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Error inequalities for a quadrature formula of open type

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Abstract. An optimal 2-point quadrature formula of open type is derived. It is shown that the optimal quadrature formula has a better error bound than the well-known 2-point Gauss quadrature formula. Various error inequalities for this formula are established. Applications in numerical integration are given.

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1. **Introduction**

In recent years a number of authors have considered an error analysis for quadrature rules of Newton-Cotes type. In particular, the mid-point, trapezoid and Simpson rules have been investigated more recently ([2], [3], [4], [5], [6], [11], [14]) with the view of obtaining bounds on the quadrature rule in terms of a variety of norms involving, at most, the first derivative. Gauss-like quadrature rules are considered in [12] and [15] from an inequalities point of view. These results enlarge the applicability of the mentioned quadrature rules.

In this paper we derive an optimal 2-point quadrature formula of open type. It is optimal in the sense that it has a minimal error bound. In Section 2 we derive the optimal quadrature formula. We show that this formula has a better estimation of error than the well-known 2-point Gaussian quadrature rule (which is also 2-point quadrature formula of open type). In section 3 we establish some error bounds for the optimal formula. Similar estimations can be found in [11], [12], [13] and [14], where some different quadrature formulas are considered. These estimations ensure that we can apply the optimal quadrature formula to different classes of functions. In Section 4 we give applications of the above mentioned results in numerical integration.

2. **An optimal quadrature formula**

Here we seek an optimal quadrature formula of the type

$$\int_{-1}^{1} f(t)dt - f(x) - f(y) = \int_{-1}^{1} K(x,y,t)f''(t)dt \tag{2.1}$$

where $x, y \in [-1,1]$, $x < y$. We define

$$K(x,y,t) = \begin{cases} \frac{1}{2}(t-\alpha)^2 + \alpha_1, & t \in [-1,x] \\ \frac{1}{2}(t-\beta)^2 + \beta_1, & t \in (x,y) \\ \frac{1}{2}(t-\gamma)^2 + \gamma_1, & t \in [x,1], \end{cases}$$

where $\alpha, \alpha_1, \beta, \beta_1, \gamma, \gamma_1 \in R$ are parameters which have to be determined such that (2.1) is optimal, i.e. that it has a minimal error bound. Integrating by parts, we obtain

$$\int_{-1}^{1} K(x,y,t)f''(t)dt = \int_{-1}^{x} \left[\frac{1}{2}(t-\alpha)^2 + \alpha_1\right] f''(t)dt$$

$$+ \int_{x}^{y} \left[\frac{1}{2}(t-\beta)^2 + \beta_1\right] f''(t)dt + \int_{y}^{1} \left[\frac{1}{2}(t-\gamma)^2 + \gamma_1\right] f''(t)dt$$

$$= -f'(-1)\left[\frac{1}{2}(1+\alpha)^2 + \alpha_1\right]$$

$$+f'(x)\left[\frac{1}{2}(x-\alpha)^2 + \alpha_1 - \frac{1}{2}(x-\beta)^2 - \beta_1\right]$$

$$+f'(y)\left[\frac{1}{2}(y-\beta)^2 + \beta_1 - \frac{1}{2}(y-\gamma)^2 - \gamma_1\right]$$

$$+f'(1)\left[\frac{1}{2}(1-\gamma)^2 + \gamma_1\right]$$

$$-\int_{-1}^{x} (t-\alpha)f'(t)dt - \int_{x}^{y} (t-\beta)f'(t)dt - \int_{y}^{1} (t-\gamma)f'(t)dt$$

$$= -f'(-1)\left[\frac{1}{2}(1+\alpha)^2 + \alpha_1\right]$$

$$+f'(x)\left[\frac{1}{2}(x-\alpha)^2 + \alpha_1 - \frac{1}{2}(x-\beta)^2 - \beta_1\right]$$

$$+f'(y)\left[\frac{1}{2}(y-\beta)^2+\beta_1-\frac{1}{2}(y-\gamma)^2-\gamma_1\right]$$
$$+f'(1)\left[\frac{1}{2}(1-\gamma)^2+\gamma_1\right]$$
$$-f(-1)(1+\alpha)+f(x)(\alpha-\beta)+f(y)(\beta-\gamma)+f(1)(1-\gamma)$$
$$+\int_{-1}^{1} f(t)dt.$$

