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Exact solutions for a new fifth-order integrable system

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Abstract. We consider a new fifth-order integrable system. The exact solutions are obtained by the generalized tanh method.

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Resumen. Nosotros consideramos un nuevo sistema integrable de quinto orden. Obtenemos las soluciones exactas de dicho sistema por el método tanh generalizado.

1. Introduction

The knowledge of closed form solutions of nonlinear PDEs facilitates the testing of numerical solvers, and aids in the stability analysis of solutions. There are a number of methods for the construction of exact solutions to differential partial differential equations of the mathematical physics [1], [2], [3], [4] [5], [6],[6], [9], [10]. Travelling wave solutions of many nonlinear PDEs, from solitons theory can by expressed as polynomials of the hyperbolic tangent and secant functions. In this brief note, we present exact solutions for the new fifth-order integrable system using the generalized tanh method.

2. Mikhailov–Novikov–Wang system

Mikhailov–Novikov–Wang system [8], [12] reads

$$\begin{cases} u_t = u_{xxxxx} - 20uu_{xxx} - 50u_xu_{xx} + 80u^2u_x + w_x \\ w_t = -6wu_{xxx} - 2u_{xx}w_x + 96wuu_x + 16w_xu^2. \end{cases} \tag{2.1}$$

We search for exact solutions of system (2.1) in the form

$$\begin{cases} u(x,t) = v(\xi) \\ w(x,t) = w(\xi) \\ \xi = x + \lambda t. \end{cases} \tag{2.2}$$

As a result system (2.1) reduces to the nonlinear system ordinary differential equation

$$\begin{cases} \lambda v'(\xi) - v^{(5)}(\xi) + 20v(\xi)v^{(3)}(\xi) + 50v'(\xi)v''(\xi) - 80v^2(\xi)v'(\xi) - w'(\xi) = 0 \\ \lambda w'(\xi) + 6w(\xi)v'''(\xi) + 2v''(\xi)w'(\xi) - 96w(\xi)v(\xi)v'(\xi) - 16w'(\xi)v^2(\xi) = 0. \end{cases} \tag{2.3}$$

We obtain

$$w'(\xi) = \lambda v'(\xi) - v^{(5)}(\xi) + 20v(\xi)v^{(3)}(\xi) + 50v'(\xi)v''(\xi) - 80v^2(\xi)v'(\xi). \tag{2.4}$$

The first equation of (2.3) can be written as follows:

$$\left(\lambda v(\xi) - v^{(4)}(\xi) + 20v(\xi)v''(\xi) + 15\left((v'(\xi))^2\right) - \frac{80}{3}(v^3(\xi)) - w(\xi)\right)' = 0. \tag{2.5}$$

Integrating (2.5) once with respect to ξ we get

$$\lambda v(\xi) - v^{(4)}(\xi) + 20v(\xi)v''(\xi) + 15\left(v'(\xi)\right)^2 - \frac{80}{3}v^3(\xi) - w(\xi) = c, \tag{2.6}$$

where c is integration constant. As we look for the exact solutions of special form, we set $c = 0$ for simplicity. Therefore

$$w(\xi) = \lambda v(\xi) - v^{(4)}(\xi) + 20v(\xi)v''(\xi) + 15\left(v'(\xi)\right)^2 - \frac{80}{3}v^3(\xi). \tag{2.7}$$

Substituting (2.4) and (2.7) into second equation of (2.3), we obtain after simplifications the following ordinary differential equation:

$$\begin{aligned} &\lambda^2 v'(\xi) - \lambda v^{(5)} + 26\lambda v(\xi)v'''(\xi) + 52\lambda v'(\xi)v''(\xi) - 192\lambda v^2(\xi)v'(\xi) \\ &\quad - 2v''(\xi)v^{(5)}(\xi) + 160v(\xi)v''(\xi)v'''(\xi) + 100v'(\xi)(v''(\xi))^2 \\ &\quad - 2880v^2(\xi)v'(\xi)v''(\xi) + 16v^2(\xi)v^{(5)} - 480v^3(\xi)v'''(\xi) + 3840v^4(\xi)v'(\xi) \\ &\quad - 6v'''(\xi)v^{(4)} + 90(v'(\xi))^2v'''(\xi) + 96v(\xi)v'(\xi)v^{(4)} - 1440v(\xi)(v'(\xi))^3 \\ &\quad = 0. \end{aligned} \tag{2.8}$$

