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Fixed points of the contractive or expansive type for multivalued mappings: toward a unified approach

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Abstract. A condition on the functions $\varphi : R^+ \to R^+ = [0, +\infty)$ which, for single valued maps, has proved useful in asserting the existence of fixed points for contractions or expansions relative to either distances or w-distances, is now used to examine the behaviour of multivalued mappings. Since it applies equally to both contracting ($\varphi(t) < t$ for $t > 0$) or expanding maps ($\varphi(t) > t$ for all $t > 0$), it also allows, to some extent, a unified approach to both types of problems.

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1. **Introduction**

Many authors have dealt with the *Banach contraction principle* for multiple maps in a metric space (X, d) within the context of *commuting mappings*, i.e., when $f \circ g(x) = g \circ f(x)$ for all $x \in X$. See, for example, Chang[3], Das and Dabata[4], Jungck[8, 10], Pant[13], Ray[14]. The problem has also been examined under less restrictive assumptions, such as those of *weak commutativity*, when $d(f \circ g(x), g \circ f(x)) \leq d(f(x), g(x))$ for all $x \in X$ (see Carbone et al.[1], Fisher and Sessa[7]), or of *compatibility*, when for any sequence (x_n) in X, from $\lim f(x_n) = \lim g(x_n)$ it follows that $\lim d(f \circ g(x_n), g \circ f(x_n)) = 0$, in which case we also say that (f, g) is *a compatible pair* (Jungck[9], Kang and Rhoades[11], Rodríguez-Montes and Charris[17]).

All of the above concepts can be extended to maps defined in a metric space (X, d) and taking as values sets of points of the same space, i.e., to *multivalued* *maps* of (X, d). In particular, if T is a multivalued map on X, a point $x \in X$ such that $x \in T(x)$ is called a *fixed point* of T.

On the other hand, the constant $0 < \alpha < 1$ of classical contraction theory can be replaced by a function $\phi : R^+ \to R^+ = [0, +\infty)$ with $\phi(t) < t$ for all $t > 0$, usually satisfying additional conditions (continuity, semicontinuity, monotonicity, etc.). See Carbone et al.[1], Dugundji and Granas[5], Kang and Rhoades[11], Rodríguez-Montes[16], in this respect. Within this framework, Chang[2] has established the following result. In all what follows, if Λ is a set of subsets of X, $[\Lambda]$ will denote the union of all sets $A \in \Lambda$.

Theorem 1.1 (Chang[2]). *Let (X, d) be a complete metric space, I and J be selfmaps of X, and $S, T : X \to \mathcal{B}(X)$, where $\mathcal{B}(X)$ is the class of bounded, non empty subsets of X, be such that $[S(X)] \subseteq J(X)$ and $[T(X)] \subseteq I(X)$. Further assume that for all $x, y \in X$,*

$$\delta(Sx, Ty) \leq \phi(max\{d(Ix, Jy), \delta(Ix, Sx), \delta(Jy, Ty), \frac{1}{2}(D(Ix, Ty) + D(Jy, Sx))\}), \tag{1.1}$$

where $\phi : R^+ \to R^+$, with $\phi(t) < t$, $t > 0$, is upper semicontinuous, that both (I, S) and (T, J) are compatible, and that at least one of I or J is continuous. Then I, J, S and T have a unique common fixed point z in X. Furthermore

$$Sz = Tz = \{Iz\} = \{Jz\} = \{z\}. \tag{1.2}$$

For the concepts and notations in the statement of Theorem 1.1, see Section 2 below.

In this paper we will follow at first ideas and techniques in Rodríguez-Montes and Charris[17] to generalize results of the contractive type in Carbone et al.[1], Chang[2], Jungck[8, 9, 10], Kang and Rhoades[11], as well as in Kubiak[12], Pant[13], Rodríguez-Montes[16], Sing and Whitfield[19]. Then we will turn our attention to expansions. Assumptions such as the compatibility of maps will be replaced by less stringent conditions. Other assumptions, such as the semicontinuity of ϕ in Theorem 1.1, will also be considerably weakened. As a matter of fact, we will only requiere the functions $\phi : R^+ \to R^+ = [0, +\infty)$ to satisfy the simple Condition (A) below. Such functions have proved very valuable in establishing results of the contractive type ($\phi(t) < t$ for all $t > 0$) or of the expansive type ($\phi(t) > t$ for all $t > 0$) for one or multiple single-valued maps on a metric space (as in Rodríguez-Montes and Charris[17]), as well as for contractions or expansions of single-valued maps relative to w-distances in uniform spaces (as in Rodríguez-Montes and Charris[18]). We will now explore the implications of Condition (A) in the case of multivalued maps.

Condition (A) for $\varphi : R^+ \to R^+$ is the following:

(A) *For any decreasing sequence* (t_n) *in* R^+ *(i.e.,* $t_{n+1} < t_n$ *for all* $n \geq 1$*) such that*

$$\lim t_n = \lim \varphi(t_n) = t, \tag{1.3}$$

it follows that $t = 0$.

Condition (A) is seen to hold for most contracting or expanding functions ϕ appearing in the fixed point theory of single valued maps, allowing to simplify arguments, and even leading to the remotion of hypothesis. Since it applies to both types of functions, it also provides a unifying approach to contractions and expansions. We shall see that this also holds for multivalued maps.

We observe that for contracting ($\varphi(t) < t$ for all $t > 0$) or expanding functions ($\varphi(t) > t$ for all $t > 0$), Condition (A) is obviously equivalent to the apparently stronger Condition (B) below.

(B) *For any non increasing sequence* (t_n) *in* $(0, +\infty)$ *such that (1.3) holds, it follows that* $t = 0$.

2. Basic definitions, results and notations

Definition 2.1. . *In a metric space* (X, d)*, we define:*
1. $2^X = \{A \subseteq X / A \neq \phi\}$.
2. $\mathcal{B}(X) = \{A \in 2^X / A$ *is bounded*$\}$, $\mathcal{B}_c(X) = \{A \in 2^X / A$ *is closed and bounded*$\}$.
3. For $A, B \in \mathcal{B}(X)$,

$$\delta(A, B) = Sup\{d(a, b)/a \in A, b \in B\}.$$

Then $\delta(A) = \delta(A, A)$ *is called the diameter of* A.
4. For $a \in X$, $S \in \mathcal{B}(X)$ *and* $r > 0$,

$$d(a, S) = Inf\{d(a, s)/s \in S\}, \qquad S_r = \{a \in X/d(a, S) < r\}.$$

5. For $A, B \in \mathcal{B}(X)$,

$$\begin{aligned} D(A, B) &= Inf\{d(a, b)/a \in A, b \in B\}, \\ H(A, B) &= Inf\{r > 0/A \subseteq B_r, B \subseteq A_r\}. \end{aligned}$$

It is easily verified that $d(a, S) = 0$ if and only if $a \in \bar{S}$, the closure of S, i.e., $\bar{S} = \cap_{r>0} S_r$. Also $D(A, B) = D(\bar{A}, \bar{B})$, $H(A, B) = H(\bar{A}, \bar{B})$. Furthermore, $H(A, B) = 0$ if and only if $\bar{A} = \bar{B}$. Since obviously $H(A, B) = H(B, A)$ and $H(A, B) \leq H(A, C) + H(C, B)$, H is a metric on $\mathcal{B}_c(X)$, and if (X, d) is complete, it can be proved that $(\mathcal{B}_c(X), H)$ is also complete. Finally observe that $D(A, B) \leq H(A, B) \leq \delta(A, B) \leq \delta(A \cup B)$.