We require that

$$\begin{gathered}
\frac{1}{2}(1+\alpha)^2+\alpha_1=0,\\
\frac{1}{2}(x-\alpha)^2+\alpha_1-\frac{1}{2}(x-\beta)^2-\beta_1=0,\\
\frac{1}{2}(y-\beta)^2+\beta_1-\frac{1}{2}(y-\gamma)^2-\gamma_1=0,\\
\frac{1}{2}(1-\gamma)^2+\gamma_1=0,\\
1+\alpha=0,\\
\alpha-\beta=-1,\\
\beta-\gamma=-1,\\
1-\gamma=0.
\end{gathered}$$

From the above equations we easily find

$$\alpha=-1,\ \gamma=1,\ \alpha_1=0,\ \gamma_1=0,\ \beta=0,\ \beta_1=x+\frac{1}{2}=-y+\frac{1}{2}$$

which implies $x=-y$. Hence, we get

$$K(x,y,t)=\begin{cases}\frac{1}{2}(t+1)^2, & t\in[-1,x]\\ \frac{1}{2}t^2+x+\frac{1}{2}, & t\in(x,y)\\ \frac{1}{2}(t-1)^2, & t\in[x,1]\end{cases}. \tag{2.2}$$

We now consider the quadrature formula

$$\int_{-1}^{1} f(t)dt-f(x)-f(y)=\int_{-1}^{1} K(x,y,t)f''(t)dt,$$

where $K(x,y,t)$ is given by (2.2). We have

$$\left|\int_{-1}^{1} K(x,y,t)f''(t)dt\right|\le\|K(x,y,\cdot)\|_2\,\|f''\|_2\,,$$

where

$$\|f''\|_2^2=\int_{-1}^{1} f''(t)^2dt.$$

We define

$$\begin{aligned} g(x) &= \|K(x,y,\cdot)\|_2^2 = \\ &= \frac{1}{4}\int_{-1}^{x}(t+1)^4dt + \int_{x}^{-x}(\frac{1}{2}t^2+x+\frac{1}{2})^2dt + \frac{1}{4}\int_{-x}^{1}(t-1)^4dt \\ &= -\frac{1}{6}x^4 - \frac{4}{3}x^3 - x^2 + \frac{1}{10} \end{aligned}$$

and seek x such that $g(x) \to \min$, i.e. we seek a global minimum of the function g on the interval $[-1,1]$. For that purpose, we calculate

$$g'(x) = -\frac{2}{3}x^3 - 4x^2 - 2x.$$

From the equation $g'(x) = 0$ we find the solutions: $x_1 = 0$, $x_2 = \sqrt{6} - 3$ and $x_3 = 3 - \sqrt{6}$. We have

$$\begin{aligned} g(0) &= \frac{1}{10}, \\ g(\sqrt{6}-3) &= \frac{98}{5} - 8\sqrt{6}, \\ g(-1) &= \frac{4}{15}, \\ g(1) &= -\frac{12}{5}. \end{aligned}$$

We conclude that $x = \sqrt{6}-3$ is the point of global minimum. For $x = \sqrt{6}-3$ we get

$$\int_{-1}^{1} f(t)dt - f(\sqrt{6}-3) - f(3-\sqrt{6}) = \int_{-1}^{1} K(\sqrt{6}-3, 3-\sqrt{6}, t)f''(t)dt$$

and

$$\left|\int_{-1}^{1} K(\sqrt{6}-3, 3-\sqrt{6}, t)f''(t)dt\right| \leq \sqrt{\frac{98}{5} - 8\sqrt{6}}\,\|f''\|_2 .$$

We now summarize the above obtained results.

