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CW-complexes with duality

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Abstract. It is the aim of this paper to provide an elementary definition of CW-complexes with duality and envisage some problems of gluing and cutting.

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Resumen. El propósito de este artículo es suministrar una definición elemental de CW-complejos con dualidad y prever algunos problemas de pegado y cortado.

1. **Introduction**

Let V be a closed, connected and oriented n-manifold. The Poincaré duality gives isomorphisms [8]

$$\cap[V] : H^k(V;\mathbb{Z}) \to H_{n-k}(V;\mathbb{Z}), \tag{1.1}$$

where $[V] \in H_n(V;\mathbb{Z})$ is the fundamental class of V. When V is nonorientable, it is advisable to introduce the orientation sheaf Ω^*_V of V, (see [2], [3]); we then have formulations of the cap and slant products which allow to establish an equivalence

$$C^k(V;\mathcal{B}) \to C_{n-k}(V;\mathcal{B}\widehat{\otimes}\,\Omega^*_V) \tag{1.2}$$

for any locally trivial sheaf $\mathcal{B}$ on V, inducing isomorphisms

$$\cap[V] : H^k(V;\mathcal{B}) \to H_{n-k}(V;\mathcal{B}\widehat{\otimes}\,\Omega^*_V),$$

where the fundamental class $[V]$ is in $H_n(V;\Omega^*_V)$.
Note that if the manifold V is triangulated, then by means of a cellular approximation of the diagonal $\Delta(V)$, one can exhibit a cycle $\{V\}$ in $C_n(\widehat{K}_1(V),\,\Omega^*_V)$ representing $[V]$, where the cellular complex $\widehat{K}_1(V)$ is obtained by barycentric subdivision of the triangulation of V, which allows us to show (cf. [3]), that the homotopy equivalence

$$\cap\{V\} : C^k(V;\mathcal{A}(V)) \to C_{n-k}(V;\mathcal{A}(V)\widehat{\otimes}\,\Omega^*_V) \tag{1.3}$$

is a simple-homotopy equivalence (i.e. the Whitehead torsion $\tau_{wh}(\cap\{V\}) = 0$ (cf. [1]). where $\mathcal{A}(V)$ denotes the fundamental sheaf of V which is the direct image of the constant sheaf $\underline{\mathbb{Z}}$ on the universal covering of V (cf [3]).

The formalism introduced in [3] to establish equivalences (2) and (3) suggests to consider spaces (not necessarily manifolds) which satisfy the Poincaré duality isomorphisms. These spaces are the analog of closed manifolds in the category of CW-complexes. Although there are several different flavors of Poincaré complexes in the literature (cf. [4], [5], [6], [7], [9]). The purpose of this article is to present a convenient definition of CW-complexes with duality, allowing us to obtain some cutting and gluing results.

2. CW-complexes with Duality (or simple-duality)

Let (X, Y) be a finite CW-pair. Write $\mathbb{A}[X] = \mathbb{Z}[\pi_1(X, x_o)]$, and $\mathbb{A}[Y] = \mathbb{Z}[\pi_1(Y, x_o)]$ for the integral group rings of the corresponding fundamental groups. Let $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ be the fundamental sheaves of X and Y, respectively. We have the identification

$$\mathcal{A}(X) \mid_Y = \mathcal{A}(Y) \otimes_{\mathbb{A}[Y]} \mathbb{A}[X].$$

Since $\pi_1(X, x_o)$ acts at the left on the universal covering $\widetilde{X}$ of X, it also acts on the left on $\mathcal{A}(X)$, endowing the fibre $\mathcal{A}(X)_{x_0}$ with an $\mathbb{A}[X]$-module structure.

Let X be a finite CW-complex, and let Ω_X^* be a sheaf on X that is locally isomorphic to the constant sheaf $\underline{\mathbb{Z}}$, $n \in \mathbb{N}$, and $[X]$ a homology class in $H_n(X; \Omega_X^*)$ represented by a cycle $\{X\} \in C_n(X; \Omega_X^*)$.