Definition 2.2 (Chang[2]). *A set valued map $S : X \to \mathcal{B}(X)$ is continuous if for any $x \in X$ and any sequence (x_n) in X converging to x,* $\lim H(Sx_n, Sx) = 0$.

Definition 2.3 (Chang[2]). *The maps $I : X \to X$ and $S : X \to \mathcal{B}(X)$ are compatible if $IS(x) \in \mathcal{B}(X)$ for any $x \in X$ and* $\lim H(SI(x_n), IS(x_n)) = 0$ *for any sequence (x_n) in X such that* $\lim \delta(Ix_n, Sx_n) = 0$*. We also say that (I, S) is a compatible pair.*

Definition 2.4. *The maps $I : X \to X$ and $S : X \to \mathcal{B}(X)$ are locally commuting if $SI(x) = IS(x)$ at any $x \in X$ such that $S(x) = \{I(x)\}$. It is also said that (I, S) is a locally commuting pair.*

Remark 2.1. *Clearly if (I, S) is compatible, it is locally commuting. For single valued maps, local commutativity reduces to commutativity at coincidence points (i.e., at points x such that $Sx = Ix$).*

For *contracting maps*, i.e., for functions $\phi : R^+ \to R^+$ such that $\phi(t) < t$ for all $t > 0$, a weaker condition than (A) is the following.

(C) *For any decreasing sequence (t_n) in R^+ such that $t_{n+1} \leq \phi(t_n)$ for all $n \geq 1$, if (1.3) holds, then $t = 0$.*

Since (t_n) is assumed decreasing, we may as well require $t_{n+1} < \phi(t_n)$ for all $n \geq 1$, in Condition (C) above.

Condition (C) was also introduced in Rodríguez-Montes and Charris[17], but to stress the unifying quality of (A), and in spite of the fact that (C) was all that was needed in many instances, most proofs were given appealing to (A).

In this paper we will use more sistematically Condition (C), with the conviction that, being a specialization of Condition (A), it does not hide the unifying power of the latter. It further simplifies many arguments, though.

The following lemma improves Lemma 3.1 in [17] or Lemma 1.3 in [18]. In what follows, Φ^* will denote the set of contracting maps $\phi : R^+ \to R^+$ satisfying Condition (C).

Lemma 2.1. *Let $\phi : R^+ \to R^+$ be contractive and for each $t \geq 0$ let*

$$\tilde{\phi}(t) = Sup\{\phi(x)/0 < x \leq t\}, \quad t > 0\ ; \quad \tilde{\phi}(0) = 0. \tag{2.1}$$

Then, for $\phi \in \Phi^*$, the following holds:

(i) *$\tilde{\phi}$ is a nondecreasing function such that $\tilde{\phi}(0) = 0$ and $\phi(t) \leq \tilde{\phi}(t) < t$ or $\phi(t) < \tilde{\phi}(t) \leq t$ for all $t > 0$.*
(ii) *For each $\epsilon > 0$ there is $0 < t \leq \epsilon$ such that $\tilde{\phi}(t) < t$.*
(iii) *$\tilde{\phi}$ satisfies Condition (C).*

Proof. Condition (i) follows at once from the definition of $\tilde{\phi}$. To prove (ii), assume that there is $\epsilon > 0$ such that $\tilde{\phi}(t) = t$ for all $t \leq \epsilon$. Let $0 < \alpha < \epsilon$. Since $\tilde{\phi}(\epsilon) = \epsilon$, there exists t_1 in R^+ such that $\alpha < \phi(t_1) < t_1 \leq \epsilon$. Since $\tilde{\phi}(\phi(t_1)) = \phi(t_1)$, also t_2 exists such that $\alpha < \phi(t_2) < t_2 \leq \phi(t_1)$. Iteration of the argument then yields a decreasing sequence (t_n) in R^+ such that $t_{n+1} \leq \phi(t_n) < t_n$ for $n \geq 1$ and $\lim \phi(t_n) = \lim t_n \geq \alpha > 0$, which is absurd. To establish (iii), let (t_n) be a decreasing sequence in R^+ with $t_{n+1} \leq \tilde{\phi}(t_n)$ and such that $\lim \tilde{\phi}(t_n) = \lim t_n = t$. The definition of $\tilde{\phi}$ yields a sequence (s_n) in R^+ such that $t_{n+1} < \phi(s_n) < s_n \leq t_n$ and $\phi(s_n) \leq \tilde{\phi}(t_n), n \geq 1$, so that (s_n) is decreasing and $\lim s_n = \lim \phi(s_n) = t$. Since ϕ verifies Condition (C) then $t = 0$, and the assertion follows. ☑

Remark 2.2. *The above definition of $\tilde{\phi}$ differs from that in Rodríguez-Montes and Charris*[17, 18]*, unless $\phi(0) = 0$, which was implicitly (but not explicitly) assumed in those papers (otherwise $\tilde{\phi}(0) = \phi(0) > 0$ and $\tilde{\phi}$ would still be nondecreasing, so that, since $\tilde{\phi}(0+) = \tilde{\phi}(0) > 0$, $\tilde{\phi}(t) \leq t$ could not hold for all $t > 0$). It may happen that Condition (A) holds for ϕ but not for $\tilde{\phi}$.*

3. Results of the contractive type in metric spaces

Let (X, d) be a metric space and $\mathcal{B}(X)$ be the set of bounded non empty subsets of X. We recall that if Λ is a set of subsets of X, $[\Lambda]$ stands for their union. Observe that if $I, J : X \to X$ and $S, T : X \to \mathcal{B}(X)$ are such that $[S(X)] \subseteq J(X)$ and $[T(X)] \subseteq I(X)$, then, starting with $x_0 \in X$ arbitrary, a sequence (x_n) in X can be found such that

$$Jx_{2n+1} \in Sx_{2n}, Ix_{2n+2} \in Tx_{2n+1}, n \geq 0. \tag{3.1}$$

Theorem 3.1. *Let (X, d) be a complete metric space and I, J be single valued selfmaps of X. Let $S, T : X \to \mathcal{B}(X)$ be multivalued maps such that:*

(i) *For some $\phi \in \Phi^*$,*

$$\delta(Sx, Ty) \leq \phi(max\{d(Ix, Jy), \delta(Ix, Sx), \delta(Jy, Ty), \frac{1}{2}(D(Ix, Ty) + D(Jy, Sx)) \tag{3.2}$$

holds for all x, y in X.

(ii) $[S(X)] \subseteq J(X)$, $[T(X)] \subseteq I(X)$.

(iii) *For any sequence (x_n) in X as in (3.1), if*

$$\lim Ix_{2n} = \lim Jx_{2n+1} = y \tag{3.3}$$

for some $y \in X$, then $y \in I(X) \cup J(X) \cup [S(X)] \cup [T(X)]$.