Theorem 1. *Let $I \subset R$ be an open interval such that $[-1,1] \subset I$ and let $f : I \to R$ be a twice differentiable function such that $f'' \in L_2(-1,1)$. Then we have*

$$\int_{-1}^{1} f(t)dt - f(\sqrt{6}-3) - f(3-\sqrt{6}) = R_2(f) \tag{2.3}$$

and

$$|R_2(f)| \leq \sqrt{\frac{98}{5} - 8\sqrt{6}}\,\|f''\|_2 . \tag{2.4}$$

Remark 1. *The quadrature formula (2.3) is optimal in the sense mentioned in Section 1.*

We now compare the above result with the 2-point Gauss formula. We have

$$\left\|K(-\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\cdot)\right\|_2^2=-\frac{34}{135}+\frac{4}{27}\sqrt{3}$$

Thus,

$$\left|\int_{-1}^{1} f(t)dt-f(-\frac{\sqrt{3}}{3})-f(\frac{\sqrt{3}}{3})\right|\le\sqrt{-\frac{34}{135}+\frac{4}{27}\sqrt{3}}\,\|f''\|_2\,. \tag{2.5}$$

Hence, the estimate (2.4) is better than the estimate (2.5), since $\sqrt{\frac{98}{5}-8\sqrt{6}}<\sqrt{-\frac{34}{135}+\frac{4}{27}\sqrt{3}}$. If we consider the above problem on the interval $[a,b]$ then we get the following result.

Theorem 2. *Let $I\subset R$ be an open interval such that $[a,b]\subset I$ and let $f:I\to R$ be a twice differentiable function such that $f''\in L_2(a,b)$. Then we have*

$$\int_a^b f(t)dt=f(x_1)+f(x_2)+R(f),$$

where

$$x_1=\frac{b-a}{2}x+\frac{a+b}{2},\ x_2=\frac{a-b}{2}x+\frac{a+b}{2},\ x=\sqrt{6}-3 \tag{2.6}$$

and

$$|R(f)|\le\sqrt{\frac{49}{80}-\frac{1}{4}\sqrt{6}}\,\|f''\|_2\,(b-a)^{5/2}.$$

3. **Error inequalities**

First we consider some basic properties of the spaces $L_p(a,b)$, for $p=1,2,\infty$. As we know, $X=(L_2(a,b),(\cdot,\cdot))$ is a Hilbert space with the inner product

$$(f,g)=\int_a^b f(t)g(t)dt. \tag{3.1}$$

In the space X the norm $\|\cdot\|_2$ is defined in the usual way,

$$\|f\|_2=\left(\int_a^b f(t)^2dt\right)^{1/2}. \tag{3.2}$$

We also consider the space $Y = (L_2(a,b), \langle\cdot,\cdot\rangle)$ where the inner product $\langle\cdot,\cdot\rangle$ is defined by

$$\langle f, g\rangle = \frac{1}{b-a}\int_a^b f(t)g(t)dt. \tag{3.3}$$

It is not difficult to see that Y is a Hilbert space, too. In the space Y the norm $\|\cdot\|$ is defined by

$$\|f\| = \sqrt{\langle f, f\rangle}. \tag{3.4}$$

We also define the Chebyshev functional

$$T(f,g) = \langle f, g\rangle - \langle f, e\rangle \langle g, e\rangle \,, \tag{3.5}$$

where $f, g \in L_2(a,b)$ and $e = 1$. This functional satisfies the pre-Grüss inequality ([9, p. 296]),

$$T(f,g)^2 \le T(f,f)T(g,g). \tag{3.6}$$

Specially, we define

$$\sigma(f) = \sigma(f; a, b) = \sqrt{(b-a)T(f,f)}. \tag{3.7}$$

The space $L_1(a,b)$ is a Banach space with the norm

$$\|f\|_1 = \int_a^b |f(t)|\, dt \tag{3.8}$$

and the space $L_\infty(a,b)$ is also a Banach space with the norm

$$\|f\|_\infty = \underset{t\in[a,b]}{ess\sup} |f(t)|\,. \tag{3.9}$$

If $f \in L_1(a,b)$ and $g \in L_\infty(a,b)$ then we have

$$|(f,g)| \le \|f\|_1 \|g\|_\infty \,. \tag{3.10}$$

More about the above mentioned spaces can be found, for example, in [1].

