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Ostrowski, Grüss, Čebyšev type inequalities for functions whose second derivatives belong to Lp(a,b) and whose modulus of second derivatives are convex

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Abstract. Ostrowski, Grüss, Čebyšev type inequalities involving functions whose second derivatives belong to $L_p(a,b)$ and whose modulus of second derivatives are convex are established. The results provide better bounds than those currently available in the literature.

Keywords. Ostrowski Grüss-Čebyšev inequalities, modulus of second derivative convex, convex function.

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Resumen. Se establecen desigualdades de tipo Ostrowski, Grüss, Čebyšev que comprenden funciones cuyas segundas derivadas pertenecen a $L_p(a,b)$ y cuyos módulos de segundas derivadas son convexos. Los resultados obtenidos proporcionan mejores cotas que las actualmente disponibles en la literatura.

1. Introduction

In 1938, A. M. Ostrowski [6] proved the following:

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b) i.e., $|f'(x)| \leq M <$*

∞, *then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)\, M, \tag{1.1}$$

for all $x \in [a,b]$, where M is constant.

For two absolutely continuous functions $f, g : [a,b] \to \mathbb{R}$, consider the functional,

$$T(f,g) = \frac{1}{b-a} \int_a^b f(x)\, g(x)\, dx - \left(\frac{1}{b-a} \int_a^b f(x)\, dx \right) \left(\frac{1}{b-a} \int_a^b g(x)\, dx \right), \tag{1.2}$$

provided, the involved integrals exist.

In 1882, P. L. Čebyšev [7] proved that, if $f', g' \in L_\infty[a,b]$, then,

$$T(f,g) \leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty . \tag{1.3}$$

In 1934, G. Grüss [7] showed that

$$T(f,g) \leq \frac{1}{4} (M-m)(N-n), \tag{1.4}$$

provided m, M, n and N are real numbers satisfying the conditions,

$$-\infty < m \leq f(x) \leq M < \infty,$$
$$-\infty < n \leq g(x) \leq N < \infty,$$

for all $x \in [a,b]$.

Pachpatte in [11] proved the following results.

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. If $|f''|$, $|g''|$ are convex on $[a,b]$ and f'', $g'' \in L_\infty[a,b]$, then,*

$$|S(f,g)| \leq \Big[|g(x)| \left(|f'(x)| + \|f'\|_\infty \right) + |f(x)| \left(|g'(x)| + \|g'\|_\infty \right) \Big] \times \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2} \right)^2}{(b-a)^2} \right] \frac{b-a}{4}, \tag{1.5}$$

for all $x \in [a,b]$.

Corollary 1.1. *Under the assumptions of theorem 1.2, we have the mid point inequality,*

$$|S_M(f,g)| \leq \frac{(b-a)}{16} \left[\left| g\left(\frac{a+b}{2} \right) \right| \left(\left| f'\left(\frac{a+b}{2} \right) \right| + \|f'\|_\infty \right) + \left| f\left(\frac{a+b}{2} \right) \right| \left(\left| g'\left(\frac{a+b}{2} \right) \right| + \|g'\|_\infty \right) \right]. \tag{1.6}$$

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. If $|f''|$, $|g''|$ are convex on $[a,b]$ and f'', $g'' \in L_\infty[a,b]$, then,*

$$|T(f,g)| \leq \frac{1}{4(b-a)^2} \int_a^b [|g(x)| (|f'(x)| + \|f'\|_\infty) + |f(x)| (|g'(x)| + \|g'\|_\infty)] E(x)dx, \tag{1.7}$$

for all $x \in [a,b]$, where $E(x) = \frac{(x-a)^2+(b-x)^2}{2}$.