(iv) *Both pairs of maps (I, S) and (J, T) are locally commuting.*

Then, I, J, S and T have a unique common fixed point y_0 in X, and

$$Sy_0 = Ty_0 = \{Iy_0\} = \{Jy_0\} = \{y_0\}. \tag{3.4}$$

Proof. From (ii), a sequence (x_n) as in (3.1) can be chosen. Let (Y_n) be the sequence of subsets of X defined by $Y_{2n} = Sx_{2n}$, $Y_{2n+1} = Tx_{2n+1}$, $n \geq 0$. It follows from (i) that

$$\begin{aligned}
\delta(Y_{2n+2}, Y_{2n+1}) &= \delta(Sx_{2n+2}, Tx_{2n+1}) \\
&\leq \phi(max\{d(Ix_{2n+2}, Jx_{2n+1}), \delta(Ix_{2n+2}, Sx_{2n+2}), \delta(Jx_{2n+1}, Tx_{2n+1}), \\
&\qquad \frac{1}{2}(D(Ix_{2n+2}, Tx_{2n+1}) + D(Jx_{2n+1}, Sx_{2n+2}))\}) \\
&= \phi(max\{\delta(Ix_{2n+2}, Y_{2n+2}), \delta(Jx_{2n+1}, Y_{2n+1}), \frac{1}{2}(D(Jx_{2n+1}, Y_{2n+2})\}) \\
&\leq \tilde{\phi}(max\{d(Y_{2n+1}, Y_{2n+2}), \delta(Y_{2n}, Y_{2n+1}), \\
&\qquad \frac{1}{2}(\delta(Y_{2n+1}, Y_{2n+2}) + \delta(Y_{2n}, Y_{2n+1}))\}).
\end{aligned}$$

Since the assumption $\delta(Y_{2n+1}, Y_{2n+2}) > \delta(Y_{2n}, Y_{2n+1})$ leads to

$$\delta(Y_{2n+2}, Y_{2n+1}) \leq \tilde{\phi}(\delta(Y_{2n+1}, Y_{2n+2})) < \delta(Y_{2n+1}, Y_{2n+2})$$

or

$$\delta(Y_{2n+2}, Y_{2n+1}) < \tilde{\phi}(\delta(Y_{2n+2}, Y_{2n+1})) \leq \delta(Y_{2n+2}, Y_{2n+1}),$$

which is contradictory, then

$$\delta(Y_{2n+1}, Y_{2n+2}) \leq \tilde{\phi}(\delta(Y_{2n}, Y_{2n+1})) \leq \delta(Y_{2n}, Y_{2n+1}), n \geq 1. \tag{3.5}$$

Similarly,

$$\delta(Y_{2n+1}, Y_{2n}) \leq \tilde{\phi}(\delta(Y_{2n}, Y_{2n-1})) \leq \delta(Y_{2n}, Y_{2n-1}), n \geq 1. \tag{3.6}$$

Let $t_n = \delta(Y_{n+1}, Y_n)$, $n \geq 0$, and assume first that $t_n > 0$ for all $n \geq 0$. Since, from Lemma 2.1 (i), $\lim t_n = \lim \tilde{\phi}(t_n)$, then $\lim \delta(Y_n, Y_{n+1}) = 0$.

Also, if m is even, n is odd and $m > n$, then

$$\begin{aligned}
\delta(Y_m, Y_n) &= \delta(Sx_m, Tx_n) \\
&\leq \tilde{\phi}(max\{d(Ix_m, Jx_n), \delta(Ix_m, Sx_m), \delta(Jx_n, Tx_n), \\
&\qquad \frac{1}{2}(D(Ix_m, Tx_n) + D(Jx_n, Sx_m))\}) \\
&\leq \tilde{\phi}(max\{\delta(Y_{m-1}, Y_{n-1}), \delta(Y_m, Y_{m-1}), \delta(Y_n, Y_{n-1}), \\
&\qquad \frac{1}{2}(\delta(Y_n, Y_{m-1}) + \delta(Y_{n-1}, Y_m))\}), \\
&\leq \tilde{\phi}(max\{\delta(Y_i, Y_j)/n - 1 \leq i \neq j \leq m\}).
\end{aligned}$$

Let $\epsilon > 0$ be such that $\tilde{\phi}(\epsilon) < \epsilon$ (Lemma 2.1), and let $\delta = (\epsilon - \tilde{\phi}(\epsilon))/2$ and $N \geq 1$ be such that $\delta(Y_{n+1}, Y_n) < \delta$, $\delta(Y_{n+2}, Y_n) < \delta$ for all $n \geq N$. We claim that $\delta(Y_m, Y_n) < \epsilon$ for all $m, n \geq 2N$, $m \neq n$. This follows from an induction argument. In fact, if we assume that $\delta(Y_{2N+i}, Y_{2N+j}) < \epsilon$ for $i, j = 0, 1, ..., k$, then, for p even, k odd and $p < k-2$,

$$\begin{aligned}\delta(Y_{2N+k+1}, Y_{2N+p}) &\leq \delta(Y_{2N+k+1}, Y_{2N+k-1}) + \delta(Y_{2N+k-1}, Y_{2N+p+1}) \\ &\quad + \delta(Y_{2N+p+1}, Y_{2N+p}) \leq 2\delta + \tilde{\phi}(max\ \delta(Y_i, Y_j)) \\ &\leq 2\delta + \tilde{\phi}(\epsilon) = \epsilon, \quad i, j = 2N+1, ..., 2N+k, \quad i \neq j.\end{aligned}$$

On the other hand, if both p, q are even and $p < k-1$, then

$$\begin{aligned}\delta(Y_{2N+k+1}, Y_{2N+p}) &\leq \delta(Y_{2N+k+1}, Y_{2N+k}) + \delta(Y_{2N+k}, Y_{2N+p+1}) \\ &\quad + \delta(Y_{2N+p+1}, Y_{2N+p}) \leq 2\delta + \tilde{\phi}(max\ \delta(Y_i, Y_j)) \\ &\leq 2\delta + \tilde{\phi}(\epsilon) = \epsilon, \quad i, j = 2N+1, ..., 2N+k, \quad i \neq j.\end{aligned}$$

Similarly, if p is odd, $\delta(Y_{2N+k+1}, Y_{2N+p}) \leq \epsilon$. Thus, since $\delta(Y_n, Y_m) \leq \epsilon$ for $m, n \geq 2N$, $m \neq n$, any sequence (y_n), $n \geq 1$, with $y_n \in Y_n$, is a Cauchy sequence, so that for some $y_0 \in X$, $\lim Iy_n = y_0$. In particular $\lim Ix_{2n} = \lim Jx_{2n+1} = y_0$, and $\lim \delta(y_0, Y_n) = 0$.

Now, resorting to (iii), assume there is $y \in X$ such that $y_0 = Iy$, i.e., $y_0 \in I(X)$. If $\delta(y_0, Sy) > 0$, by letting

$$\begin{aligned}t_{2n+1} = max\{&d(Iy, Jx_{2n+1}), \delta(Iy, Sy), \delta(Jx_{2n+1}, Y_{2n+1}), \\ &\frac{1}{2}(D(Iy, Y_{2n+1}) + D(Jx_{2n+1}, Sy))\},\end{aligned}$$

we obtain $t_{2n+1} = \delta(Iy, Sy)$ for all n large enough, so that

$$\delta(Sy, Tx_{2n+1}) \leq \phi(t_{2n+1}) = \phi(\delta(Iy, Sy)) < \delta(Iy, Sy)$$

for all such $n's$. Since $\delta(Sy, Tx_{2n+1}) \to \delta(Sy, Iy)$, this is a contradiction. Thus $\delta(y_0, Sy) = 0$, and then $\{y_0\} = \{Iy\} = Sy$.

