} \in L^1_{loc}$, we obtain that for any compact set $K \subset R \times R^+$,

$$\int\int_K (u^\varepsilon - h(v^\varepsilon))^2 dxdt \to 0, \quad \text{as } \varepsilon \to 0,$$

which implies that there is a subsequence u^ε converging to $u = h(v)$ almost everywhere. So we end the proof of the theorem. ☑

3. The Le Roux System with Relaxation

Adding a relaxation term to the Le Roux system, we get

$$\begin{cases} u_t + (u^2 + v)_x + \frac{u - h(v)}{\tau} = \varepsilon u_{xx}, \\ v_t + (uv)_x = \varepsilon v_{xx}. \end{cases} \tag{3.1}$$

By simple calculations, the two eigenvalues of system (3.1) are

$$\lambda_1 = \frac{3u}{2} - \frac{D}{2}, \quad \lambda_2 = \frac{3u}{2} + \frac{D}{2}$$

and the two Riemann invariants are

$$W(u,v) = u + D, \quad Z(u,v) = u - D.$$

Here and below $D = \sqrt{u^2 + 4v}$.

The main result in this section is given as follows:

Theorem 3.1. *Let $\tau = o(\varepsilon)$ as $\varepsilon \to 0$, $h(v) \in C^2(R)$ and meas$\{v : g''(v) = 0\} = 0$, where $g(v) = vh(v)$. Suppose that there exists a region*

$$\Sigma_2 = \{(u,v) : W(u,v) \le N, Z(u,v) \ge -L, \quad v \ge 0\}$$

for some N, $L > 0$ such that the curve $u = h(v)$ and the initial data $(u_0(x), v_0(x) + \varepsilon)$ are inside the region Σ_2, and $u = h(v)$ passes $(0, h(0))$ and the intersection $(\bar{v}, \bar{u})$ of the curves $w = N$ and $z = -L$ (see Figure 2). Then, the solutions

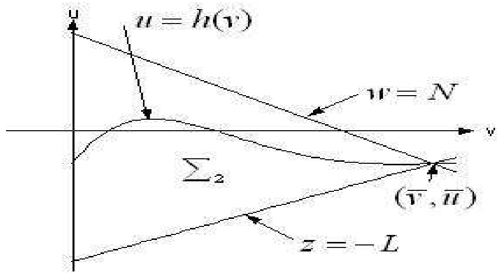

FIGURE 2

$(u^\varepsilon, v^\varepsilon) = (u^{\varepsilon,\tau(\varepsilon)}, \ v^{\varepsilon,\tau(\varepsilon)})$ *of the Cauchy problem (1.3)-(1.4) satisfying*

$$|u^\varepsilon(x,t)| \le M, |v^\varepsilon(x,t)| \le M, \quad (x,t) \in R \times R^+,$$

where M is independent of ε. Moreover, there exists a subsequence (still labeled) $(u^\varepsilon, v^\varepsilon)$ converging strongly to the functions (u, v) as $\varepsilon \to 0$, which are the equilibrium states uniquely determined by $(E_1) - (E_2)$:

(E_1) $u(x,t) = h(v(x,t))$, for almost all $(x,t) \in R \times R^+$;

(E_2) $v(x,t)$ is the L^∞ entropy solution of the Cauchy problem

$$v_t + (vh(v))_x = 0, \quad v(x,0) = v_0(x).$$

To prove the theorem, we need the following lemma.

Lemma 3.1. *Let $(u^\varepsilon, v^\varepsilon) \in C^\infty(R \times (0,T])$ be the local solution of the Cauchy problem (1.3)-(1.4). Then $v^\varepsilon(x,t) > 0, \quad (x,t) \in R \times (0,T]$.*

Proof. We rewrite the second equation in system (1.3) as

$$w_t + uw_x + u_x u = \varepsilon(w_{xx} + w_x^2), \tag{3.2}$$

where $w = \log v$. Then

$$w_t = \varepsilon w_{xx} + \varepsilon(w_x - \frac{u}{2\varepsilon})^2 - u_x - \frac{u^2}{4\varepsilon}.$$