To obtain exact solution for equation (2.8), we use the generalized tanh method [10] which consist of the following five steps:

step 1
We consider solutions of (2.8) in the form

$$v(\xi) = \sum_{k=0}^{m} a_k \phi^k(\xi), \tag{2.9}$$

where $\phi(\xi)$ satisfies the Riccati equation

$$\phi'(\xi) = \phi^2(\xi) + k \tag{2.10}$$

whose solutions are given by

$$\phi(\xi) = \begin{cases} -\frac{1}{\xi}, & k = 0 \\ \sqrt{k}\tan\left(\sqrt{k}\xi\right) & k > 0 \\ -\sqrt{k}\cot\left(\sqrt{k}\xi\right) & k > 0 \\ -\sqrt{-k}\tanh\left(\sqrt{-k}\xi\right) & k < 0 \\ -\sqrt{-k}\coth\left(\sqrt{-k}\xi\right) & k < 0. \end{cases} \tag{2.11}$$

The exponent m must be determined by balancing the linear term of the highest order with the nonlinear term in (2.8). We obtain $m = 2$.

step 2
Substituting (2.9) into (2.8) and using (2.10) we obtain a polynomial in $\phi(\xi)$. Equating the coefficients of this polynomial to zero, we get the following system of algebraic equations:

(1) $-16\,k^3\,\lambda\,a_1 + k\,\lambda^2\,a_1 + 52\,k^2\,\lambda\,a_0\,a_1 + 256\,k^3\,{a_0}^2\,a_1 - 192\,k\,\lambda\,{a_0}^2\,a_1 - 960\,k^2\,{a_0}^3\,a_1 + 3840\,k\,{a_0}^4\,a_1 + 180\,k^4\,{a_1}^3 - 1440\,k^3\,a_0\,{a_1}^3 - 256\,k^5\,a_1\,a_2 + 104\,k^3\,\lambda\,a_1\,a_2 + 2176\,k^4\,a_0\,a_1\,a_2 - 5760\,k^3\,{a_0}^2\,a_1\,a_2 + 400\,k^5\,a_1\,{a_2}^2 = 0.$

(2) $-136\,k^2\,\lambda\,a_1 + \lambda^2\,a_1 + 208\,k\,\lambda\,a_0\,a_1 + 2176\,k^2\,{a_0}^2\,a_1 - 192\,\lambda\,{a_0}^2\,a_1 - 3840\,k\,{a_0}^3\,a_1 + 3840\,{a_0}^4\,a_1 + 3912\,k^3\,{a_1}^3 - 192\,k\,\lambda\,{a_1}^3 - 18720\,k^2\,a_0\,{a_1}^3 + 23040\,k\,{a_0}^2\,{a_1}^3 - 5824\,k^4\,a_1\,a_2 + 1196\,k^2\,\lambda\,a_1\,a_2 + 37120\,k^3\,a_0\,a_1\,a_2 - 1152\,k\,\lambda\,a_0\,a_1\,a_2 - 66240\,k^2\,{a_0}^2\,a_1\,a_2 + 46080\,k\,{a_0}^3\,a_1\,a_2 - 15840\,k^3{a_1}^3a_2 + 22048\,k^4\,a_1\,{a_2}^2 - 51840\,k^3\,a_0\,a_1\,{a_2}^2 = 0.$

(3) $-256\,k^4{a_1}^2 + 156\,k^2\lambda\,{a_1}^2 + 2688\,k^3a_0\,{a_1}^2 - 384\,k\,\lambda\,a_0{a_1}^2 - 8640\,k^2{a_0}^2{a_1}^2 + 15360\,k\,{a_0}^3\,{a_1}^2 - 1440\,k^3\,{a_1}^4 - 272\,k^3\,\lambda\,a_2 + 2\,k\,\lambda^2\,a_2 + 416\,k^2\,\lambda\,a_0a_2 + 4352\,k^3\,{a_0}^2\,a_2 - 384\,k\lambda\,{a_0}^2\,a_2 - 7680\,k^2\,{a_0}^3a_2 + 7680\,k\,{a_0}^4a_2 + 5136\,k^4\,{a_1}^2\,a_2 - 20160\,k^3\,a_0\,{a_1}^2\,a_2 - 2624\,k^5\,{a_2}^2 + 208\,k^3\,\lambda\,{a_2}^2 + 8192\,k^4\,a_0\,{a_2}^2 - 11520\,k^3\,{a_0}^2\,{a_2}^2 + 800\,k^5\,{a_2}^3 = 0.$

