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Local convergence for the curve tracing of the homotopy method

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Abstract. The local convergence of a Newton-method for the tracing of an implicitly defined smooth curve is analyzed. The domain of attraction is shown to be larger than in [6]. Moreover finer error bounds on the distances involved are obtained and quadratic instead of geometrical order of convergence is established. A numerical example is also provided where our results compare favourably with the corresponding ones in [6].

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Resumen. Se analiza la convergencia local de un método de Newton para trazado de una curva suave definida implícitamente. Se muestra que el dominio de atracción es más grande que en [6]. Además se obtienen errores mas finos para las cotas de las distancias involucradas y se establece orden cuadrático en lugar de lineal para la convergencia. Se da un ejemplo numérico donde nuestro resultado se compara favorablemente con los resultados correspondientes en [6].

1. Introduction

We are concerned with the following problem: Suppose that a smooth curve $\Gamma \subset \mathbb{R}^{n+1}$ is implicitly defined by

$$F(x,t) = 0\,, \tag{1.1}$$

where $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a C^2 function. We intend to numerically trace curve Γ from the point (x_0, t_0) to the point (x^*, t^*). We assume the $n \times (n+1)$ Jacobian matrix $DF(x,t)$ has full rank at every point in Γ. A survey of such techniques can be found in [1], [8] and the references there.

We will use the following algorithmic form:

(a) Let $y_i = (x_i, t_i) \in \mathbb{R}^{n+1}$ be an approximation for Γ. Use the predictor

$$z_0 = y_i + h_i \tau_i \tag{1.2}$$

for the next approximating point, where h_i is an appropriate step length and τ_i is the tangent vector of Γ at y_i;

(b) Starting from z_0, take a sequence of Newton iterations by requiring z_k to lie on the hyperplane normal to a certain vector (usually the tangent vector τ_i);

(c) Set $y_{i+1} = z$ where z is the point of convergence for the sequence $\{z_k\}$.

We need some preliminaries:

A point (x, t) in $\mathbb{R}^{n+1}$ will be denoted by y. Let σ be the arc length, along the curve Γ, then an initial value problem is implicitly defined by

$$DF(y) \cdot \dot{y} = 0; \qquad y(0) = y_0, \tag{1.3}$$

where $\cdot = \frac{d}{d\sigma}$. It is known that vector field $\dot{y}$ is locally Lipschitzian [8].

We assume $DF(y)$ is full rank along the solution curve, then equation

$$DF(y)\, y' = -F(y) \tag{1.4}$$

can be reduced to

$$y' = -DF^{+}(y)\, F(y) \tag{1.5}$$

where $DF^{+}(y) = DF^{T}(y)\left[DF(y)\, DF^{T}(y)\right]^{-1}$ is the Moore-Penrose generalized inverse of $DF(y)$. By the result

$$\operatorname{rang}\left(DF^{+}\right) = \operatorname{rang}\left(DF^{T}\right) = \ker(DF)^{\perp} \tag{1.6}$$

and equation

$$F(y(\tau)) = e^{-\tau} F(y(0)) \tag{1.7}$$

we conclude a solution $y(\tau)$ of (1.5) is such that the magnitude of $F(y)$ is reduced and also remains perpendicular to the 1−dimensional kernel space of $F(y)$.

Consider the Euler step of (1.5). This corresponds to the Newton method in the form

$$y_{k+1} = y_k - DF^{+}(y_k)\, F(y_k). \tag{1.8}$$

In the next section we analyze the local convergence of method (1.8).

We state a result whose proof can be found in [6, p. 327]:

Theorem 1.1. *Let $F : D \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a C^2 function such that*

$$\|DF(x) - DF(y)\| \le \ell \|x - y\|, \qquad \textit{for all } x, y \in D. \tag{1.9}$$

Suppose that $F(x^*)$ and $DF(x^*)$ is full rank. Let $\delta \in \left(0, \frac{3-\sqrt{5}}{2}\right)$ and define

$$M = \min\left\{\frac{2}{3\|DF^{+}(x^*)\|\,\ell},\ dist(x^*, \partial D)\right\}. \tag{1.10}$$

If $r \in (0, \delta M = r_0)$ *is such that for every* $x \in U(x^*, r) = \{x \in \mathbb{R}^{n+1} : \|x - x^*\| \leq r\}$ *we have*

$$\|F(x)\| \leq \frac{\delta \ell M^2}{2}, \tag{1.11}$$

then for any $x_0 \in U(x^*, r) \subseteq D$*, method (1.8) is well defined and converges geometrically to a point in* $\Gamma \cap U(x^*, M)$.

Remark 1.1. *Under the hypotheses of Theorem 1.1 method (1.8) converges only geometrically and condition (1.1) should hold. To do so we first introduce the center Lipschitz condition*

$$\|DF(x) - DF(x^*)\| \leq \ell_0 \|x - x^*\|, \qquad \text{for all } x \in D. \tag{1.12}$$

We note that in general

$$\ell_0 \leq \ell \tag{1.13}$$

holds and $\frac{\ell}{\ell_0}$ can be arbitrarily large. In practice the computation of ℓ requires that of ℓ_0.