Finally, we define the functional

$$\begin{aligned} Q(f) &= Q(f; a, b) \qquad (3.11)\\ &= \int_a^b f(t)dt - \frac{b-a}{2}\left[f(x_1) + f(x_2)\right], \end{aligned}$$

where x_1, x_2 are given by (2.6). We also need the following lemma.

Lemma 1. *Let*

$$f(t) = \begin{cases} f_1(t), & t \in [a, x_1] \\ f_2(t), & t \in (x_1, x_2] \\ f_3(t), & t \in (x_2, b] \end{cases}\,, \tag{3.12}$$

where $x_1, x_2 \in [a,b]$, $x_1 < x_2$, $f_1 \in C^1[a, x_1]$, $f_2 \in C^1[x_1, x_2]$, $f_3 \in C^1[x_2, b]$. *If* $f_1(x_1) = f_2(x_1)$ *and* $f_2(x_2) = f_3(x_2)$ *then* f *is an absolutely continuous function.*

A variant of this lemma can be found in [15].

Theorem 3. *Let $f : [-1,1] \to R$ be a function such that $f' \in L_1(-1,1)$. If there exists a real number γ_1 such that $\gamma_1 \le f'(t)$, $t \in [-1,1]$, then*

$$|Q(f;-1,1)| \le 2(3-\sqrt{6})(S-\gamma_1), \tag{3.13}$$

and if there exists a real number Γ_1 such that $f'(t) \le \Gamma_1$, $t \in [-1,1]$, then

$$|Q(f;-1,1)| \le 2(3-\sqrt{6})(\Gamma_1 - S), \tag{3.14}$$

where $Q(f;-1,1)$ is defined by (3.11) and $S = [f(1) - f(-1)]/2$. If there exist real numbers γ_1, Γ_1 such that $\gamma_1 \le f'(t) \le \Gamma_1$, $t \in [-1,1]$, then

$$|Q(f;-1,1)| \le \left(\frac{25}{2} - 5\sqrt{6}\right)(\Gamma_1 - \gamma_1). \tag{3.15}$$

Proof. We first prove that (3.15) holds. We define the function

$$p_1(t) = \begin{cases} t+1, & t \in [-1,x] \\ t, & t \in (x,y] \\ t-1, & t \in (y,1] \end{cases}. \tag{3.16}$$

where $x = \sqrt{6} - 3$ and $y = -x$. It is easy to verify that

$$(p_1, f') = -Q(f;-1,1). \tag{3.17}$$

On the other hand, we have

$$\left(f' - \frac{\Gamma_1 + \gamma_1}{2}, p_1\right) = (f', p_1), \tag{3.18}$$

since $(p_1, e) = 0$. From (3.10) we get

$$\left|\left(f' - \frac{\Gamma_1 + \gamma_1}{2}, p_1\right)\right| \le \left\|f' - \frac{\Gamma_1 + \gamma_1}{2}\right\|_\infty \|p_1\|_1 \le \left(25 - 10\sqrt{6}\right)\frac{\Gamma_1 - \gamma_1}{2}, \tag{3.19}$$

since

$$\left\|f' - \frac{\Gamma_1 + \gamma_1}{2}\right\|_\infty \le \frac{\Gamma_1 - \gamma_1}{2}$$

and

$$\|p_1\|_1 = 25 - 10\sqrt{6}.$$

From (3.17)-(3.19) we see that (3.15) holds. We now prove that (3.13) holds. We have

$$|(f' - \gamma_1, p_1)| \le \|p_1\|_\infty \|f' - \gamma_1\|_1 = 2(3-\sqrt{6})(S - \gamma_1),$$

since

$$\|p_1\|_\infty = 3 - \sqrt{6}$$

and

$$\begin{aligned} \|f' - \gamma_1\|_1 &= \int_{-1}^{1} (f'(t) - \gamma_1)dt = f(1) - f(-1) - 2\gamma_1 \\ &= 2(S - \gamma_1). \end{aligned}$$