Corollary 1.2. *Under the assumptions of theorem 1.3, we have the mid point inequality,*

$$|T_M(f,g)| \leq \frac{b-a}{16} \left[\left| g\left(\frac{a+b}{2}\right) \right| \left(\left| f'\left(\frac{a+b}{2}\right) \right| + \|f'\|_\infty \right) + \left| f\left(\frac{a+b}{2}\right) \right| \left(\left| g'\left(\frac{a+b}{2}\right) \right| + \|g'\|_\infty \right) \right]. \tag{1.8}$$

Theorem 1.4. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. If $|f''|$, $|g''|$ are convex on $[a,b]$ and f'', $g'' \in L_\infty[a,b]$, then,*

$$|T(f,g)| \leq \frac{1}{4(b-a)^3} \int_a^b \left[(|f'(x)| + \|f'\|_\infty) \times (|g'(x)| + \|g'\|_\infty) \right] E^2(x)dx, \tag{1.9}$$

for all $x \in [a,b]$, where $E(x) = \frac{(x-a)^2+(b-x)^2}{2}$.

Corollary 1.3. *Under the assumptions of theorem 1.4, we have the mid point inequality*

$$|T_M(f,g)| \leq \frac{(b-a)^2}{64} \left[\left(\left| f'\left(\frac{a+b}{2}\right) \right| + \|f'\|_\infty \right) \times \left(\left| g'\left(\frac{a+b}{2}\right) \right| + \|g'\|_\infty \right) \right]. \tag{1.10}$$

During the past few years, many researchers have given considerable attention to the above inequalities and various generalizations, extensions and variants of them have appeared in the literature, see $[1-12]$, and the references cited therein. Motivated by results given in $[8-11]$, we establish here some inequalities similar to those given by Ostrowski, Grüss and Čebyšev involving functions whose derivatives belong to $L_p(a,b)$ space and whose modulus of second derivatives are convex. The analysis used in the proofs is elementary and based on integral identities proved in $[1-2]$.

2. Statement of results

Let I be a suitable interval of the real line $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$$

for all $x, y \in I$ and $\lambda \in [0,1]$ (see [12]).

We need the following identities proved by Mir et al. in [5]:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \left(x - \frac{a+b}{2}\right)f'(x)$$
$$- \frac{1}{b-a}\int_a^b (x-t)^2 \left[\int_0^1 (1-\lambda) f''((1-\lambda)x + \lambda t)\, d\lambda\right] dt,$$

for all $x \in [a,b]$, where $f : I \to \mathbb{R}$ is an absolutely continuous function on $[a,b]$ and $\lambda \in [0,1]$.

We use the following notation to simplify the details of presentation,

$$S(f,g) = f(x)g(x) - \frac{1}{2(b-a)}\left(f(x)\int_a^b g(t)dt + g(x)\int_a^b f(t)dt\right)$$
$$- \frac{1}{2}\left(x - \frac{a+b}{2}\right)(f(x)g'(x) + g(x)f'(x)).$$

At the mid-point we denote this by $S_M(f,g)$, noting that the last term on the RHS vanishes.

$$T(f,g) = \frac{1}{b-a}\int_a^b S(f,g)dx = \frac{1}{b-a}\int_a^b f(x)g(x)dx$$
$$- \frac{1}{2(b-a)^2}\left(\int_a^b f(x)dx\int_a^b g(t)dt + \int_a^b g(x)dx\int_a^b f(t)dt\right)$$
$$- \frac{1}{2(b-a)}\int_a^b \left(x - \frac{a+b}{2}\right)\big(f(x)g'(x) + g(x)f'(x)\big)dx$$
$$= \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)$$
$$- \frac{1}{2(b-a)}\int_a^b \left(x - \frac{a+b}{2}\right)\big(f(x)g'(x) + g(x)f'(x)\big)dx,$$

at the mid-point we denote this by $T_M(f,g)$.

$$\begin{aligned}\widetilde{S}(f,g) &= f(x)g(x) - \left(x - \frac{a+b}{2}\right)\left(f(x)g'(x) + g(x)f'(x)\right) \\ &\quad - \frac{1}{b-a}\left(f(x)\int_a^b g(t)dt + g(x)\int_a^b f(t)dt\right) \\ &\quad + \left(x - \frac{a+b}{2}\right)^2 f'(x)g'(x) \\ &\quad + \left(x - \frac{a+b}{2}\right)\left(\frac{f'(x)}{b-a}\int_a^b g(t)dt + \frac{g'(x)}{b-a}\int_a^b f(t)dt\right) \\ &\quad + \left(\frac{1}{b-a}\int_a^b f(t)dt\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right),\end{aligned}$$