Now let $y' \in X$ be such that $\{Jy'\} = Sy$. The existence of y' follows from $[S(X)] \subseteq J(X)$ in (ii). From (i) it also follows, provided $\delta(Sy, Ty') > 0$, that

$$\begin{aligned}\delta(Sy, Ty') &\leq \phi(max\{d(Iy, Jy'), \delta(Iy, Sy), \delta(Jy', Ty') \\ &\qquad \frac{1}{2}(D(Iy, Ty') + D(Jy', Sy))\}) \\ &= \phi(\delta(Jy', Ty')) = \phi(\delta(Sy, Ty')) < \delta(Sy, Ty'),\end{aligned}$$

which is clearly absurd. Thus, $\{y_0\} = \{Iy\} = Sy = \{Jy'\} = Ty'$, and by (i) and (iv) we have, inasmuch as $\delta(Sy_0, y_0) > 0$, that

$$\begin{aligned}\delta(Sy_0, y_0) &= \delta(Sy_0, Ty') \\ &\leq \phi(max\{d(Iy_0, Jy'), \delta(Iy_0, Sy_0), \delta(Jy', Ty'), \\ &\qquad \frac{1}{2}(D(Iy_0, Ty') + D(Jy', Sy_0))\}) \\ &= \phi(\delta(Sy_0, y_0)) < \delta(Sy_0, y_0),\end{aligned}$$

which is again a contradiction. Therefore $\{y_0\} = Sy_0 = \{Iy_0\}$.

An entirely symmetrical argument shows that $\{y_0\} = \{Jy_0\} = Ty_0$, and completely demonstrates that if $y_0 \in I(X)$ then y_0 is a fixed point of I, J, S and T satisfying (3.4). Since an entirely analogous argument applies if $y_0 \in J(X)$, and because of (ii), also if $y_0 \in [S(X)]$ or $y_0 \in [T(X)]$, the existence of a common fixed point y_0 is granted.

To establish the uniqueness of y_0, assume $z = I(z) = J(z) \in S(z) \cap T(z)$ and $d(y_0, z) > 0$. From (i) we obtain

$$\begin{aligned}\delta(z, Sz) &\leq \delta(Sz, Tz) \\ &\leq \phi(max\{d(Iz, Jz), \delta(Iz, Sz), \delta(Jz, Tz), \\ &\qquad \frac{1}{2}(D(Iz, Tz) + D(Jz, Sz))\}) \\ &= \phi(max\{0, \delta(z, Sz), \delta(z, Tz), 0\}) \\ &= \phi(max\{\delta(z, Sz), \delta(z, Tz)\}),\end{aligned}$$

and, symmetrically,

$$\delta(z, Tz) \leq \phi(max\{\delta(z, Sz), \delta(z, Tz)\}),$$

which in any possibility for $max\{\delta(z, Sz), \delta(z, Tz)\}$ leads, provided we assume $\delta(z, Sz) > 0$ or $\delta(z, Tz) > 0$, to a contradiction with $\phi(t) < t$.

Hence $\delta(z, Sz) = \delta(z, Tz) = 0$, and $\{z\} = \{Iz\} = \{Jz\} = Sz = Tz$. From

$$\begin{aligned}d(y_0, z) &= \delta(Sy_0, Tz) \\ &\leq \phi(max\{d(Iy_0, Jz), \delta(Iy_0, Sy_0), \delta(Jz, Tz), \\ &\qquad \frac{1}{2}(D(Iy_0, Tz) + D(Jz, Sy_0))\}) \\ &= \phi(d(y_0, z))\end{aligned}$$

it follows that the assumption $d(y_0, z) > 0$ is contradictory, and we conclude that $y_0 = z$. Hence, provided $t_n = \delta(Y_n, Y_{n+1}) > 0$ for all $n \geq 1$, the existence and uniqueness of a common fixed point of I, J, S and T is ensured.

Now, if $t_m = \delta(Y_m, Y_{m+1}) = 0$ for some m, it follows from (3.5) and (3.6) that $\delta(Y_n, Y_{n+1}) = 0$ for $n \geq m$, so that for some $y_0 \in X$, $Y_n = \{y_0\}$ for all $n \geq m$. This implies that also $y_0 = \lim Ix_{2n} = \lim Jx_{2n+1}$. Since the argument from here on is exactly as above, the proof is complete in all details. ☑

Remark 3.1. *If ϕ satisfies $\phi(t) < t$ for $t > 0$ and is upper semicontinuous, and if (t_n) is a decreasing sequence in R^+ verifying (1.3) with $t > 0$, then $t = \lim sup\ \phi(t_n) = \phi(t) < t$, which is absurd. Thus $t = 0$, and Condition (C) holds for ϕ.*

Remark 3.2. *Assume I, J, S, T are as in Theorem 3.1, but instead of conditions (iii) and (iv) assume either I or J is continuous and $(I, S), (J, T)$ are compatible , the other assumptions remaining unchanged. Then, the proof of Theorem 4 in* [2], *p. 680, ensures that if (x_n) is as in (3.1), and (3.3) holds, then $y_0 = Iy_0$ if I is continuous or $y_0 = Jy_0$ if J is. Thus, Condition (iii) of Theorem 3.1 actually holds. This and Remark 3.1 show that Theorem 1.1 is a consequence of Theorem 3.1.*

Theorem 3.2. *Let (X, d) be a complete metric space and I, J be selfmaps of X. Let $(S_\alpha)_{\alpha\in\Lambda}$ and $(T_\beta)_{\beta\in\Lambda'}$ be families of multivalued maps of X into $\mathcal{B}(X)$, and assume that*

(i)

$$\delta(S_\alpha x, T_\beta y) \leq \phi(max\{d(Ix, Jy), \delta(Ix, S_\alpha x), \delta(Jy, T_\beta y), \frac{1}{2}(D(Ix, T_\beta y) + D(Jy, S_\alpha x))\}), \tag{3.7}$$

for all $x, y \in X$ and all $\alpha \in \Lambda, \beta \in \Lambda'$, where $\phi \in \Phi^*$ is fixed.

Also assume that there exist α_0 in Λ and β_0 in Λ' such that:

(ii) $[S_{\alpha_0}(X)] \subseteq I(X)$, $[T_{\beta_0}(X)] \subseteq J(X)$.
(iii) *For any sequence (x_n) in X such that $Jx_{2n+1} \in S_{\alpha_0}x_{2n}$ and $Ix_{2n+2} \in T_{\beta_0}x_{2n+1}$, $n \geq 0$, if $\lim Ix_{2n} = \lim Jx_{2n+1} = y \in X$, it follows that $y \in I(X) \cup J(X) \cup [S_{\alpha_0}(X)] \cup [T_{\beta_0}(X)]$.*
(iv) *The maps I and S_{α_0} as well as J and T_{β_0} are locally commuting.*

Then, I, J, S_α and T_β, have a unique common fixed point y_0 in X for all $\alpha \in \Lambda, \beta \in \Lambda'$. Furthermore

$$S_\alpha y_0 = \{y_0\} = \{Iy_0\} = \{Jy_0\} = T_\beta y_0 \tag{3.8}$$

for all $\alpha \in \Lambda, \beta \in \Lambda'$.