In a similar way we can prove that (3.14) holds. ☑

Remark 2. *Note that we can apply the estimate (3.15) only if the first derivative f' is bounded. It means that we cannot use (3.15) to estimate directly the error when approximating the integral of such a well-behaved function as $f(t)=\sqrt{t}$ on $[0,1]$, (since $f'(t)=1/(2\sqrt{t})$ is unbounded on $[0,1]$). On the other hand, we can use the estimation (3.13), (since $\gamma_1 = 1/2$ on $[0,1]$ for the given function).*

Remark 3. *In* [12] *we can find the following result for the 2-point Gaussian quadrature formula,*

$$\left| f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}) - \int_{-1}^{1} f(t)dt \right| \leq \frac{\Gamma-\gamma}{6}(5-2\sqrt{3}). \tag{3.20}$$

We see that (3.15) is better than (3.20), since $\frac{25}{2} - 5\sqrt{6} < \frac{5-2\sqrt{3}}{6}$.

Theorem 4. *Let $f : [a,b] \to R$ be a function such that $f' \in L_1(a,b)$. If there exists a real number γ_1 such that $\gamma_1 \leq f'(t)$, $t \in [a,b]$, then*

$$|Q(f;a,b)| \leq \frac{3-\sqrt{6}}{2}(S-\gamma_1)(b-a)^2, \tag{3.21}$$

and if there exists a real number Γ_1 such that $f'(t) \leq \Gamma_1$, $t \in [a,b]$, then

$$|Q(f;a,b)| \leq \frac{3-\sqrt{6}}{2}(\Gamma_1 - S)(b-a)^2, \tag{3.22}$$

where $Q(f;a,b)$ is defined by (3.11) and $S = (f(b)-f(a))/(b-a)$. If there exist real numbers γ_1, Γ_1 such that $\gamma_1 \leq f'(t) \leq \Gamma_1$, $t \in [a,b]$, then

$$|Q(f;a,b)| \leq \left(\frac{25}{8} - \frac{5}{4}\sqrt{6}\right)(\Gamma_1 - \gamma_1)(b-a)^2. \tag{3.23}$$

Theorem 5. *Let $f : [-1,1] \to R$ be an absolutely continuous function such that $f' \in L_2(-1,1)$. Then*

$$|Q(f;-1,1)| \leq \sqrt{\frac{74}{3} - 10\sqrt{6}}\ \sigma(f';-1,1), \tag{3.24}$$

where $\sigma(f;-1,1)$ is defined by (3.7). The inequality (3.24) is sharp in the sense that the constant $\sqrt{\frac{74}{3} - 10\sqrt{6}}$ cannot be replaced by a smaller one.

Proof. Let p_1 be defined by (3.16). We have

$$\langle p_1, f' \rangle = -\frac{1}{2}Q(f;0,1),$$

since (3.17) holds and $\langle f,g \rangle = \frac{1}{2}(f,g)$ if $[a,b]=[-1,1]$. On the other hand, we have

$$\langle p_1, f' \rangle = T(f', p_1),$$

since $\langle p_1, e\rangle = 0$. From (3.6) it follows

$$\begin{aligned}|T(f', p_1)| &\leq \sqrt{T(p_1, p_1)}\sqrt{T(f', f')} = \frac{1}{2}\|p_1\|_2\,\sigma(f'; -1, 1)\\ &= \frac{1}{2}\sqrt{\frac{74}{3} - 10\sqrt{6}}\sigma(f'; -1, 1),\end{aligned}$$

since

$$\|p_1\|_2 = \sqrt{\frac{74}{3} - 10\sqrt{6}}.$$

Hence, the inequality (3.24) is proved. We have to prove that this inequality is sharp. For that purpose, we define the function