Then we can show the following improvement over Theorem 1.1.

Theorem 1.2. *Suppose hypotheses of Theorem 1.1 and (1.12) hold but* M *is defined as*

$$M_0 = \min\left\{\frac{2}{(2\ell_0 + \ell)\|DF^+(x^*)\|},\ dist(x^*, \partial D)\right\}, \tag{1.14}$$

then the conclusions of Theorem 1.1 hold with M_0 *replacing* M.

Proof. For any $x \in U(x^*, M_0)$, we get using Lemma 3.1 in [6, p. 326] and (1.12):

$$\|DF(x) - DF(x^*)\| \left\|DF^+(x^*)\right\| \leq \ell_0 \|x - x^*\| \left\|DF^+(x^*)\right\| < \frac{2}{3} < 1. \tag{1.15}$$

The rest of the proof follows exactly as in Theorem 1 in [6, p. 326] (with M_0 replacing M). That completes the proof of the theorem. ☑

Remark 1.2. *If equality holds in (1.13) then Theorem 1.2 reduces to Theorem 1.1. Otherwise*

$$M < M_0 \tag{1.16}$$

holds and the bounds on the distances $\|y_{n+1} - y_n\|$, $\|y_{n+1} - x^*\|$ $(n \geq 0)$ *are finer in Theorem 1.2. This improvement allows a wider choice of initial guesses* x_0*. Such an observation is important in computational mathematics. By comparing (1.10) and (1.14) we see that* M_0 *can be (at most) three times larger than* M *(if* $\ell_0 = \ell$*).*

In order to show that it is possible to achieve quadratic convergence and drop strong condition (1.11) we use a modification of our Theorem 2 in [3] *(where we have replaced* $F'(x)^{-1}$ *by* $DF(x)^+$ *and use Lemma 3.1 in* [6] *instead of Banach Lemma on invertible operators in the proof of Theorem 2 in* [3] *to obtain the proof of Theorem 1.3 that follows:*

Theorem 1.3. *Assume conditions of Theorem 1.2 hold excluding (1.11). If*

$$U_1\left(x^*, r_1\right) \subseteq D\,, \tag{1.17}$$

where

$$r_1 = \frac{1}{\ell_0 \left\| DF\left(x^*\right)^+ \right\|}\,, \tag{1.18}$$

then for all $x_0 \in U_2\left(x^*, r_2\right)$, *where*

$$r_2 = \frac{2+\gamma-\sqrt{\gamma^2+2\gamma}}{\left(2+\gamma\right)\ell_0 \left\| DF\left(x^*\right)^+ \right\|}, \qquad \text{for } \gamma \geq 2,\ \ell = \frac{\gamma}{2}\ell_0\,, \tag{1.19}$$

the following hold:

Newton-Kantorovich hypothesis

$$h = 2\ell \left\| DF\left(x_0\right)^+ \right\| \left\| DF\left(x_0\right)^+ F\left(x_0\right) \right\| \leq 1 \tag{1.20}$$

holds as strict inequality, and consequently the Newton-Kantorovich theorem guarantees method (1.8) is well-defined and converges quadratically to a point in $\Gamma \cap U\left(x^*, r_1\right)$.

Remark 1.3. *Even if equality holds in (1.13) we can set* $\gamma = 2$ *and* r_2 *can be written as*

$$r_2 = \frac{2-\sqrt{2}}{2\ell_0 \left\| DF\left(x^*\right)^+ \right\|}\,, \tag{1.21}$$

which is larger than r_0 *since*

$$\delta < \frac{2-\sqrt{2}}{2}\,. \tag{1.22}$$

If strict inequality holds in (1.13) then r_2 *is enlarged even further (see also Example 1.4 as follows).*

Convergence radius r_2 *can be extended even further by using Theorem 3 in* [3] *based on an even weaker hypotheses than (1.20) found by us in Section 1.2:*

$$h_0 = \left(\ell + \ell_0\right) \left\| DF\left(x_0\right)^+ \right\| \left\| DF\left(x_0\right)^+ F\left(x_0\right) \right\| \leq 1. \tag{1.23}$$

However we do not pursue this here, leaving it for the motivated reader.

Instead we provide an example where strict inequality holds in (1.13).

Example 1.4. *Let* $D = U\left(0,1\right)$ *and define function* F *on the real line by*

$$F\left(x\right) = e^x - 1\,. \tag{1.24}$$

For simplicity we take $x_0 = x^*$. *We obtain*

$$\begin{aligned} \ell &= e\,, \\ \ell_0 &= e - 1\,, \\ \left\| DF\left(x^*\right)^+ \right\| &= 1\,, \\ \gamma &= 3.163953415\,, \\ \delta &= .381966011\,, \\ M &= .245252961\,, \\ M_0 &= .324947231\,, \\ r_0 &= \delta M = .093678295\,, \\ \bar{r}_0 &= \delta M_0 = .124118798\,, \\ r_1 &= .581976707\,, \\ r_2 &= .126433594\,. \end{aligned}$$

Therefore we conclude

$$M < M_0 < r_1$$

and

$$r_0 < \bar{r}_0 < r_2,$$

which demonstrate the superiority of our results over the ones in [6].

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