and

$$\begin{aligned}\widetilde{T}(f,g) &= \frac{1}{b-a}\int_a^b \widetilde{S}(f,g)dx = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\right) \\ &\quad \times \left(\int_a^b \left(x - \frac{a+b}{2}\right)\left(f(x)g'(x) + g(x)f'(x)\right)dx\right) \\ &\quad + \frac{1}{b-a}\int_a^b \left(x - \frac{a+b}{2}\right)^2 f'(x)g'(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right) \\ &\quad \times \left(\frac{1}{b-a}\int_a^b g(x)dx\right) + \frac{1}{(b-a)^3}\int_a^b \left(x - \frac{a+b}{2}\right) \\ &\quad \times \left(f'(x)\int_a^b g(t)dt + g'(x)\int_a^b f(t)dt\right)dx,\end{aligned}$$

and at the mid-point we denote this by $\widetilde{T}(f,g)$. We also use

$$Q(x) = \left[\frac{(b-x)^{2q+1} + (x-a)^{2q+1}}{2q+1}\right]^{1/q}.$$

We define $\|.\|_p$ as the usual Lebesgue norm on $L_p[a,b]$,; in other words, $\|h\|_p = \left(\int_a^b |h(t)|^p\, dt\right)^{1/p}$ for $h \in L_p[a,b]$ and $\left(p, q > 1, \frac{1}{p} + \frac{1}{q} = 1\right)$.

The following theorem deals with Ostrowski type inequalities involving two functions.

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous on $[a, b]$. If $|f''|$, $|g''|$ are convex on $[a, b]$ and f'', $g'' \in L_p[a, b]$, then,*

$$|S(f,g)| \le \frac{Q(x)}{12(b-a)}\left[2|b-a|^{1/p}\left(|f(x)||g''(x)|+|g(x)||f''(x)|\right) + \left(|g(x)|\,\|f''\|_p + |f(x)|\,\|g''\|_p\right)\right], \tag{2.1}$$

for all $x \in [a, b]$.

Proof. From the hypothesis of theorem 2.1, the following identities hold:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \left(x - \frac{a+b}{2}\right)f'(x) - \frac{1}{b-a}\int_a^b (x-t)^2\left[\int_0^1 (1-\lambda)f''((1-\lambda)x+\lambda t)\,d\lambda\right]dt, \tag{2.2}$$

$$g(x) = \frac{1}{b-a}\int_a^b g(t)dt + \left(x - \frac{a+b}{2}\right)g'(x) - \frac{1}{b-a}\int_a^b (x-t)^2\left[\int_0^1 (1-\lambda)g''\big((1-\lambda)x+\lambda t\big)d\lambda\right]dt, \tag{2.3}$$

for all $x \in [a, b]$.

Multiplying both sides of (2.2) and (2.3) by $g(x)$ and $f(x)$ respectively, adding the resulting identities and rewriting, we have,

$$S(f,g) = -\frac{1}{2(b-a)}\left[g(x)\int_a^b (x-t)^2\left(\int_0^1 (1-\lambda)f''\big((1-\lambda)x+\lambda t\big)d\lambda\right)dt + f(x)\int_a^b (x-t)^2\left(\int_0^1 (1-\lambda)g''\big((1-\lambda)x+\lambda t\big)d\lambda\right)dt\right]. \tag{2.4}$$

Since $|f''|$, $|g''|$ are convex on $[a, b]$ then, from (2.4), and using properties of modulus, we have,