$$f(t) = \begin{cases} \frac{1}{2}(t+1)^2, & t \in [-1, x] \\ \frac{1}{2}t^2, & t \in (x, y] \\ \frac{1}{2}(t-1)^2, & t \in (y, 1] \end{cases} \tag{3.25}$$

such that $f'(t) = p_1(t)$. From Lemma 1 we see that the function f, defined by (3.25), is an absolutely continuous function. For this function the left-hand side of (3.24) becomes

$$L.H.S.(3.24) = \frac{74}{3} - 10\sqrt{6}.$$

The right-hand side of (3.24) becomes

$$R.H.S.(3.24) = \frac{74}{3} - 10\sqrt{6}.$$

We see that $L.H.S.(3.24) = R.H.S.(3.24)$. Thus, (3.24) is sharp. ☑

Theorem 6. *Let $f : [a, b] \to R$ be an absolutely continuous function such that $f' \in L_2(a, b)$. Then*

$$|Q(f; a, b)| \leq \sqrt{\frac{37}{12} - \frac{5}{4}\sqrt{6}}\;\sigma(f'; a, b)(b-a)^{3/2}, \tag{3.26}$$

where $\sigma(f; a, b)$ is defined by (3.7). The inequality (3.26) is sharp in the sense that the constant $\sqrt{\frac{37}{12} - \frac{5}{4}\sqrt{6}}$ cannot be replaced by a smaller one.

Remark 4. *The estimate (3.23) is better than the estimate (3.26). However, note that the estimate (3.23) can be applied only if f' is bounded. On the other hand, the estimate (3.26) can be applied for an absolutely continuous function if $f' \in L_2(a, b)$.*

There are many examples where we cannot apply the estimate (3.23) but we can apply (3.26).

Example 1. *Let us consider the integral* $\int\limits_0^1 \sqrt[3]{\sin t^2}dt$. *We have*

$$f(t) = \sqrt[3]{\sin t^2} \text{ and } f'(t) = \frac{2t\cos t^2}{3\sqrt[3]{\sin^2 t^2}}$$

such that $f'(t) \to \infty$, $t \to 0$ *and we cannot apply the estimate (3.23). On the other hand, we have*

$$\int\limits_0^1 [f'(t)]^2 \, dt \le \frac{4}{9} \max_{t\in[0,1]} \frac{t^2\cos t^2}{\sin t^2} \int\limits_0^1 \frac{dt}{\sqrt[3]{\sin t^2}} \le \frac{16}{9},$$

i.e. $\|f'\|_2 \le \frac{4}{3}$ *and we can apply the estimate (3.26).*

4. Applications in numerical integration

Let $\pi = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a given subdivision of the interval $[a, b]$ such that $h_i = x_{i+1} - x_i = h = (b-a)/n$. From (3.11) we get

$$\begin{aligned} & Q(f; x_i, x_{i+1}) \\ = & \int_{x_i}^{x_{i+1}} f(t)dt - \frac{h}{2}\left[f(x_{1i}) + f(x_{2i})\right], \end{aligned}$$

where

$$x_{1i} = \frac{h}{2}x + \frac{x_i + x_{i+1}}{2},\ x_{2i} = -\frac{h}{2}x + \frac{x_i + x_{i+1}}{2},\ x = \sqrt{6} - 3.$$

If we now sum the above relation over i from 0 to $n-1$ then we get

$$\begin{aligned} & \sum_{i=0}^{n-1} Q(f; x_i, x_{i+1}) \\ = & \int_a^b f(t)dt - \frac{h}{2}\sum_{i=0}^{n-1}\left[f(x_{1i}) + f(x_{2i})\right]. \end{aligned}$$

We introduce the notation

$$S(f; a, b) = \sum_{i=0}^{n-1} Q(f; x_i, x_{i+1}). \tag{4.1}$$

We also define

$$\sigma_n(f) = \sum_{i=0}^{n-1} \sqrt{\frac{b-a}{n}\left\|f'\right\|_2^2 - \left[f(x_{i+1}) - f(x_i)\right]^2} \tag{4.2}$$

and

$$\omega_n(f) = \left[(b-a)\left\|f'\right\|_2^2 - \frac{1}{n}(f(b) - f(a))^2\right]^{1/2}. \tag{4.3}$$

