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Spinor formulation of the differential geometry of curves

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Abstract. It is shown that the Frenet equations for curves in $\mathbb{R}^3$ can be reduced to a single equation for a vector with two complex components and some examples of the usefulness of this representation are given.

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1. Introduction

In differential geometry the curves and surfaces in $\mathbb{R}^3$ are usually studied employing the vector formalism. In the case of a differentiable curve, at each point a triad of mutually orthogonal unit vectors (called tangent, normal and binormal) is constructed and the rates of change of these vectors along the curve define the curvature and torsion of the curve. These two functions characterize the curve completely except for its position and orientation in space. In a similar manner, at each point of a smooth surface a triad of mutually orthogonal unit vectors can be defined in such a way that one of these vectors is normal to the surface. Then the rate of variation of the normal unit vector to the surface along the directions of the other two vectors determines the curvature of the surface.

Given the importance of the triads of mutually orthogonal unit vectors in differential geometry, it is of interest that each such triad can be expressed in terms of a single vector with two complex components, called a *spinor* [1–3]. The aim of this paper is to show that the basic equations of the differential geometry of curves (the Frenet equations) can be expressed in a compact and useful way making use of spinors. In Section 2 the basic elements about spinors are briefly presented (a more complete elementary treatment can be found in [1,3]); in Section 3 the spinor equivalent of the Frenet equations for a curve is obtained and in Sec. 4 some examples of its application are given.

2. **Orthonormal bases and spinors**

The group of rotations about the origin in $\mathbb{R}^3$ (denoted as SO(3)) is known to be homomorphic to the group of unitary complex 2×2 matrices with unit determinant (denoted as SU(2)). In fact, there is a two-to-one homomorphism of SU(2) onto SO(3). Whereas the elements of SO(3) act on points of $\mathbb{R}^3$ (that is, vectors with three real components), the elements of SU(2) act on vectors with two complex components which are called *spinors* (see also, for example, [4,5]).

An explicit way of exhibiting this homomorphism consists in noticing that each spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{1}$$

defines three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$ by means of

$$\mathbf{a} + i\mathbf{b} = \psi^{\mathrm{t}} \boldsymbol{\sigma} \psi, \qquad \mathbf{c} = -\widehat{\psi}^{\mathrm{t}} \boldsymbol{\sigma} \psi, \tag{2}$$

where $\boldsymbol{\sigma}$ is a vector whose Cartesian components are the complex symmetric 2×2 matrices

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{3}$$

the superscript t denotes transposition and $\widehat{\psi}$ is the *mate* [3] (or conjugate [1]) of ψ,

$$\widehat{\psi} \equiv -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\psi} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\psi_1} \\ \overline{\psi_2} \end{pmatrix} = \begin{pmatrix} -\overline{\psi_2} \\ \overline{\psi_1} \end{pmatrix}, \tag{4}$$

where the bar denotes complex conjugation. Thus, the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are explicitly given by

$$\begin{aligned} \mathbf{a} + i\mathbf{b} &= (\psi_1^2 - \psi_2^2, i(\psi_1^2 + \psi_2^2), -2\psi_1\psi_2), \\ \mathbf{c} &= (\overline{\psi_2}\psi_1 + \overline{\psi_1}\psi_2, i\overline{\psi_2}\psi_1 - i\overline{\psi_1}\psi_2, |\psi_1|^2 - |\psi_2|^2) \end{aligned}$$

and by means of an explicit computation one finds that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are mutually orthogonal and $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = \overline{\psi}^{\mathrm{t}}\psi$. Furthermore $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} > 0$.

It may be pointed out that $\widehat{\psi}$ transforms under the SU(2) transformations in exactly the same way as ψ does, that is, $\widehat{U\psi} = U\widehat{\psi}$, for any $U \in \mathrm{SU}(2)$.

Conversely, given three mutually orthogonal vectors of the same magnitude, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$, such that $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} > 0$, there exists a spinor, defined up to sign, such that (2) holds.

For any $U \in \mathrm{SU}(2)$, the spinor $\psi' = U\psi$ satisfies $\overline{\psi'}^{\mathrm{t}}\psi' = \overline{\psi}^{\mathrm{t}}\psi$; therefore, the magnitudes of the three vectors $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ defined by ψ' are equal to those of the three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ corresponding to ψ. Hence, each element of SU(2) induces a transformation that sends the right-handed orthogonal basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ of $\mathbb{R}^3$ into the right-handed orthogonal basis $\{\mathbf{a}', \mathbf{b}', \mathbf{c}'\}$, that is, an element of SO(3). This correspondence between elements of SU(2) and elements of SO(3) is two to one since U and $-U$ yield the same element of SO(3).