$$
\begin{aligned}
|S(f,g)| &\leq \frac{1}{2\,(b-a)} \\
&\times \left\{ |g(x)| \int_a^b |x-t|^2 \left[|f''(x)| \int_0^1 (1-\lambda)^2 \, d\lambda + |f''\,(t)| \int_0^1 \lambda\,(1-\lambda)\, d\lambda \right] dt \right. \\
&\left. + |f(x)| \int_a^b |x-t|^2 \left[|g'' x| \int_0^1 (1-\lambda)^2 \, d\lambda + |g''\,(t)| \int_0^1 \lambda\,(1-\lambda)\, d\lambda \right] dt \right\} \\
&\leq \frac{1}{12\,(b-a)} \left[|g(x)| \int_a^b |x-t|^2 \left(2\,|f'' x| + |f''\,(t)|\right) dt \right. \\
&\left. + \,|f(x)| \int_a^b |x-t|^2 \left[2\,|g'' x| + |g''\,(t)|\right] dt \right] \\
&\leq \frac{Q(x)}{12\,(b-a)} \left[|g(x)| \left(2\,|f''(x)|\,|b-a|^{1/p} + \|f''\|_p \right) \right. \\
&\left. + \,|f(x)| \left(2\,|g''(x)|\,|b-a|^{1/p} + \|g''\|_p \right) \right] \\
&= \frac{Q(x)}{12\,(b-a)} \left[2\,|b-a|^{1/p} \left(|f(x)|\,|g''(x)| + |g(x)|\,|f''(x)|\right) \right. \\
&\left. + \,|g(x)|\,\|f''\|_p + |f(x)|\,\|g''\|_p \right].
\end{aligned}
$$

We therefore have the desired inequality (2.1). ☑

Corollary 2.1. *Under the assumptions of theorem* 2.1*, we have the mid point inequality,*

$$
\begin{aligned}
&|S_M\,(f,g)| \\
&\leq \frac{(b-a)^{2q}}{12(2)^{2q}(2q+1)} \left[\left(\left| g\left(\frac{a+b}{2}\right) \right| \|f''\|_p + \left| f\left(\frac{a+b}{2}\right) \right| \|g''\|_p \right) \right. \\
&\quad \left. + \,2\,|b-a|^{1/p} \left(\left| f\left(\frac{a+b}{2}\right) \right| \left| g''\left(\frac{a+b}{2}\right) \right| + \left| g\left(\frac{a+b}{2}\right) \right| \left| f''\left(\frac{a+b}{2}\right) \right| \right) \right].
\end{aligned} \tag{2.5}
$$

Remark 2.1. *As we know that in the above inequality* $q > 1$ *and so* $(2)^{2q}(2q+1) > 12$*, then clearly bounds obtained in* (2.5) *are at least* 9 *times better than the bounds obtained in* (1.6)*.*

The Grüss type inequalities are embodied in the following theorem.

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. If $|f''|$, $|g''|$ are convex on $[a,b]$ and f'', $g''\in L_p[a,b]$, then,*

$$|T(f,g)|\leq\frac{1}{12(b-a)^2}\int_a^b\Big[2|b-a|^{1/p}\big(|f(x)||g''(x)|+|g(x)||f''(x)|\big)+\Big(|g(x)|\|f''\|_p+|f(x)|\|g''\|_p\Big)\Big]Q(x)dx,\tag{2.6}$$

for all $x\in[a,b]$.

Proof. From the proof of theorem 2.1, we have,

$$S(f,g)=-\frac{1}{2(b-a)}\Bigg\{g(x)\int_a^b(x-t)^2\left[\int_0^1(1-\lambda)f''\big((1-\lambda)x+\lambda t\big)d\lambda\right]dt+f(x)\int_a^b(x-t)^2\left[\int_0^1(1-\lambda)g''\big((1-\lambda)x+\lambda t\big)d\lambda\right]dt\Bigg\}.\tag{2.7}$$

Integrating (2.7) with respect to x over $[a,b]$ and dividing by $(b-a)$, we get,

$$T(f,g)=\frac{-1}{2(b-a)^2}\int_a^b\Bigg\{g(x)\int_a^b(x-t)^2\left[\int_0^1(1-\lambda)f''\big((1-\lambda)x+\lambda t\big)d\lambda\right]dt+f(x)\int_a^b(x-t)^2\left[\int_0^1(1-\lambda)g''\big((1-\lambda)x+\lambda t\big)d\lambda\right]dt\Bigg\}dx.$$