(4) ${a_1}^4 - 1232\,k^2\,\lambda\,a_2 + 2\,\lambda^2\,a_2 + 1040\,k\,\lambda\,a_0\,a_2 + 19712\,k^2\,{a_0}^2\,a_2 - 384\,\lambda\,{a_0}^2\,a_2 - 19200\,k\,{a_0}^3\,a_2 + 7680\,{a_0}^4\,a_2 + 51344\,k^3\,{a_1}^2\,a_2 - 768\,k\,\lambda\,{a_1}^2\,a_2 - 146880\,k^2\,a_0\,{a_1}^2\,a_2 + 92160\,k\,{a_0}^2\,{a_1}^2\,a_2 - 26176\,k^4\,{a_2}^2 + 1456\,k^2\,\lambda\,{a_2}^2 + 71168\,k^3\,a_0\,{a_2}^2 - 768\,k\,\lambda\,a_0\,{a_2}^2 - 80640\,k^2\,{a_0}^2\,{a_2}^2 + 30720\,k\,{a_0}^3\,{a_2}^2 - 48960\,k^3\,{a_1}^2\,{a_2}^2 + 21152\,k^4\,{a_2}^3 - 34560\,k^3\,a_0\,{a_2}^3 = 0.$

(5) $-240\,k\,\lambda\,a_1 + 156\,\lambda\,a_0\,a_1 + 3840\,k\,{a_0}^2\,a_1 - 2880\,{a_0}^3\,a_1 + 14112\,k^2\,{a_1}^3 - 192\,\lambda\,{a_1}^3 - 38880\,k\,a_0\,{a_1}^3 + 23040\,{a_0}^2\,{a_1}^3 + 3840\,k\,{a_1}^5 - 27008\,k^3\,a_1\,a_2 + 2392\,k\,\lambda\,a_1\,a_2 + 122624\,k^2\,a_0\,a_1\,a_2 - 1152\,\lambda\,a_0\,a_1\,a_2 - 132480\,k\,{a_0}^2\,a_1\,a_2 + 46080\,{a_0}^3\,a_1\,a_2 - 92640\,k^2\,{a_1}^3\,a_2 + 76800\,k\,a_0\,{a_1}^3\,a_2 + 151168\,k^3\,a_1\,{a_2}^2 - 960\,k\,\lambda\,a_1\,{a_2}^2 - 296640\,k^2\,a_0\,a_1\,{a_2}^2 + 115200\,k\,{a_0}^2\,a_1\,{a_2}^2 - 57600\,k^3\,a_1\,{a_2}^3 = 0.$

(6) $-1856\,k^3\,{a_1}^2 + 416\,k\,\lambda\,{a_1}^2 + 12928\,k^2\,a_0\,{a_1}^2 - 384\,\lambda\,a_0\,{a_1}^2 - 23040\,k\,{a_0}^2\,{a_1}^2 + 15360\,{a_0}^3\,{a_1}^2 - 11040\,k^2\,{a_1}^4 + 15360\,k\,a_0 - 120\,\lambda\,a_1 + 1920\,{a_0}^2\,a_1 +$

$17464\,k\,a_1{}^3 - 21600\,a_0\,a_1{}^3 + 3840\,a_1{}^5 - 49664\,k^2\,a_1\,a_2 + 1300\,\lambda\,a_1\,a_2 + 144128\,k\,a_0\,a_1\,a_2 - 72000\,a_0{}^2\,a_1\,a_2 - 147360\,k\,a_1{}^3\,a_2 + 76800\,a_0\,a_1{}^3\,a_2 + 356320\,k^2\,a_1\,a_2{}^2 - 960\,\lambda\,a_1\,a_2{}^2 - 466560\,k\,a_0\,a_1\,a_2{}^2 + 115200\,a_0{}^2\,a_1\,a_2{}^2 + 53760\,k\,a_1{}^3\,a_2{}^2 - 265920\,k^2\,a_1\,a_2{}^3 + 107520\,k\,a_0\,a_1\,a_2{}^3 = 0.$