Proof. It follows from Theorem 3.1 that there is a unique y in X which is a common fixed point of I, J, S_{α_0} and T_{β_0}. Let $\gamma \in \Lambda'$, $\gamma \neq \beta_0$. From (i) it also follows, if $\delta(y, T_\gamma y) > 0$, that

$$\begin{aligned}\delta(y, T_\gamma y) &= \delta(S_{\alpha_0} y, T_\gamma y) \\ &\leq \phi(max\{d(Iy, Jy), \delta(Iy, S_{\alpha_0} y), \delta(Jy, T_\gamma y), \\ &\qquad \frac{1}{2}(D(Iy, T_\gamma y) + D(Jy, S_{\alpha_0} y))\}) \\ &= \phi(max\{\delta(y, T_\gamma y), \frac{1}{2} D(y, T_\gamma y)\}) \\ &= \phi(\delta(y, T_\gamma y) < \delta(y, T_\gamma y),\end{aligned}$$

which is absurd. Hence, $T_\gamma y = \{y\} = \{Iy\} = \{Jy\}$ for all $\gamma \in \Lambda'$. The proof is entirely similar for $\gamma \in \Lambda, \gamma \neq \alpha_0$. ☑

Taking into account Remarks 3.1 and 3.2, the following corollaries hold.

Corollary 3.1 (Chang[2])**.** *If (X, d) is a complete metric space, I, J are self-maps of X and $(S_\alpha)_{\alpha \in \Lambda}$, $(T_\alpha)_{\alpha \in \Lambda}$ are two families of maps of X into $\mathcal{B}(X)$ with*

$$\cup_{\alpha \in \Lambda}[S_\alpha X] \subseteq J(X), \quad \cup_{\alpha \in \Lambda}[T_\alpha X] \subseteq I(X) \tag{3.9}$$

and such that

$$\begin{aligned}\delta(S_\alpha x, T_\beta y) \leq \phi(max\{&d(Ix, Jy), \delta(Ix, S_\alpha x), \delta(Jy, T_\beta y), \\ &\frac{1}{2}(D(Ix, T_\beta y) + D(Jy, S_\alpha x))\})\end{aligned} \tag{3.10}$$

for all $x, y \in X$ and all $\alpha, \beta \in \Lambda$, where $\phi : R^+ \to R^+$, with $\phi(t) < t, t > 0$, is upper semicontinuous, then, if for all $\alpha, \beta \in \Lambda$, (T_α, J) and (S_β, I) are compatible, and one of I or J is continuous, a unique $y_0 \in X$ exists such that

$$S_\alpha y_0 = T_\beta y_0 = \{Iy_0\} = \{Jy_0\} = \{y_0\} \tag{3.11}$$

for all $\alpha, \beta \in \Lambda$.

Proof. The compatibility of pairs implies their local commutativity (Remark 2.1). ☑

Corollary 3.2. *If (X, d) is complete and $S_\alpha : X \to \mathcal{B}(X)$, $\alpha \in \Lambda$, is a family of multivalued maps such that*

$$\begin{aligned}\delta(S_\alpha x, S_\beta y) \leq \phi(max\{&d(x, y), \delta(x, S_\alpha x), \delta(y, S_\beta y), \\ &\frac{1}{2}(D(x, S_\beta y) + D(y, S_\alpha x))\})\end{aligned} \tag{3.12}$$

for all $x, y \in X$ and all $\alpha, \beta \in \Lambda$, where $\phi \in \Phi^*$ is a fixed, then the maps S_α, $\alpha \in \Lambda$, have a unique common fixed point y_0 in X, and $S_\alpha y_0 = \{y_0\}$ for all $\alpha \in \Lambda$.

Proof. Just let $I = J$ be the identity map of X. ☑

Remark 3.3. *Corollary 3.2 above generalizes Corollary 6 in Chang*[2].

Corollary 3.3. *Let (X, d) be a complete metric space and I, J be selfmaps of X. Also let $S, T : X \to \mathcal{B}(X)$ be such that $[S(X)] \subseteq J(X)$ and $[T(X)] \subseteq I(X)$ and that (3.2) holds with $\phi \in \Phi^*$. Also assume that*

$$\delta(Sx, Sx) \leq \delta(x, Sx) \tag{3.13}$$

holds for all $x \in X$. If the pairs (S, I) and (T, J) are compatible and if S is continuous, then S, T, I and J have a unique fixed point $y_0 \in X$. Furthermore,

$$Sy_0 = Ty_0 = \{Iy_0\} = \{Jy_0\} = \{y_0\}. \tag{3.14}$$

Proof. Condition (3.13), the compatibility of I and S and the continuity of S guarantee (Chang[2], p.682) that if $Jx_{2n+1} \in Sx_{2n}$ and $Ix_{2n+2} \in Tx_{2n+1}$, $n \geq 0$ and if $y = \lim Ix_{2n} = \lim Jx_{2n+1}$, then $y \in S(X)$. The conclusion then follows from Theorem 3.1. ☑

Remark 3.4. *Corollary 3.3 above improves Theorem 7 in Chang*[2].

Theorem 3.3. *Let (X, d) be a complete metric space and I, J, S and T be selfmaps of X such that:*

(i)
$$d(Sx, Ty) \leq \phi(max\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), \frac{1}{2}(d(Ix, Ty) + d(Jy, Sx))\}) \tag{3.15}$$
for all $x, y \in X$, where $\phi \in \Phi^*$ is fixed.

(ii) $S(X) \subseteq J(X)$, $T(X) \subseteq I(X)$.

(iii) *For any sequence (x_n) in X such that $Jx_{2n+1} = Sx_{2n}$, $Ix_{2n+2} = Tx_{2n+1}$, $n \geq 0$, and $\lim Ix_{2n} = \lim Jx_{2n+1} = y$, it follows that $y \in I(X) \cup J(X) \cup S(X) \cup T(X)$.*

(iv) *The maps I, S as well as J, T commute at their coincidence points; i.e., (I, S) and (J, T) are locally commuting.*

Then, S, T, I, J have a unique common fixed point in X.

Proof. Theorem 3.1 and its proof obviously hold when S, T are also single-valued. ☑

Remark 3.5. *Since Theorem 8 in Chang*[2] *is an easy consequence of Theorem 3.3, many results in Fisher*[6, 7]*, Kubiak*[12]*, Rodríguez-Montes and Charris*[17] *and Sing and Whitfield*[19] *are special cases of Theorems 3.1 and 3.3. For example:*

Corollary 3.4 (Rodríguez-Montes and Charris[17])**.** *Let X be a complete metric space and let f, g be selfmaps of X, at least one of them being continuous, such that*

$$d(g(x), g(y)) \leq Q(max\{d(f(x), f(y)), d(f(x), g(x)), d(f(y), g(y)), \frac{1}{2}(d(f(x), g(y)) + d(f(y), g(x)))\}) \tag{3.16}$$

for all $x, y \in X$, where $Q : R^+ \to R^+$ satisfies

(a) $0 < Q(t) < t$ *for* $t > 0$, $Q(0) = 0$
(b) $q(t) = t/(t - Q(t))$ *is non increasing on* $(0, +\infty)$.

Also assume that

(i) f *and* g *are compatible, and*
(ii) $g(X) \subseteq f(X)$.

Then, f *and* g *have a unique common fixed point.*

Proof. If (t_n) is a sequence in R^+ such that $t_{n+1} \leq Q(t_n) < t_n$, $n \geq 1$, and $\lim t_n = \lim Q(t_n) = t$, assuming $t > 0$ leads from (b) to $q(t) = t/(t - Q(t)) \geq t_n/(t_n - Q(t_n)) = q(t_n)$, $n \geq 1$, and letting $n \to \infty$, to $q(t) = +\infty$, which is absurd. Hence, Q satisfies Condition (C). Since the compatibility of f and g implies their local conmutativity, and the continuity of either f or g ensures that for any sequence (x_n) in X such that $g(x_n) = f(x_{n+1})$ and $y_0 = \lim g(x_n) = \lim f(x_{n+1})$ it follows that $y_0 \in f(X) \cup g(X)$, the corollary is a consequence of either Theorem 3.1 (when f is continuous) or of Corollary 3.3 (when g is continuous), with $S = T = g$ and $I = J = f$. ☑

The above corollary was also proved in Rodríguez-Montes and Charris[17] by a different procedure, and then used to establish or generalize results of Carbone et al.[1], sometimes removing redundant assumptions.