Since $|f''|$, $|g''|$ are convex on $[a,b]$ and using the properties of modulus, we have,

$$|T(f,g)|\leq\frac{1}{2(b-a)^2}\int_a^b\Bigg\{|g(x)|\int_a^b|x-t|^2\left[|f''(x)|\int_0^1(1-\lambda)^2d\lambda+|f''(t)|\int_0^1\lambda(1-\lambda)d\lambda\right]dt+|f(x)|\int_a^b|x-t|^2\left[|g''x|\int_0^1(1-\lambda)^2d\lambda+|g''(t)|\int_0^1\lambda(1-\lambda)d\lambda\right]dt\Bigg\}dx$$

$$
\begin{aligned}
&\le \frac{1}{12\,(b-a)^2}\int_a^b \Bigg\{ |g(x)| \Bigg[2\,|f''(x)| \left(\int_a^b |x-t|^{2q}\,dt\right)^{1/q} \left(\int_a^b 1^p dt\right)^{1/p} \\
&+ \left(\int_a^b |x-t|^{2q}\,dt\right)^{1/q} \left(\int_a^b |f''(t)|^p\,dt\right)^{1/p} \Bigg] \\
&+ |f(x)| \Bigg[2\,|g''(x)| \left(\int_a^b |x-t|^{2q}\,dt\right)^{1/q} \left(\int_a^b 1^p dt\right)^{1/p} \\
&+ \left(\int_a^b |x-t|^{2q}\,dt\right)^{1/q} \left(\int_a^b |g''(t)|^p\,dt\right)^{1/p} \Bigg] \Bigg\} dx \\
&= \frac{1}{12\,(b-a)^2}\int_a^b \Big[2\,|b-a|^{1/p} \left(|f(x)|\,|g''(x)| + |g(x)|\,|f''(x)|\right) \\
&+ \Big(|g(x)|\,\|f''\|_p + |f(x)|\,\|g''\|_p\Big)\Big]\, Q(x)dx.
\end{aligned}
$$

Hence we get desired inequality (2.6). ☑

Corollary 2.2. *Under the assumptions of theorem* 2.2*, we have the mid point inequality,*

$$
\begin{aligned}
&|T_M\,(f,g)| \le \\
&\frac{(b-a)^{2q}}{12(2)^{2q}(2q+1)} \Bigg[\left(\left| g\left(\frac{a+b}{2}\right)\right| \|f''\|_p + \left| f\left(\frac{a+b}{2}\right)\right| \|g''\|_p \right) \\
&+ 2\,|b-a|^{1/p} \left(\left| f\left(\frac{a+b}{2}\right)\right| \left| g''\left(\frac{a+b}{2}\right)\right| + \left| g\left(\frac{a+b}{2}\right)\right| \left| f''\left(\frac{a+b}{2}\right)\right| \right) \Bigg].
\end{aligned}
\tag{2.8}
$$

Remark 2.2. *As we know that in the above inequality* $q > 1$ *and so* $(2)^{2q}(2q+1) > 12$*, then clearly bounds obtained in* (2.8) *are at least* 9 *times better than the bounds obtained in* (1.8)*.*

The next theorem contains Čebyšev type inequalities.

Theorem 2.3. *Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous on $[a, b]$. If $|f''|$, $|g''|$ are convex on $[a, b]$ and f'', $g'' \in L_p[a, b]$, then,*

$$\left|\widetilde{T}(f,g)\right| \leq \frac{1}{36(b-a)^3} \int_a^b \left\{ \left[|g(x)| \left(2|f''(x)| |b-a|^{1/p} + \|f''\|_p \right) \right] \times \left[|f(x)| \left(2|g''(x)| |b-a|^{1/p} + \|g''\|_p \right) \right] \right\} Q^2(x) dx \tag{2.9}$$

for all $x \in [a, b]$.

Proof. From the hypothesis of theorem 2.3 the identities (2.2) and (2.3) hold. Multiplying both sides of these by each other, we have:

$$\begin{aligned}
&\left[f(x) - \frac{1}{b-a} \int_a^b f(t)dt - \left(x - \frac{a+b}{2} \right) f'(x) \right] \\
&\times \left[g(x) - \frac{1}{b-a} \int_a^b g(t)dt - \left(x - \frac{a+b}{2} \right) g'(x) \right] \\
&= \frac{1}{b-a} \int_a^b (x-t)^2 \left[\int_0^1 (1-\lambda) f''((1-\lambda)x + \lambda t) d\lambda \right] dt \\
&\times \frac{1}{b-a} \int_a^b (x-t)^2 \left[\int_0^1 (1-\lambda) g''((1-\lambda)x + \lambda t) d\lambda \right] dt,
\end{aligned}$$