(7) $-4288\,k^2\,a_1{}^2 + 260\,\lambda\,a_1{}^2 + 18304\,k\,a_0\,a_1{}^2 - 14400\,a_0{}^2\,a_1{}^2 - 19680\,k\,a_1{}^4 + 15360\,a_0\,a_1{}^4 - 1680\,k\,\lambda\,a_2 + 624\,\lambda\,a_0\,a_2 + 26880\,k\,a_0{}^2\,a_2 - 11520\,a_0{}^3\,a_2 + 142928\,k^2\,a_1{}^2\,a_2 - 768\,\lambda\,a_1{}^2\,a_2 - 256320\,k\,a_0\,a_1{}^2\,a_2 + 92160\,a_0{}^2\,a_1{}^2\,a_2 + 23040\,k\,a_1{}^4\,a_2 - 87680\,k^3\,a_2{}^2 + 2496\,k\,\lambda\,a_2{}^2 + 185856\,k^2\,a_0\,a_2{}^2 - 768\,\lambda\,a_0\,a_2{}^2 - 138240\,k\,a_0{}^2\,a_2{}^2 + 30720\,a_0{}^3\,a_2{}^2 - 247680\,k^2\,a_1{}^2\,a_2{}^2 + 138240\,k\,a_0\,a_1{}^2 a_2{}^2 + 116736\,k^3\,a_2{}^3 - 384\,k\,\lambda\,a_2{}^3 - 172800\,k^2\,a_0\,a_2{}^3 + 46080\,k\,a_0{}^2\,a_2{}^3 - 23040\,k^3\,a_2{}^4 = 0.$

(8) $-4032\,k\,a_1{}^2 + 8064\,a_0\,a_1{}^2 - 10080\,a_1{}^4 - 720\,\lambda\,a_2 + 11520\,a_0{}^2\,a_2 + 153392\,k\,a_1{}^2\,a_2 - 129600\,a_0\,a_1{}^2\,a_2 + 23040\,a_1{}^4\,a_2 - 133248\,k^2\,a_2{}^2 + 1248\,\lambda\,a_2{}^2 + 192000\,k\,a_0\,a_2{}^2 - 69120\,a_0{}^2\,a_2{}^2 - 365760\,k\,a_1{}^2\,a_2{}^2 + 138240\,a_0\,a_1{}^2\,a_2{}^2 + 246144\,k^2\,a_2{}^3 - 384\,\lambda\,a_2{}^3 - 253440\,k\,a_0\,a_2{}^3 + 46080\,a_0{}^2\,a_2{}^3 + 61440\,k\,a_1{}^2\,a_2{}^3 - 99840\,k^2\,a_2{}^4 + 30720\,k\,a_0\,a_2{}^4 = 0.$

(9) $-12096\,a_1\,a_2 + 119808\,a_1\,a_2{}^2 - 164160\,a_1\,a_2{}^3 + 34560\,a_1\,a_2{}^4 = 0.$

(10) $7084\,a_1{}^3 - 40320\,k\,a_1\,a_2 + 56448\,a_0\,a_1\,a_2 - 70560\,a_1{}^3\,a_2 + 346608\,k\,a_1\,a_2{}^2 - 221760\,a_0\,a_1\,a_2{}^2 + 53760\,a_1{}^3\,a_2{}^2 - 372480\,k\,a_1\,a_2{}^3 + 107520\,a_0\,a_1\,a_2{}^3 + 34560\,k\,a_1\,a_2{}^4 = 0.$

(11) $-25920\,a_2{}^2 + 73440\,a_2{}^3 - 57600\,a_2{}^4 + 7680\,a_2{}^5 = 0$

(12) $-1344\,a_1{}^2 + 56672\,a_1{}^2\,a_2 - 95040\,k\,a_2{}^2 + 69120\,a_0\,a_2{}^2 - 167040\,a_1{}^2\,a_2{}^2 + 223200\,k\,a_2{}^3 - 115200\,a_0\,a_2{}^3 + 61440\,a_1{}^2\,a_2{}^3 - 134400\,k\,a_2{}^4 + 30720\,a_0\,a_2{}^4 + 7680\,k\,a_2{}^5 = 0.$

step 3:
Upon solving the above system with respect to the unknowns k, λ, a_0, a_1, a_2, and using the elimination method of Wu [11] and the Mathematica software [7], we consider only the following solutions:

$$a_1 = 0, \quad a_2 = \frac{3}{4}, \quad k = \pm\sqrt{\lambda}, \quad a_0 = \pm\frac{\sqrt{\lambda}}{2}.$$