Definition 2.1. *The triple $([X], n, \Omega_X^*)$ is said to be a duality (resp. simple-duality) on X if*

$$\cap\{X\} : C^k(X; \mathcal{A}(X)) \to C_{n-k}(X; \mathcal{A}(X) \widehat{\otimes}\, \Omega_X^*)$$

is an equivalence (resp. simple-homotopy equivalence) for each $0 \le k \le n$.

Example 2.2. *If X is a n-manifold, Ω_X^* its orientation sheaf [2], and $[X]$ its fundamental class, then $([X], n, \Omega_X^*)$ is a simple-duality on X.*

Remark 2.1. *There are CW-complexes with duality having nontrivial torsion. For example, let X be a CW-complex with a simple-duality $([X], n, \Omega_X^*)$, and consider a homotopy equivalence $f : X \to Y$ such that $\tau_{wh}(f) = \tau \neq 0$. The fundamental class $[X]$ gives a class $[Y]$, and in the commutative diagram*

$$\begin{array}{ccc} C^*(X; \mathcal{A}(X)) & \overset{\cap\{X\}}{\longrightarrow} & C_*(X; \mathcal{A}(X) \widehat{\otimes}\, \Omega_X^*) \\ f^* \uparrow & & \downarrow f_* \\ C^*(Y; \mathcal{A}(Y)) & \underset{\cap\{Y\}}{\rightarrow} & C_*(Y; \mathcal{A}(Y) \widehat{\otimes}\, \Omega_Y^*) \end{array}$$

we have $\tau_{wh}(\cap\{Y\}) = \tau_{wh}(f^*) + \tau_{wh}(f_*) = \tau + \bar{\tau}$, where $\tau \to \bar{\tau}$ is the Whitehead homomorphism corresponding to transposition. Then it suffices to take a CW-complex X such that the homomorphism from $Wh(\pi_1)$ to it self, defined by $\tau \to \tau + \bar{\tau}$ is trivial.

Definition 2.3. *Let (X,Y) be a finite CW pair, Ω^*_X a sheaf on X locally isomorphic to the constant sheaf $\underline{\mathbb{Z}}$, and $[X] \in H_n(X,Y;\Omega^*_X)$. We say that $([X],n;\Omega^*_X)$ is a duality (resp.simple-duality) if*

$$C^k(X;\mathcal{A}(X)) \xrightarrow{\cap\{X\}} C_{n-k}(X,Y;\mathcal{A}(X)\widehat{\otimes}\,\Omega^*(X))$$
$$C^k(X,Y;\mathcal{A}(X)) \xrightarrow{\cap\{X\}} C_{n-k}(X;\mathcal{A}(X)\widehat{\otimes}\,\Omega^*(X))$$

are homotopy equivalences (resp.simple-homotopy equivalences) for each $0 \le k \le n$.

3. **Gluing of spaces with duality**

We consider a triad of finite CW-complexes $(Z;X,X')$ and $Y = X \cap X'$. Suppose an isomorphism

$$f : \Omega^*_X\mid_Y \to \Omega^*_{X'}\mid_Y$$

is given. We construct a sheaf Ω^*_Z on Z such that $\Omega^*_Z\mid_X = \Omega^*_X$ and $\Omega^*_Z\mid_{X'} = \Omega^*_{X'}$ as follows.

Consider $[X] \in H_n(X,Y;\Omega^*_X)$ and $[X'] \in H_n(X',Y;\Omega^*_{X'})$, two homology classes such that $\partial[X]+\partial'[X'] = 0$, where $\partial : H_n(X,Y;\Omega^*_X) \to H_{n-1}(Y;\Omega^*_X)$ and $\partial' : H_n(X',Y;\Omega^*_{X'}) \to H_{n-1}(Y;\Omega^*_{X'})$ are the boundary maps of the long exact sequences of the pairs (X,Y) and (X',Y), respectively.