Making use of the foregoing definitions it can be shown that if ϕ and ψ are two arbitrary spinors

$$\overline{\phi^{\mathrm{t}}\boldsymbol{\sigma}\psi} = -\widehat{\phi}^{\mathrm{t}}\boldsymbol{\sigma}\widehat{\psi}, \tag{5}$$

and for any pair of complex numbers, a, b,

$$(a\phi + b\psi)\widehat{\ } = \overline{a}\widehat{\phi} + \overline{b}\widehat{\psi}. \tag{6}$$

Furthermore,

$$\widehat{\widehat{\psi}} = -\psi. \tag{7}$$

Making use of these properties it can be readily seen that the vector $\mathbf{c}$, given by (2), is real.

The correspondence between spinors and orthogonal bases given by (2) is two to one; the spinors ψ and $-\psi$ correspond to the same ordered orthonormal basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, with $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$ and $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} > 0$. It is important to notice that the ordered triads $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, $\{\mathbf{b}, \mathbf{c}, \mathbf{a}\}$ and $\{\mathbf{c}, \mathbf{a}, \mathbf{b}\}$ correspond to different spinors.

The symmetry of the matrices (3) amounts to $\phi^{\mathrm{t}}\boldsymbol{\sigma}\psi = \psi^{\mathrm{t}}\boldsymbol{\sigma}\phi$ for any pair of spinors ϕ and ψ. (The matrices (3) are the products of the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ by the Pauli matrices employed in physics [2,3].) With the conventions chosen in (2)–(4), taking $\psi = (1, 0)^{\mathrm{t}}$ one finds that $\widehat{\psi} = (0, 1)^{\mathrm{t}}$ and the triad $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is the canonical basis of $\mathbb{R}^3$.

If ψ is a spinor different from zero, the set $\{\psi, \widehat{\psi}\}$ is linearly independent (using the complex numbers as scalars), which follows from the fact that the determinant of the matrix formed by the components of ψ and $\widehat{\psi}$ is

$$\begin{vmatrix} \psi_1 & -\overline{\psi_2} \\ \psi_2 & \overline{\psi_1} \end{vmatrix} = |\psi_1|^2 + |\psi_2|^2$$

and this sum is equal to zero if and only if ψ_1 and ψ_2 are simultaneously equal to zero.

3. **Frenet equations**

For any curve $\alpha : I \to \mathbb{R}^3$ such that $d\alpha(t)/dt \neq 0$, there exists a function $s = s(t)$, called the arclength, such that if the curve is parametrized by s, $|d\alpha/ds| = 1$. Then, $T \equiv d\alpha/ds$ is a unit vector, called the tangent of α and if $dT/ds \neq 0$, the curvature of α, κ, is defined by $\kappa \equiv |dT/ds|$; thus, $dT/ds = \kappa N$, where N is some unit vector, called normal of α, and T and N are orthogonal to each other. The binormal vector, B, is defined by $B \equiv T \times N$. The derivative dB/ds is also proportional to N and therefore $dB/ds = -\tau N$, where τ is some real-valued function called torsion. Using that $\{T, N, B\}$ is an orthonormal set, from the foregoing relations one deduces that $dN/ds = -\kappa T + \tau B$. The formulas

$$\begin{aligned}\frac{dT}{ds} &= \kappa N, \\ \frac{dN}{ds} &= -\kappa T + \tau B, \\ \frac{dB}{ds} &= -\tau N,\end{aligned}$$

that express the derivatives of T, N and B in terms of the same vectors constitute the Frenet equations (see, for example, [6,7]).

According to the results presented in the preceding section, there exists a spinor, ψ, defined up to sign, such that

$$N + iB = \psi^{\mathrm{t}} \boldsymbol{\sigma} \psi, \qquad T = -\widehat{\psi}^{\mathrm{t}} \boldsymbol{\sigma} \psi \tag{8}$$

with $\overline{\psi}^{\mathrm{t}} \psi = 1$. Hence, the spinor ψ represents the triad $\{N, B, T\}$ and the variations of this triad along the curve, given by the Frenet equations, must correspond to some expression for $d\psi/ds$.