(The other solutions of the above system are given by $a_1 = 0$, $a_2 = 0$, k, λ, a_0 arbitrary constants. But this case is not interesting).

step 4:
According to (2.11) and taking into account the results of the previous step (solutions of the system of algebraic equations) after simplifications, we restrict ourselves to considering only the following solutions for (2.8) :
For $\lambda > 0$,

(1) $v_1(\xi) = \frac{\sqrt{\lambda}}{2} + \frac{3}{4}\sqrt{\lambda}\cot^2\left(\lambda^{\frac{1}{4}}\xi\right).$

(2) $v_2(\xi) = -\frac{\sqrt{\lambda}}{2} + \frac{3}{4}\sqrt{\lambda}\coth^2\left(\lambda^{\frac{1}{4}}\xi\right).$

(3) $v_3(\xi) = \frac{\sqrt{\lambda}}{2} + \frac{3}{4}\sqrt{\lambda}\tan^2\left(\lambda^{\frac{1}{4}}\xi\right)$.

(4) $v_4(\xi) = -\frac{\sqrt{\lambda}}{2} + \frac{3}{4}\sqrt{\lambda}\tanh^2\left(\lambda^{\frac{1}{4}}\xi\right)$.

step 5 :
We obtain solutions for system (2.1). By (2.2) and (2.7), the solutions for $u(x,t)$ and $w(x,t)$ are given by (for $\lambda > 0$):

(1) $u_1(x,t) = \frac{\sqrt{\lambda}}{2} + \frac{3}{4}\sqrt{\lambda}\cot^2\left(\lambda^{\frac{1}{4}}(x+\lambda t)\right)$

$w_1(x,t) = \frac{\lambda^{\frac{3}{2}}}{6}$

(2) $u_2(x,t) = -\frac{\sqrt{\lambda}}{2} + \frac{3}{4}\sqrt{\lambda}\coth^2\left(\lambda^{\frac{1}{4}}(x+\lambda t)\right)$.

$w_2(x,t) = -\frac{\lambda^{\frac{3}{2}}}{6}$

(3) $u_3(x,t) = \frac{\sqrt{\lambda}}{2} + \frac{3}{4}\sqrt{\lambda}\tan^2\left(\lambda^{\frac{1}{4}}(x+\lambda t)\right)$.

$w_3(x,t) = \frac{\lambda^{\frac{3}{2}}}{6}$

(4) $u_4(x,t) = -\frac{\sqrt{\lambda}}{2} + \frac{3}{4}\sqrt{\lambda}\tanh^2\left(\lambda^{\frac{1}{4}}(x+\lambda t)\right)$.

$w_4(x,t) = -\frac{\lambda^{\frac{3}{2}}}{6}$

Remark 1: One also can consider solutions with $k = -\sqrt{\lambda}$.
Remark 2: In the case $w(x,t) = 0$, system (2.1) is reduced to well-known Kaup–Kupershmidt equation [6], [8], [9], [12]

$$u_t = u_{xxxxx} - 20uu_{xxx} - 50u_xu_{xx} + 80u^2u_x \tag{2.12}$$

which is a particular case of a class of fifth order PDEs with four parameters

$$u_t + \rho u_{xxxxx} + \gamma uu_{xxx} + \beta u_xu_{xx} + \alpha u^2u_x = 0 \tag{2.13}$$

which has known solutions in some cases:

- For $\alpha = 30$, $\beta = 20$, $\gamma = 10$, $\rho = 1$ [6].
- For $\alpha = 5$, $\beta = 5$, $\gamma = 5$, $\rho = 1$ (2.13) reduces to Sawada and Kotera (SK) equation [6].
- For $\alpha = 20$, $\beta = 25$, $\gamma = 10$, $\rho = 1$ [6].

Remark 3: A plenty of exact solutions for the Kaup–Kupershmidt equation can be found using the inverse scattering transform for the corresponding spectral problem [13]. In [8], Sergyeyev present a *zero curvature representation* **ZCR** for the M–N–W system. Knowing a **ZCR** involving an essential (spectral) parameter enables one to use the inverse scattering transform and construct plenty of exact solutions for the system in question, including multisoliton and finite-gap solutions [8].

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