The homology class $[X]$ is represented by a relative cycle $\{X\} \in C_n(X;\Omega^*_X)$, and so $d\{X\} \in C_{n-1}(X;\Omega^*_X)$ where d is the boundary operator of the complex $C_*(X;\Omega^*_X)$. Although $d\{X\} \in C_{n-1}(X;\Omega^*_X)$ is not necessarily zero, we have that $d\{X\} \in C_{n-1}(Y;\Omega^*_X)$. Similarly, we have $d\{X'\} \in C_{n-1}(Y;\Omega^*_{X'})$. Furthermore, $d\{X\}$ and $-d\{X'\}$ are null homologous in Y, so there is an $(n-1)$-chain $\{Y\} \in C_{n-1}(Y;\Omega^*_Y)$ such that $d\{X\}+d\{X'\} = d\{Y\}$. Hence we obtain a cycle $\{Z\} = \{X\}+\{Y\}+\{X'\}$ representing a class $[Z] \in H_n(Z;\Omega^*_Z)$ which we also denote by $[X \cup_Y X']$. Note that the image of $[X \cup_Y X']$ in $H_n(Z,X';\Omega^*_Z) \simeq H_n(X,Y;\Omega^*_X)$ is $[X]$, and that the image of $[X \cup_Y X']$ in $H_n(Z,X;\Omega^*_Z) \simeq H_n(X',Y;\Omega^*_{X'})$ is $[X']$.

Consider now a locally trivial sheaf $\mathcal{F}$ on $Z = X \cup X'$. By using the long exact sequence of the pair (Z,X') and the naturality of the cap product we obtain the commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
C^*(Z,X';\mathcal{F}) & \overset{\cap\{Z\}}{\to} & C_{n-*}\left(Z-X';\mathcal{F}\widehat{\otimes}\Omega^*_Z\right) \\
\downarrow & & \downarrow \\
C^*(Z;\mathcal{F}) & \overset{\cap\{Z\}}{\to} & C_{n-*}\left(Z;\mathcal{F}\widehat{\otimes}\Omega^*_Z\right) \\
\downarrow & & \downarrow \\
C^*(X';\mathcal{F}) & \overset{\cap\{Z\}}{\to} & C_{n-*}\left(Z,Z-X';\mathcal{F}\widehat{\otimes}\Omega^*_Z\right) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

We also have an excision which induces the isomorphism

$$\xi^* : C^*\left(Z, X^{'}; \mathcal{F}\right) \overset{\approx}{\to} C^*\left(X, Y; \mathcal{F}\mid_X\right).$$

Thus we obtain the exact sequence

$$0 \to C^*(X, Y; \mathcal{F}\mid_X) \simeq C^*\left(Z, X^{'}; \mathcal{F}\right) \overset{k}{\to} C^*(Z; F) \to C^*\left(X^{'}; \mathcal{F}\mid_{X^{'}}\right) \to 0.$$

Lemma 3.1. *For a locally trivial sheaf $\mathcal{F}$ on Z, we have the commutative diagram*

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
C^*(X, Y; \mathcal{F}\mid_X) & \overset{\cap\{X\}}{\to} & C_{n-*}(X; \mathcal{F}\mid_X \widehat{\otimes}\, \Omega_X^*) \\
\downarrow k & & \downarrow i \\
C^*(Z; \mathcal{F}) & \overset{\cap\{Z\}}{\to} & C_{n-*}(Z; \mathcal{F}\widehat{\otimes}\, \Omega_Z^*) \\
\downarrow & & \downarrow \\
C^*(X'; \mathcal{F}\mid_{X'}) & \overset{\cap\{X'\}}{\to} & C_{n-*}(X', Y; \mathcal{F}\mid_{X'} \widehat{\otimes}\, \Omega_{X'}^*) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

where the rows are exact, and i is induced by the inclusion $X \hookrightarrow Z$