Differentiating the first equation (8) with respect to s and using the Frenet equations one finds

$$-\kappa T + \tau B - i\tau N = (d\psi/ds)^{\mathrm{t}} \boldsymbol{\sigma} \psi + \psi^{\mathrm{t}} \boldsymbol{\sigma} (d\psi/ds). \tag{9}$$

Since $\{\psi, \widehat{\psi}\}$ is a basis for the two-component spinors, there exist two (possibly complex-valued) functions, f and g, such that

$$\frac{d\psi}{ds} = f\psi + g\widehat{\psi}$$

and substituting this relation into (9), using again (8), we have

$$-\kappa T - i\tau(N + iB) = f(N + iB) - gT + f(N + iB) - gT$$

which amounts to $f = -i\tau/2$, $g = \kappa/2$. The second equation in (8) does not give additional relations. Thus, we have proved the following

Proposition 1. *If the two-component spinor ψ represents the triad $\{N, B, T\}$ of a curve parametrized by its arclength s, according to (8), the Frenet equations are equivalent to the single spinor equation*

$$\frac{d\psi}{ds} = \frac{1}{2}(-i\tau\psi + \kappa\widehat{\psi}), \tag{10}$$

where τ and κ denote the torsion and curvature of the curve, respectively.

4. **Applications**

A basic theorem of differential geometry establishes that if two curves in $\mathbb{R}^3$ have the same curvature and torsion functions then, after translating and rotating appropriately one of them, the two curves coincide at all their points. In order to prove this theorem we shall consider two spinors ϕ and ψ such that $\overline{\phi}^{\rm t}\phi = 1$ and $\overline{\psi}^{\rm t}\psi = 1$, then

$$\overline{(\psi \pm \phi)}^{\rm t}(\psi \pm \phi) = 2 \pm \overline{\psi}^{\rm t}\phi \pm \overline{\phi}^{\rm t}\psi = 2 \pm \overline{\psi}^{\rm t}\phi \pm \psi^{\rm t}\overline{\phi} = 2 \pm 2\,{\rm Re}\,(\overline{\psi}^{\rm t}\phi).$$

Assuming that the spinors ϕ and ψ correspond to two curves with the same functions κ and τ, from (10) it follows that

$$\begin{aligned} \frac{d}{ds}(\overline{\psi}^{\rm t}\phi) &= \frac{1}{2}\overline{(-i\tau\psi + \kappa\widehat{\psi})}^{\rm t}\phi + \frac{1}{2}\overline{\psi}^{\rm t}(-i\tau\phi + \kappa\widehat{\phi}) \\ &= \frac{1}{2}(i\tau\overline{\psi}^{\rm t}\phi + \kappa\overline{\widehat{\psi}}^{\rm t}\phi - i\tau\overline{\psi}^{\rm t}\phi + \kappa\overline{\psi}^{\rm t}\widehat{\phi}) = \frac{\kappa}{2}(\overline{\widehat{\psi}}^{\rm t}\phi + \overline{\psi}^{\rm t}\widehat{\phi}) \end{aligned}$$

which is pure imaginary, and therefore $\overline{(\psi \pm \phi)}^{\rm t}(\psi \pm \phi)$ is a constant. Hence, if the triads $\{N, B, T\}$ corresponding to the two curves coincide for some value of s (which is achieved by appropriately translating and rotating one of them), at that point ϕ coincides with ψ or with $-\psi$ and this coincidence is maintained for all s. This implies that the tangent vectors to the two curves coincide for all s and, since for a curve α, $T = d\alpha/ds$, the two curves can only differ by a constant vector, but the previously mentioned translation makes the curves coincide for some value of s, and therefore they coincide for all s, thus proving the theorem.

Another application of the spinor form of the Frenet equations, (10), can be given in the case where the quotient τ/κ is some constant. There exists a constant angle θ ($0 \leq \theta \leq \pi$) such that $\tau/\kappa = \cot\theta$. Then, from (10) we have that $d\psi/ds = -[i\kappa/(2\sin\theta)](\cos\theta\,\psi + i\sin\theta\,\widehat{\psi})$ or, equivalently, $d\widehat{\psi}/ds = [i\kappa/(2\sin\theta)](\cos\theta\,\widehat{\psi} + i\sin\theta\,\psi)$. By combining these equations one finds that

$$\frac{d}{ds}\Big(\cos(\theta/2)\,\psi + i\sin(\theta/2)\,\widehat{\psi}\Big) = -\frac{i\kappa}{2\sin\theta}\Big(\cos(\theta/2)\,\psi + i\sin(\theta/2)\,\widehat{\psi}\Big)$$

which leads to

$$\cos(\theta/2)\,\psi + i\sin(\theta/2)\,\widehat{\psi} = \exp\left(-\frac{i}{2\sin\theta}\int^s \kappa(s')ds'\right)\phi, \qquad (11)$$

where ϕ is some spinor independent of s, with $\overline{\phi}^{\rm t}\phi = 1$. Hence,

$$\cos(\theta/2)\,\widehat{\psi} + i\sin(\theta/2)\,\psi = \exp\left(\frac{i}{2\sin\theta}\int^s \kappa(s')ds'\right)\widehat{\phi},$$