implying

$$\begin{aligned}
&f(x)g(x) - \left(x - \frac{a+b}{2} \right) [f(x)g'(x) + g(x)f'(x)] \\
&- \frac{1}{b-a} \left[f(x) \int_a^b g(t)dt + g(x) \int_a^b f(t)dt \right] + \left(x - \frac{a+b}{2} \right)^2 f'(x)g'(x) \\
&+ \left(x - \frac{a+b}{2} \right) \left[\frac{f'(x)}{b-a} \int_a^b g(t)dt + \frac{g'(x)}{b-a} \int_a^b f(t)dt \right] \\
&+ \left(\frac{1}{b-a} \int_a^b f(t)dt \right) \left(\frac{1}{b-a} \int_a^b g(t)dt \right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{b-a}\int_a^b (x-t)^2 \left[\int_0^1 (1-\lambda)\, f''\left((1-\lambda)\,x+\lambda t\right) d\lambda\right] dt \\
&\times \frac{1}{b-a}\int_a^b (x-t)^2 \left[\int_0^1 (1-\lambda)\, g''\left((1-\lambda)\,x+\lambda t\right) d\lambda\right] dt
\end{aligned}
$$

which gives

$$
\begin{aligned}
\widetilde{S}(f,g) = {} & \frac{1}{b-a}\int_a^b (x-t)^2 \left[\int_0^1 (1-\lambda)\, f''\left((1-\lambda)\,x+\lambda t\right) d\lambda\right] dt \\
& \times \frac{1}{b-a}\int_a^b (x-t)^2 \left[\int_0^1 (1-\lambda)\, g''\left((1-\lambda)\,x+\lambda t\right) d\lambda\right] dt
\end{aligned} \tag{2.10}
$$

and, consequently, we obtain

$$
\begin{aligned}
&\left|\widetilde{T}(f,g)\right| \leq \\
&\frac{1}{(b-a)^3}\int_a^b \left\{\int_a^b (x-t)^2 \left[\int_0^1 \left((1-\lambda)^2 \left|f''(x)\right| + \lambda\,(1-\lambda)\left|f''(t)\right|\right) d\lambda\right] dt\right. \\
&\left.\times \int_a^b (x-t)^2 \left[\int_0^1 \left[(1-\lambda)^2 \left|g''(x)\right| + \lambda\,(1-\lambda)\left|g''(t)\right|\right] d\lambda\right] dt\right\} dx \\
&= \frac{1}{36\,(b-a)^3}\int_a^b \left\{\left[\left|g(x)\right| \left(2\left|f''(x)\right| \left|b-a\right|^{1/p} + \left\|f''\right\|_p\right)\right]\right. \\
&\left.\times \left[\left|f(x)\right| \left(2\left|g''(x)\right| \left|b-a\right|^{1/p} + \left\|g''\right\|_p\right)\right]\right\} Q^2(x)dx.
\end{aligned}
$$

This completes the proof. ☑

Corollary 2.3. *Under the assumptions of theorem* 2.3*, we have the mid-point inequality,*

$$
\begin{aligned}
|T_M(f,g)| \le{}& \frac{(b-a)^{4q}}{36(2)^{4q}(2q+1)^2} \\
&\times \left[2\,|b-a|^{1/p} \left(\left| f\left(\frac{a+b}{2}\right)\right| \left| g''\left(\frac{a+b}{2}\right)\right| + \left| g\left(\frac{a+b}{2}\right)\right| \left| f''\left(\frac{a+b}{2}\right)\right| \right) \right. \\
&+ \left. \left(\left| g\left(\frac{a+b}{2}\right)\right| \|f''\|_p + \left| f\left(\frac{a+b}{2}\right)\right| \|g''\|_p \right) \right]. \qquad (2.11)
\end{aligned}
$$

Remark 2.3. *As we know that in the above inequality* $q > 1$ *and so* $(2)^{4q}(2q+1)^2 > 144$*, then clearly bounds obtained in* (2.11) *are at least* 81 *times better than the bounds obtained in* (1.10)*.*

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