Proof. For the square to the left, it suffices to observe that we have the two commutative diagrams

$$\begin{array}{ccc}
C^*\left(Z, X^{'}; \mathcal{F}\right) & \overset{\cap\{Z\}^{'}}{\to} & C_*\left(Z; \mathcal{F}\widehat{\otimes}\, \Omega_Z^*\right) \\
\downarrow & & \| \\
C^*(Z; \mathcal{F}) & \overset{\cap\{Z\}}{\to} & C_*\left(Z; \mathcal{F}\widehat{\otimes}\, \Omega_Z^*\right)
\end{array}$$

$$\begin{array}{ccc}
C^*(X, Y; \mathcal{F}\mid_X) & \overset{\cap\{X\}}{\to} & C_*\left(X; \mathcal{F}\mid_X \widehat{\otimes}\, \Omega_X^*\right) \\
\uparrow & & \downarrow \\
C^*\left(Z, X^{'}; \mathcal{F}\right) & \overset{\cap\{Z\}^{'}}{\to} & C_*\left(Z; \mathcal{F}\widehat{\otimes}\, \Omega_Z^*\right)
\end{array}$$

where $\{Z\}^{'}$ is the image of $\{Z\}$ under the map

$$C_*\left(Z; \Omega_Z^*\right) \quad \to \quad C_*\left(Z, X^{'}; \Omega_Z^*\right),$$

which agrees with $\{X\}$ by identifying $C_*\left(Z, X^{'}; \Omega_Z^*\right)$ with $C_*\left(X, Y; \Omega_X^*\right)$. We use a similar argument for the right square. ☑

Now, when $\mathcal{F}$ is the fundamental sheaf $\mathcal{A}(Z)$ of Z, if $([X], n, \Omega_X^*)$ is a duality (resp. a simple-duality) on (X, Y), and $([X'], n, \Omega_X^*)$ is a duality (resp. a simple-duality) on (X', Y), we have isomorphisms

$$\begin{array}{rcl} C^*(X;\mathcal{A}(X)) & \overset{\cap\{X\}}{\to} & C_*(X,Y;\mathcal{A}(X)\widehat{\otimes}\,\Omega^*_X) \\ C^*(X,Y;\mathcal{A}(X)) & \overset{\cap\{X\}}{\to} & C_*(X;\mathcal{A}(X)\widehat{\otimes}\,\Omega^*_X) \\ C^*(X';\mathcal{A}(X')) & \overset{\cap\{X'\}}{\to} & C_*(X',Y;\mathcal{A}(X')\widehat{\otimes}\,\Omega^*_{X'}) \\ C^*(X',Y;\mathcal{A}(X')) & \overset{\cap\{X'\}}{\to} & C_*(X';\mathcal{A}(X')\widehat{\otimes}\,\Omega^*_{X'}) \end{array}$$

by application of the functors $\cdot \rightsquigarrow \otimes_{\mathbf{A}[X]}\mathbf{A}[Z]$ and $\cdot \rightsquigarrow \otimes_{\mathbf{A}[X']}\mathbf{A}[Z]$, we obtain the equivalences

$$\begin{array}{rcl} C^*(X;\mathcal{A}(Z)\mid_X) & \overset{\cap\{X\}}{\to} & C_*(X,Y;\mathcal{A}(Z)\mid_X\widehat{\otimes}\,\Omega^*_X) \\ C^*(X,Y;\mathcal{A}(Z)\mid_X) & \overset{\cap\{X\}}{\to} & C_*(X;\mathcal{A}(Z)\mid_X\widehat{\otimes}\,\Omega^*_X) \\ C^*(X';\mathcal{A}(Z)\mid_{X'}) & \overset{\cap\{X'\}}{\to} & C_*(X',Y;\mathcal{A}(Z)\mid_X\widehat{\otimes}\,\Omega^*_{X'}) \\ C^*(X',Y;\mathcal{A}(Z)\mid_{X'}) & \overset{\cap\{X'\}}{\to} & C_*(X';\mathcal{A}(Z)\mid_{X'}\widehat{\otimes}\,\Omega^*_{X'}) \end{array}$$

Furthermore, if in the diagram of Lemma 3.1 the vertical arrows $\{X\}$ and $\cap\{X'\}$ are equivalences, so is the vertical arrow $\cap\{Z\}$. We obtain the following result.