In [15], Rhoades et al. state a contraction result of the Meir-Keeler type, but their conclusion is erroneous (see Chang[2]). Chang[2] proposes additional assumptions to validate it. In what follows we adapt techniques of Chang[2] to our point of view and establish results which improve some of those in [2] and in Pant[13].

Lemma 3.1. *let* Y *be a set and* $f, g : Y \to R^+$ *be such that* $f(x) = 0$ *whenever* $g(x) = 0$. *Assume there is* $\hat{\delta} : R^+ \to R^+$ *which is either non decreasing or left lower semicontinuous and such that* $\hat{\delta}(t) > 0$ *when* $t > 0$ *and for any* $\epsilon > 0$, $f(x) < \epsilon$ *whenever* $\epsilon \leq g(x) < \epsilon + \hat{\delta}(\epsilon)$. *Then, there is a nondecreasing function* $\phi \in \Phi^*$ *such that* $f(x) \leq \phi(g(x))$ *for all* $x \in Y$.

Proof. The conditions on f and g ensure that $f(x) \leq g(x)$ for all $x \in Y$ and if $f(x) > 0$ then $g(x) \notin \Big[f(x), f(x) + \hat{\delta}(f(x))\Big)$. Thus, $f(x) + \hat{\delta}(f(x)) \leq g(x)$ whenever $f(x) > 0$. For each $t \geq 0$, let $\phi(t) = sup\{f(x)/g(x) \leq t\}$ provided $\{f(x)/g(x) \leq t\} \neq \phi$ and $\phi(t) = 0$ otherwise. By definition, $0 \leq \phi(t) \leq t$ for all $t \geq 0$. If $\phi(t) = t$ for some $t > 0$, there is a sequence (x_n) in Y such that $0 < f(x_n) \leq t$, $0 < g(x_n) \leq t$, $(f(x_n))$ and $(g(x_n))$ are nondecreasing,

$$\lim f(x_n) = \lim g(x_n) = t \tag{3.17}$$

and

$$f(x_n) + \hat{\delta}(f(x_n)) \leq g(x_n). \tag{3.18}$$

Thus, if $t_n = f(x_n)$, $n \geq 1$, then $\lim \hat{\delta}(t_n) = 0$. Now, if $\hat{\delta}$ is nondecreasing, $\hat{\delta}(t_n) \geq \hat{\delta}(t_1) > 0$ for all $n \geq 1$, which is contradictory. Also, if $\hat{\delta}$ is left lower semicontinuous then, since $t_n \leq t$, $0 < \hat{\delta}(t) \leq \lim inf\ \hat{\delta}(t_n) = 0$, which is equally contradictory. Thus, $\phi(t) < t$ for all $t > 0$. Now let (t_n) be a decreasing sequence in R^+ such that $t_{n+1} < \phi(t_n) < t_n$ for all $n \geq 1$ and $\lim \phi(t_n) = \lim t_n = t$. From the definition of ϕ, there is a sequence (x_n) in Y such that $t < t_{n+1} < f(x_n) < g(x_n) \leq t_n$ and $f(x_n) \leq \phi(t_n)$ for all $n \geq 1$. Then $\lim f(x_n) = \lim g(x_n) = t$. If $t > 0$, there exists $N \geq 1$ such that $t \leq g(x_n) < t + \hat{\delta}(t)$, and therefore $f(x_n) < t$ for all $n \geq N$, which is absurd. Then $t = 0$, and ϕ satisfies Condition (C). ☑

Remark 3.6. *Lemma 3.1 above is a simpler alternative to Proposition 11 in Chang*[2].

Theorem 3.4. *Let (X, d), S, T, I and J be as in Theorem 3.1, but instead of Condition (i) assume that*

(i') *there is a function $\hat{\delta} : (0, +\infty) \to (0, +\infty)$ which is either nondecreasing or left lower semicontinuous and such that, for all $x, y \in X$ and $\epsilon > 0$, $\delta(Sx, Ty) < \epsilon$ whenever*

$$\epsilon \leq max\{d(Ix, Jy), \delta(Ix, Sx), \delta(Jy, Ty), \frac{1}{2}(D(Ix, Ty) + D(Jy, Sx))\} < \epsilon + \hat{\delta}(\epsilon), \tag{3.19}$$

the other conditions remaining unchanged. Then, the conclusions of Theorem 3.1 hold.

Proof. Follows from Lemma 3.1 with $Y = X \times X$, $f(x, y) = \delta(Sx, Ty)$ and $g(x, y) = max\{d(Ix, Jy), \delta(Ix, Sx), \delta(Jy, Ty), \frac{1}{2}(D(Ix, Ty) + D(Jy, Sx))\}$, and from Theorem 3.1. ☑

Remark 3.7. *Theorems 3.2 and 3.3, as well as Corollary 3.2, also remain valid if condition (i) in each of them is replaced by condition (i') in Theorem 3.4 above. This shows that the results in Chang*[2] *and Pant*[13] *follow from results in the present paper.*

4. Results of the expansive type

Now we explore the use of Condition (A) for expanding maps $\psi : R^+ \to R^+$ ($\psi(t) > t$ for $t > 0$) in the context of multi-valued maps in metric spaces.

For a contracting function ϕ, necessarily $\phi(0+) = \lim_{t\to 0+} \phi(t) = 0$. For an expanding map ψ this may not hold and has to be assumed when needed. We denote by Ψ the set of expanding maps satisfying Condition (A), and by Ψ_0 the set of those $\psi \in \Psi$ verifying $\psi(0+) = 0$.

For $\psi \in \Psi$ and $t > 0$, let $\hat{\psi}(t) = Sup\{x/\psi(x) < t\}$ provided $\{x/\psi(x) < t\} \neq \phi$, $\hat{\psi}(t) = 0$ otherwise. Then $0 \leq \hat{\psi}(t) \leq t$ for $t > 0$.

The following lemma states two useful properties of $\psi \in \Psi_0$. It appears in Rodríguez-Montes and Charris[17, 18], but for completeness we also include its proof here.

Lemma 4.1. *For $\psi \in \Psi_0$, the following holds:*

(a) $0 < \hat{\psi}(t) \leq t$ *for all* $t > 0$.

(b) *For any* $\epsilon > 0$ *there is* $0 < t \leq \epsilon$ *such that* $\hat{\psi}(t) < t$.

Proof. From $\psi(0+) = 0$ it follows that $\hat{\psi}(t) > 0$ for all $t > 0$. If (b) were not satisfied, $\epsilon > 0$ could be found such that $\hat{\psi}(t) = t$ for all $0 < t \leq \epsilon$. Let $0 < \alpha < \epsilon$. Since $\hat{\psi}(\epsilon) = \epsilon$ then, for some t_1, $\alpha < t_1 < \psi(t_1) < \epsilon$, and having selected t_n such that $\alpha < t_n < \psi(t_n)$, we could also choose t_{n+1} such that $\alpha < t_{n+1} < \psi(t_{n+1}) < t_n$, from which it follows that $t \geq \alpha > 0$ exists such that $\lim t_n = \lim \psi(t_n) = t > 0$. This is absurd for ψ satisfying Condition (A). ☑

Theorem 4.1. *Let (Y, d) be a metric space, X be a subspace of Y and I, J be maps of X into Y. Let $S, T : X \to \mathcal{B}(Y)$ be such that:*

(i)

$$\psi(\delta(Sx, Ty)) \leq max\{d(Ix, Jy), \frac{1}{2}(D(Ix, Ty) + D(Jy, Sx)) \tag{4.1}$$

for all $x, y \in X$, where $\psi \in \Psi_0$ is fixed.