The solution w of (3.2) with initial data $w_0(x) = \log(v_0(x) + \varepsilon)$ can be represented by a Green function $G^\varepsilon(x-y,t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \exp\{-\frac{(x-y)^2}{4\varepsilon t}\}$:

$$\begin{aligned} w = &\int_{-\infty}^{\infty} G^\varepsilon(x-y,t) w_0(y) dy \\ &+ \int_0^t \int_{-\infty}^{\infty} [\varepsilon(w_x - \tfrac{u}{2\varepsilon})^2 - u_x - \tfrac{u^2}{4\varepsilon}] G^\varepsilon(x-y, t-s) dyds. \end{aligned} \tag{3.3}$$

Since

$$\int_{-\infty}^{\infty} G^\varepsilon(x-\xi,t) d\xi = 1, \ \int_0^t \int_{-\infty}^{+\infty} |G_y^\varepsilon(x-y,t-s)| dyds = 2\sqrt{\frac{t}{\pi\varepsilon}} \ (t > 0),$$

it follows from (3.3) that

$$\begin{aligned} w &\geq \log\varepsilon + \int_0^t \int_{-\infty}^{\infty} (-u_x - \tfrac{u^2}{4\varepsilon}) G(x-y,t-s) dyds \\ &= \log\varepsilon + \int_0^t \int_{-\infty}^{\infty} (uG_y(x-y,t-s) - \tfrac{u^2}{4\varepsilon} G(x-y,t-s)) dyds \\ &\geq \log\varepsilon - 2K\sqrt{\tfrac{t}{\pi\varepsilon}} - K_1 t \geq -C(t,\delta,\varepsilon) > -\infty. \end{aligned}$$

Thus $v^\varepsilon(x,t)$ has a positive lower bound $c(t,\varepsilon)$. ☑

Proof of Theorem 3.1. By simple calculation, we have

$$W_u = 1 + \frac{u}{D}, W_v = \frac{2}{D}, \ W_{uu} = \frac{4v}{D^3}, \ W_{uv} = -\frac{2u}{D^3}, \ W_{vv} = -\frac{4}{D^3};$$

$$Z_u = 1 - \frac{u}{D}, \ Z_v = -\frac{2}{D}, \ Z_{uu} = -\frac{4v}{D^3}, \ Z_{uv} = \frac{2u}{D^3}, \ Z_{vv} = \frac{4}{D^3}.$$

Thus, multiplying system (1.3) by $\nabla W(u,v), \ \nabla Z(u,v)$ respectively, we obtain

$$W(u,v)_t + \lambda_2 W(u,v)_x + (1 + \frac{u}{D})\frac{u - h(v)}{\tau} = \varepsilon W_{xx} - \frac{\varepsilon}{D} W(u,v)_x Z(u,v)_x;$$

and

$$Z(u,v)_t + \lambda_1 Z(u,v)_x + (1 - \frac{u}{D})\frac{u-h(v)}{\tau} = \varepsilon W_{xx} + \frac{\varepsilon}{D} W(u,v)_x Z(u,v)_x.$$

Clearly, on the intersection of $\partial\Sigma_2$ and the curve $W(u,v) = N$, $(1+\frac{u}{D})\frac{u-h(v)}{\tau} \geq 0$; on the intersection of $\partial\Sigma_2$ and the curve $Z(u,v) = -L$, $(1-\frac{u}{D})\frac{u-h(v)}{\tau} \leq 0$. Therefore, by Theorem 4.4 of [1], the region Σ_2 is an invariant region and hence $|u^\varepsilon(x,t)| \leq M, \quad 0 < v^\varepsilon(x,t) \leq M$ by Lemma 3.2, where M is a positive constant depending only on the L^∞ norm of the initial data.

Using the same technique as in the proof of Theorem 2.1, we can complete the rest of the proof for Theorem 3.1. ☑

We conclude this paper with the following remark.

Remark 3.1. *If the relaxation terms in (1.1),(1.3) both are $\alpha(u,v)\frac{u-h(v)}{\tau}$, where $\alpha(u,v) > 0$ is lipchitz continuous, then from the proof of the theorems, we have the same conclusions.*

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