Let $(Z;X,X')$ be a triad of CW-complexes, and let $Y = X\cap X'$. Let $\{X\}\in C_n\,(X,Y;\,\Omega^*_X)$ and $\{X'\}\in C_n\,(X',Y;\,\Omega^*_{X'})$ be such that $\partial[X]+\partial'[X']=0$. If $([X],n,\,\Omega^*_X)$ is a duality (resp. a simple-duality) on (X,Y), and $\left([X^{'}],n,\,\Omega^*_{X'}\right)$ is a duality (resp. a simple-duality) on (X',Y), then there is a cycle $[Z]\in C_n\,(Z;\,\Omega^*_Z)$ such that $([Z],n,\,\Omega^*_Z)$ is a duality (resp. a simple-duality) on Z.

4. **Cutting lemma**

Let (Z,X) and (Z,X') be finite CW-pairs such that $Z = X\cup_Y X'$ with $Y = X\cap X'$. Suppose a duality (resp. a simple-duality) $([Z],n,\,\Omega^*_Z)$ on Z is given. Let us define $[X]$ to be the image of $[Z]$ in $C_n(X,Y;\,\Omega^*_Z\mid_X)$ and $[X']$ to be the image of $[Z]$ in $C_n(X',Y;\,\Omega^*_Z\mid_{X'})$. Then we have

Lemma 4.1. *If $([X],n,\,\Omega^*_X)$ is a duality (resp. a simple-duality) on (X,Y), and $\pi_1\left(X^{'}\right)\to\pi_1(Z)$ is an isomorphism, then $\left([X^{'}],n,\,\Omega^*_{X'}\right)$ is a duality (resp. a simple-duality) on (X',Y).*

Proof. It can be easily verified that the gluing of $[X]$ and $[X']$ is $[Z]$, by using the five lemma in the diagrams of the lemma 3.1. Note that the hypothesis on π_1 ensures that $\mathcal{A}\,(X')$ agrees with $\mathcal{A}(Z)\mid_{X'}$. ☑

Remark 4.1. *If the hypothesis on π_1 is not verified, we obtain isomorphisms*

$$\begin{array}{l} C^*(X',Y;\mathcal{A}(Z)\mid_{X'})\to C_*(X';\mathcal{A}(Z)\mid_{X'}\widehat{\otimes}\,\Omega^*_{X'}) \\ C^*(X';\mathcal{A}(Z)\mid_{X'})\to C_*(X^{'},Y;\mathcal{A}(Z)\mid_{X'}\widehat{\otimes}\,\Omega^*_{X'}) \end{array}$$

which are equivalent to isomorphisms

$$C^*\left(X',Y;\mathcal{A}\left(X'\right)\right)\otimes_{\mathcal{A}[X']}\mathbf{A}[Z]\to C_*\left(X';\mathcal{A}\left(X'\right)\widehat{\otimes}\,\Omega^*_{X'}\right)\otimes_{\mathbf{A}[X']}\mathbf{A}[Z]$$
$$C^*\left(X';\mathcal{A}\left(X'\right)\right)\otimes_{\mathbf{A}}[X']\mathbf{A}[Z]\to C_*\left(X',Y;\mathcal{A}(X')\widehat{\otimes}\,\Omega^*_{X'}\right)\otimes_{\mathbf{A}}[X']\mathbf{A}[Z]$$

and the maps

$$C^*\left(X^{'};\mathcal{A}(X')\right)\to C_*\left(X',Y;\mathcal{A}(X')\widehat{\otimes}\,\Omega^*_{X'}\right)$$

and

$$C^*(X',Y;\mathcal{A}(X'))\to C_*(X';\mathcal{A}(X')\widehat{\otimes}\,\Omega^*_{X'})$$

are not necessarily homotopy equivalences.

However, if $\pi_1(X')$ is a direct factor of $\pi_1(Z)$ then Lemma 3.1 is still true. Indeed, it suffices to note that $\mathbf{A}[X']$ is a $\mathbf{A}[Z]$-module, and for each left $\mathbf{A}[X']$-module $\mathcal{M}$, we have $\left(\mathcal{M}\otimes_{\mathbf{A}[X']}\mathbf{A}[Z]\right)\otimes_{\mathbf{A}[Z]}\mathbf{A}[X']\approx\mathcal{M}$.