(ii) *There exists a sequence (x_n) in X such that*

$$Jx_{2n+1} \in Sx_{2n}, Ix_{2n+2} \in Tx_{2n+1}, n \geq 0. \tag{4.2}$$

(iii) *Either $I(X)$ or $J(X)$ is a complete subspace of (Y, d), and for any sequence (x_n) as in (4.2) such that* $\lim Ix_{2n} = \lim Jx_{2n+1} = y$ *for some $y \in Y$, it follows that $y \in I(X) \cap J(X)$.*

(iv) *The maps I, S as well as J, T are locally commuting.*

Then I, J, S and T have a unique common fixed point $y_0 \in X$. Furthermore

$$Sy_0 = Ty_0 = \{Iy_0\} = \{Jy_0\} = \{y_0\}, \tag{4.3}$$

and $y_0 = \lim Ix_{2n} = \lim Jx_{2n+1}$ *for any sequence (x_n) as in (4.2).*

Proof. Observe that condition (iv) implies that for any $x \in X$ such that $Sx = \{Ix\}$ (resp. $Tx = \{Jx\}$) it follows that $Ix \in X$ (resp. $Jx \in X$), which occurs in particular if $X = Y$. From (i) we obtain that if $Y_{2n} = Sx_{2n}$, $Y_{2n+1} = Tx_{2n+1}$ then

$$\begin{aligned}\psi(\delta(Y_{2n+2}, Y_{2n+1})) = \psi(\delta(Sx_{2n+2}, Tx_{2n+1})) &\leq max\{d(Ix_{2n+2}, Jx_{2n+1}), \\ &\frac{1}{2}(D(Ix_{2n+2}, Tx_{2n+1}) + D(Jx_{2n+1}, Sx_{2n+2}))\} \\ &\leq \max\{\delta(Y_{2n+1}, Y_{2n}), \frac{1}{2}(\delta(Y_{2n}, Y_{2n+1}) + \delta(Y_{2n+1}, Y_{2n+2}))\}.\end{aligned}$$

We first assume that $\delta(Y_n, Y_{n+1}) > 0$ for all $n \geq 1$. If it were $\delta(Y_{2n}, Y_{2n+1}) \leq \delta(Y_{2n+1}, Y_{2n+2})$ then $\psi(\delta(Y_{2n+1}, Y_{2n+2})) \leq \delta(Y_{2n+1}, Y_{2n+2})$, which is absurd. Hence, $\delta(Y_{2n+1}, Y_{2n+2}) < \delta(Y_{2n}, Y_{2n+1})$, and therefore

$$\delta(Y_{2n+1}, Y_{2n+2}) < \psi(\delta(Y_{2n+1}, Y_{2n+2})) \leq \delta(Y_{2n}, Y_{2n+1}), n \geq 0. \tag{4.4}$$

Similarly,

$$\delta(Y_{2n}, Y_{2n+1}) < \psi(\delta(Y_{2n}, Y_{2n+1})) \leq \delta(Y_{2n-1}, Y_{2n}), n \geq 0. \tag{4.5}$$

Since $\psi \in \Psi_0$, then $\lim \delta(Y_n, Y_{n+1}) = \lim \psi(\delta(Y_n, Y_{n+1})) = 0$.

On the other hand, if m is even, n is odd and $m > n$, then

$$\begin{aligned}\psi(\delta(Y_m, Y_n)) = \psi(\delta(Sx_m, Tx_n)) &\leq max\{d(Ix_m, Jx_n),\\ &\qquad \frac{1}{2}(D(Ix_m, Tx_n) + D(Jx_n, Sx_m))\}\\ &\leq max\{\delta(Y_{m-1}, Y_{n-1}), \frac{1}{2}(\delta(Y_n, Y_{m-1}) + \delta(Y_{n-1}, Y_m))\}.\\ &\leq max\{\delta(Y_i, Y_j)/n - 1 \leq i \neq j \leq m\}.\end{aligned}$$

The same holds if m is odd, n is even and $m > n$.

Let $\epsilon > 0$ be such that $\hat{\psi}(\epsilon) < \epsilon$ (Lemma 4.1), let $\delta = (\epsilon - \hat{\psi}(\epsilon))/2$ and let $N > 0$ be such that $\delta(Y_{n+1}, Y_n) < \delta$ and $\delta(Y_{n+2}, Y_n) < \delta$ for all $n \geq N$. We claim that $\delta(Y_m, Y_n) < \epsilon$ for all $m, n \geq 2N$, $m \neq n$. This follows from an induction argument. In fact, if we assume that $\delta(Y_{2N+i}, Y_{2N+j}) < \epsilon$, $i, j = 0, 1, ..., k$, $i \neq j$, then, if k is odd, p is even and $p < k - 2$, we have that

$$\begin{aligned}\delta(Y_{2N+k+1}, Y_{2N+p}) \leq \delta(Y_{2N+k+1}, Y_{2N+k-1}) + \delta(Y_{2N+K-1}, Y_{2N+p+1})\\ + \delta(Y_{2N+p+1}, Y_{2N+p}) \leq 2\delta + \delta(Y_{2N+k-1}, Y_{2N+p+1}).\end{aligned}$$

Since $\psi(\delta(Y_{2N+k-1}, Y_{2N+p+1})) \leq max\{\delta(Y_i, Y_j) : i, j = 2N, ..., 2N + k, i \neq j\} < \epsilon$ then $\delta(Y_{2N+k-1}, Y_{2N+p+1}) \leq \hat{\psi}(\epsilon)$, and therefore $\delta(Y_{2N+k+1}, Y_{2N+p}) \leq 2\delta + \hat{\psi}(\epsilon) = \epsilon$. If p is odd then

$$\begin{aligned}\delta(Y_{2N+k+1}, Y_{2N+p}) \leq \delta(Y_{2N+k+1}, Y_{2N+k-1}) + \delta(Y_{2N+K-1}, Y_{2N+p+2})\\ \delta(Y_{2N+p+2}, Y_{2N+p}) \leq 2\delta + \delta(Y_{2N+k-1}, Y_{2N+p+2}),\end{aligned}$$

and since $\psi(\delta(Y_{2N+k-1}, Y_{2N+p})) \leq max\{\delta(Y_i, Y_j) : i, j = 2N, ..., 2N + k, i \neq j\} < \epsilon$, then $\delta(Y_{2N+k-1}, Y_{2N+p+2}) \leq \hat{\psi}(\epsilon)$, and therefore $\delta(Y_{2N+k+1}, Y_{2N+p}) < 2\delta + \hat{\psi}(\epsilon) = \epsilon$. The proof is similar if k is even.

Since either $I(X)$ or $J(X)$ is complete and $\delta(Y_n, Y_m) \to 0$ when $m, n \to \infty$, there is y_0 in Y such that $y_0 = \lim Ix_{2n} = \lim Jx_{2n+1}$ and, since $y_0 \in I(X) \cap J(X)$, also $\tilde{x}_1, \tilde{x}_2$ in X such that $y_0 = I(\tilde{x}_1) = J(\tilde{x}_2)$. Furthermore, $\lim \delta(y_0, Y_n) = 0$.