Theorem 7. *Under the assumptions of Theorem 2 we have*

$$\begin{aligned}&\left|\int_a^b f(t)dt - \frac{h}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]\right|\\ \leq\ &\sqrt{\frac{49}{80}-\frac{1}{4}\sqrt{6}}\frac{\|f''\|_2}{n\sqrt{n}}(b-a)^{5/2}.\end{aligned}$$

Proof. Apply Theorem 2 to the intervals $[x_i, x_{i+1}]$ and sum. ☑

Theorem 8. *Under the assumptions of Theorem 4 we have*

$$|S(f;a,b)| \leq \left(\frac{25}{8}-\frac{5}{4}\sqrt{6}\right)\frac{\Gamma_1-\gamma_1}{n}(b-a)^2,$$

$$|S(f;a,b)| \leq \left(3-\sqrt{6}\right)\frac{S-\gamma_1}{2n}(b-a)^2,$$

$$|S(f;a,b)| \leq \left(3-\sqrt{6}\right)\frac{\Gamma_1-S}{2n}(b-a)^2,$$

where $S(f;a,b)$ is defined by (4.1) and $\{a = x_0 < x_1 < \cdots < x_n = b\}$ is a uniform subdivision of $[a,b]$, i.e. $x_i = a+ih$, $h = (b-a)/n$, $i = 0,1,...,n$.

Proof. Apply Theorem 4 to the intervals $[x_i, x_{i+1}]$ and sum. Note that

$$\sum_{i=0}^{n-1}[f(x_{i+1})-f(x_i)] = f(b)-f(a).$$

☑

Theorem 9. *Under the assumptions of Theorem 6 we have*

$$|S(f;a,b)| \leq \sqrt{\frac{37}{12}-\frac{5}{4}\sqrt{6}}\frac{b-a}{n}\sigma_n(f) \leq \sqrt{\frac{37}{12}-\frac{5}{4}\sqrt{6}}\frac{b-a}{\sqrt{n}}\omega_n(f), \qquad (4.4)$$

where $S(f;a,b)$, $\sigma_n(f)$ and $\omega_n(f)$ are defined by (4.1), (4.2) and (4.3), respectively and $\{a = x_0 < x_1 < \cdots < x_n = b\}$ is a uniform subdivision of $[a,b]$, i.e. $x_i = a+ih$, $h = (b-a)/n$, $i = 0,1,...,n$.

Proof. We apply Theorem 6 to the interval $[x_i, x_{i+1}]$ and sum. Then we have

$$\begin{aligned}&|S(f;a,b)|\\ \leq\ &\sqrt{\frac{37}{12}-\frac{5}{4}\sqrt{6}}\,h^{3/2}\sum_{i=0}^{n-1}\left[\|f'\|_2^2-\frac{1}{h}\left(f(x_{i+1})-f(x_i)\right)^2\right]^{1/2}.\end{aligned}$$

From the above relation and the fact $h = (b-a)/n$ we see that the first inequality in (4.4) holds.

Using the Cauchy inequality we get

$$\begin{aligned}
& \sum_{i=0}^{n-1}\left[\|f'\|_2^2 - \frac{1}{h}\left(f(x_{i+1}) - f(x_i)\right)^2\right]^{1/2} && (4.5)\\
\leq\ & n\left[\|f'\|_2^2 - \frac{1}{b-a}\sum_{i=0}^{n-1}\left(f(x_{i+1}) - f(x_i)\right)^2\right]^{1/2}\\
\leq\ & n\left[\|f'\|_2^2 - \frac{1}{b-a}\frac{1}{n}\left(f(b) - f(a)\right)^2\right]^{1/2}.
\end{aligned}$$

Thus the second inequality in (4.4) holds, too. ☑

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