Note that there exists a space X, a sheaf Ω^*_X, and a cycle $\{X\}\in C_n(X,\,\Omega^*_X)$ such that: $\bigcap\{X\}:C^k(X,Z)\to C_{n-k}(X,Z\widehat{\otimes}\,\Omega^*_X)$ is a homotopy equivalence but $\bigcap\{X\}:C^k(X;\mathcal{A}(X))\to C_{n-k}\left(X,\mathcal{A}(X)\widehat{\otimes}\,\Omega^*_X\right)$ is not. For example, Let $X_\circ$ be a $2n$-manifold with n large and $\pi_1(X_\circ)=\mathbb{Z}/5\mathbb{Z}$. $X_\circ$ can therefore be oriented. Consider the matrix M with coefficients belonging to $\mathbb{Z}[\mathbb{Z}/5\mathbb{Z}]$

$$M=\begin{pmatrix}1&1&0&0&0\\ \alpha&1&1&0&0\\ \alpha^2&0&1&1&0\\ \alpha^3&0&0&1&0\\ \alpha^4&0&0&0&1\end{pmatrix},\qquad \det(M)=\alpha^4-\alpha^3+\alpha^2-\alpha+1\ ,$$

and μ the matrix with coefficients in $\mathbb{Z}$ that is the image of M under the augmentation map $\mathbb{Z}[\mathbb{Z}/5\mathbb{Z}]\to\mathbb{Z}$. μ is invertible since $\det(\mu)=1$. On the other hand, M is not invertible because $\det(M)$ is not. Indeed, the product by $\det(M)$ in $\mathbb{Z}[\mathbb{Z}/5\mathbb{Z}]$ has, relative to the basis $(1,\alpha,\alpha^2,\alpha^3,\alpha^4)$ the matrix

$$D=\begin{pmatrix}1&1&-1&1&-1\\ -1&1&1&-1&1\\ 1&-1&1&1&-1\\ -1&1&-1&1&1\\ 1&-1&1&-1&1\end{pmatrix}$$

which is not invertible in $\mathbb{Z}$.

Let $X_1=X_\circ\cup5\{n\text{-cells}\}\cup5\{(n+1)\text{-cells}\}$. The n-cells are attached to a point of $X_\circ$ and the $(n+1)$-cells are attached to the attached n-cells by an application having M as matrix. Then, $H_k(X_\circ;\mathbb{Z})=H_k(X_1;\mathbb{Z})$ and $H^k(X_\circ;\mathbb{Z})=H^k(X_1;\mathbb{Z})$, for $k\neq n\ ,\ n+1$. But,

$$H_n(X_1;\mathcal{A}(X_1))=H_n(X_\circ;\mathcal{A}(X_\circ))\oplus\ker(M)$$
$$H_{n+1}(X_1;\mathcal{A}(X_1))=H_{n+1}(X_\circ;\mathcal{A}(X_\circ))\oplus co\ker(M)$$

Furthermore, $C_*\left(X_\circ;\,\Omega^*_{X_\circ}\right) \to C_*\left(X_1;\,\Omega^*_{X_1}\right)$ is a homotopy equivalence ($\Omega^*_{X_\circ}$ and $\Omega^*_{X_1}$ are trivial), and therefore $[X_\circ]$ is sent to $[X_1]$. Finally, the commutative diagram

$$\begin{array}{ccc} C^k(X_1;Z) & \xrightarrow{\cap[X_1]} & C_{2n-k}(X_1;Z \overset{\wedge}{\otimes} \Omega^*_{X_1}) \\ \downarrow & & \uparrow \\ C^k(X_\circ;Z) & \xrightarrow{\cap[X_\circ]} & C_{2n-k}(X_\circ;Z \overset{\wedge}{\otimes} \Omega^*_{X_\circ}) \end{array}$$

one deduces that $\cap[X_1] : C^k(X_1;Z) \to C_{2n-k}\left(X_1;Z\widehat{\otimes}\,\Omega^*_{X_1}\right)$ is an equivalence, but $\cap[X_1] : C^k(X_1;\mathcal{A}(X_1)) \to C_{2n-k}\left(X_1;\mathcal{A}(X_1)\widehat{\otimes}\,\Omega^*_{X_1}\right)$ is not.

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