Now, it follows from (i) that for all $x, y \in X$,

$$\begin{aligned}\delta(Sx, Ix_{2n+2}) \leq \delta(Sx, Tx_{2n+1}) < \psi(\delta(Sx, Tx_{2n+1}))\\ \leq \max\{d(Ix, Jx_{2n+1}), \frac{1}{2}(D(Ix, Tx_{2n+1}) + D(Jx_{2n+1}, Sx))\}\end{aligned}$$

and

$$\begin{aligned}\delta(Jx_{2n+1}, Ty) &\leq \delta(Sx_{2n}, Ty) < \psi(\delta(Sx_{2n}, Ty))\\ &\leq \max\{d(Ix_{2n}, Jy), \frac{1}{2}(D(Ix_{2n}, Ty) + D(Jy, Sx_{2n}))\}.\end{aligned}$$

Letting $n \to \infty$ we obtain

$$\delta(Sx, y_0) \leq max\{d(Ix, y_0), \frac{1}{2}(\delta(Ix, y_0) + \delta(y_0, Sx))\} \tag{4.6}$$

and

$$\delta(y_0, Ty) \leq max\{d(y_0, Jy), \frac{1}{2}(\delta(y_0, Ty) + \delta(Jy, y_0))\} \tag{4.7}$$

Thus, from (4.6) and assuming $\delta(S\tilde{x}_1, y_0) > 0$, we get that

$$\delta(S\tilde{x}_1, y_0) = \delta(S\tilde{x}_1, I\tilde{x}_1) < \frac{1}{2}\delta(S\tilde{x}_1, I\tilde{x}_1),$$

and from (4.7) and assuming $\delta(T\tilde{x}_2, y_0) > 0$, that $\delta(T\tilde{x}_2, J\tilde{x}_2) < \frac{1}{2}\delta(T\tilde{x}_2, J\tilde{x}_2)$. Both conclusions are absurd, so that $\{y_0\} = \{I\tilde{x}_1\} = S\tilde{x}_1 = \{J\tilde{x}_2\} = T\tilde{x}_2$.

Now, from (iv) it follows that $y_0 \in X$ and $Sy_0 = \{Iy_0\}$, $Ty_0 = \{Jy_0\}$. Hence, if we assume $\delta(Sy_0, y_0) > 0$ then, by (i),

$$\begin{aligned}\psi(\delta(Sy_0, y_0)) = \psi(\delta(Sy_0, T\tilde{x}_2)) &\leq max\{d(Iy_0, J\tilde{x}_2),\\ &\qquad \frac{1}{2}(\delta(Sy_0, T\tilde{x}_2) + \delta(J\tilde{x}_2, Sy_0))\}\\ &\leq max\{d(Sy_0, y_0), \frac{1}{2}(\delta(Sy_0, y_0) + \delta(Sy_0, y_0))\}\\ &= \delta(Sy_0, y_0),\end{aligned}$$

which is absurd and ensures that $Sy_0 = \{y_0\}$. Similarly $\{y_0\} = Ty_0$, and therefore $\{y_0\} = \{Iy_0\} = \{Jy_0\} = Sy_0 = Ty_0$; i.e., y_0 is a common fixed point of I, J, S and T.

Now assume z is another common fixed point of these maps, i.e., $z \in Tz \cap Sz$, $z = Iz = Jz$ and $d(z, y_0) > 0$. Then, by (i), $d(z, y_0) \leq \delta(Sz, Ty_0) < \psi(\delta(Sz, Ty_0)) \leq max\{d(z, y_0), \frac{1}{2}(d(z, y_0) + \delta(y_0, Sz))\} \leq \delta(y_0, Sz)$, which contradicts $\delta(y_0, Sz) = \delta(Sz, Ty_0) > 0$. Thus $y_0 = z$, and the fixed point y_0 is unique.

If we now assume $\delta(Y_m, Y_{m+1}) = 0$ for some m, it follows from the arguments preceding (4.4) and (4.5) that also $\delta(Y_{m+1}, Y_{m+2}) = 0$, so that $Y_n = \{y_0\}$ for some $y_0 \in Y$ and all $n \geq m$. This implies as before that $y_0 = \lim Ix_{2n} = \lim Jx_{2n+1}$ and shows, as above, that y_0 is a unique fixed point of I, J, S and T. ☑

Remark 4.1. *We observe that if* $[T(X)] \subseteq I(X)$ *and* $[S(X)] \subseteq J(X)$, *condition (ii) and relation (4.2) in Theorem 4.1 are automatically satisfied. In fact, choosing arbitrarily* $x_0 \in X$, x_1 *and* x_2 *in* X *can be chosen such that* $Jx_1 \in Sx_0$ *and* $Ix_2 \in Tx_1$; *and having selected* x_{2n}, *also* x_{2n+1} *and* x_{2n+2} *can be picked out such that* $Jx_{2n+1} \in Sx_{2n}$ *and* $Ix_{2n+2} \in Tx_{2n+1}$.

We have the following Corollary of Theorem 4.1.

Corollary 4.1. *Let (X, d) be a complete metric space, I and J be selfmaps of X, $S, T : X \to \mathcal{B}(X)$. Assume that $[S(X)] \subseteq J(X)$, $[T(X)] \subseteq I(X)$, that one of $I(X)$ or $J(X)$ is closed in X, and that conditions (i) and (iv) of the theorem hold true. Then I, J, S and T have a unique common fixed point y_0, and $y_0 = \lim Ix_{2n} = \lim Jx_{2n+1}$ for any sequence (x_n) in X as in (4.2).*

Proof. From Remark 4.1, a sequence (x_n) exists as in the statement of the corollary. From (i) it follows, as in the proof of the theorem, that $y_0 \in X$ exists such that $y_0 = \lim Ix_{2n} = \lim Jx_{2n+1}$ and $\delta(y_0, Sx_{2n}) \to 0$, $\delta(y_0, Tx_{2n+1}) \to 0$ when $n \to \infty$. If $I(X)$ is closed, there is $\tilde{x}_1 \in X$ such that $y_0 = I\tilde{x}_1$, and again, as in the proof of the theorem, $\{y_0\} = \{I\tilde{x}_1\} = S\tilde{x}_1$. Since $[S(X)] \subseteq J(X)$, $y_0 = J\tilde{x}_2$ for some $\tilde{x}_2 \in X$, and again $\{y_0\} = \{J\tilde{x}_2\} = T\tilde{x}_2$. This shows that $y_0 \in I(X) \cap J(X)$, and herefrom the proof is that of the theorem. The argument is entirely similar if $J(X)$ is closed. ☑

Remark 4.2. *It can be shown that if condition (i) in Theorem 4.1 is changed to*

$$\psi(\delta(Sx, Ty)) \leq max\{d(Ix, Jy), \delta(Ix, Sx), \delta(Jy, Ty) \\ \frac{1}{2}(D(Ix, Ty) + D(Jy, Sx))\} \tag{4.8}$$

for all $x, y \in X$, where ψ is an expanding map of R^+ into R^+ such that for any increasing or decreasing sequence (t_n) in R^+ from $\lim t_n = \lim \psi(t_n) = t$ it follows that $t = 0$, then the conclusions of the theorem still hold, but we do not know if Condition (A) alone yields the same result. We observe that a lower semicontinuous expanding map ψ of R^+ into R^+ satisfies the above condition.

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