which implies that

$$\psi = \exp\left(-\frac{i}{2\sin\theta}\int^s \kappa(s')ds'\right)\cos(\theta/2)\,\phi - i\exp\left(\frac{i}{2\sin\theta}\int^s \kappa(s')ds'\right)\sin(\theta/2)\,\widehat{\phi} \quad (12)$$

and, therefore,

$$\widehat{\psi} = -i\exp\left(-\frac{i}{2\sin\theta}\int^s \kappa(s')ds'\right)\sin(\theta/2)\,\phi + \exp\left(\frac{i}{2\sin\theta}\int^s \kappa(s')ds'\right)\cos(\theta/2)\,\widehat{\phi}, \quad (13)$$

then, according to (8), denoting by $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ the triad of *constant* unit vectors corresponding to the spinor ϕ (i.e., $\mathbf{a} + i\mathbf{b} = \phi^{\mathrm{t}}\boldsymbol{\sigma}\phi, \mathbf{c} = -\widehat{\phi}^{\mathrm{t}}\boldsymbol{\sigma}\phi$),

$$T = \cos\theta\,\mathbf{c} + \sin\theta\left[\sin\left(\frac{1}{\sin\theta}\int^s \kappa(s')ds'\right)\mathbf{a} - \cos\left(\frac{1}{\sin\theta}\int^s \kappa(s')ds'\right)\mathbf{b}\right]. \quad (14)$$

This last equation proves the validity of the following

Proposition 2. *The tangent vector of a curve with τ/κ constant forms a constant angle (θ) with a constant vector ($\mathbf{c}$).*

Recalling that $T = d\alpha/ds$, integrating (14) with respect to s one obtains the expression for any curve of this class (called cylindrical helices [6,7]).

The preceding equations are also valid when $\tau = 0$, making $\theta = \pi/2$; equation (14) reduces then to

$$T = \sin\left(\int^s \kappa(s')ds'\right)\mathbf{a} - \cos\left(\int^s \kappa(s')ds'\right)\mathbf{b}, \quad (15)$$

which shows explicitly that the curve lies on a plane (normal to $\mathbf{c}$) and allows us to find the parametric form of any plane curve given its curvature (cf. [7], p. 38). For instance, in the simple case where κ is constant, equation (15) gives

$$T = \sin(\kappa s)\,\mathbf{a} - \cos(\kappa s)\,\mathbf{b}$$

and integrating this expression with respect to s one finds that

$$\alpha(s) = \frac{1}{\kappa}(-\cos(\kappa s)\,\mathbf{a} - \sin(\kappa s)\,\mathbf{b}) + \mathbf{d},$$

where $\mathbf{d}$ is a constant vector, thus showing that the curve is a circle of radius $1/\kappa$.

As pointed out in Sec. 2, there is a two-to-one homomorphism of the group SU(2) onto the rotation group in three dimensions SO(3). Assuming that $\overline{\psi}^{\rm t}\psi = 1$, as in (8) and (10), the 2×2 matrix

$$Q \equiv \begin{pmatrix} \psi_1 & \widehat{\psi}_1 \\ \psi_2 & \widehat{\psi}_2 \end{pmatrix}$$

belongs to SU(2) and corresponds to the rotation that carries the canonical basis into the orthonormal basis $\{N, B, T\}$. Making use of (6), (7), and (10), we find that the Frenet equations take the form

$$\frac{dQ}{ds} = QH(s), \qquad \text{with} \qquad H(s) \equiv -\frac{i}{2}\begin{pmatrix} \tau(s) & -i\kappa(s) \\ i\kappa(s) & -\tau(s) \end{pmatrix}. \tag{16}$$

There are two cases in which the integration of (16) is relatively simple; the first one corresponds to the trivial case where H does not depend on s (and the solution of (16) is given by $Q(s) = Q(0)\exp(sH)$) and the second one corresponds to the case where $H(s)$ commutes with $H(s')$ for all s'. This last condition amounts to $\tau(s)/\kappa(s) = \tau(s')/\kappa(s')$ and the solution of (16) can be expressed as

$$Q(s) = Q(0)\exp\int_0^s H(s')ds',$$

which is equivalent to the solution given by (12) and (13).

5. Final remarks

The fact that the Frenet equations reduce to a single spinor equation [equation (10)], equivalent to the three usual vector equations, is a consequence of the relationship between spinors and orthogonal triads of vectors and to the use of complex quantities. It may be noticed that the derivations presented in the preceding section are not a "word by word" translation of the corresponding ones in the vector formalism but there exist an independent procedure in the spinor formalism which is very useful as shown by the result given in (14).

It may also be noticed that it has not been necessary to have an explicit expression for the spinor corresponding to the triad $\{N, B, T\}$ of a given curve.

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