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(∗)-groups and pseudo-bad groups

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Abstract. We give an example of an infinite simple Frobenius group G without involutions, with a trivial kernel and a nilpotent complement. Nevertheless, this group is not ω- stable (not even superstable), this is the "only" property missing in order to be a counterexample to the Cherlin-Zil'ber Conjecture which says that simple ω- stable groups are algebraic groups.

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1. Introduction

In [Co2] we define a bad group to be a connected non-solvable group of finite Morley-rank in which all the definable proper subgroups are nilpotent-by-finite (i.e. all definable proper connected subgroups are nilpotent). We proved there that a bad group has a definable quotient which is a simple bad group. The Cherlin-Zil'ber conjecture states that every simple group of finite Morley rank is an algebraic group over an algebraically closed field, and we know that the maximal connected solvable subgroups (called Borel subgroups) of a non-solvable algebraic group are non-nilpotent. Therefore, the existence of bad groups would be in contradiction with the conjecture above.

The structure of a simple bad group is well understood (see [Co2]): if G is such a group, it has a proper definable subgroup B, which is selfnormalizing, such that the union of all its conjugates is G and the intersection of any pair of them is trivial. B is in fact a Borel subgroup of G, it is nilpotent and we can also prove that G does not have involutions. We call such a pair (G,B), where G is not necessarily of finite Morley rank, a pseudo-bad group.

Definition. A *pseudo-bad group* is an infinite simple group G, without involutions, with a nilpotent proper subgroup B which is selfnormalizing, such that the union of its conjugates is G and the intersection of any pair of them is trikvial.

Equivalently a pseudo-bad group is an infinite simple Frobenius group G, without involutions, with a nilpotent complement B and with trivial kernel. It was unknown whether such a group existed.

In this article we construct a pseudo-bad group G, or rather a family of them, where the complement B is an infinite cyclic and definable subgroup and the maximal solvable subgroups of G are B and their conjugates. Since $B \cong \mathbb{Z}$ is definable, G is not of finite Morley rank (or $\omega - stable$). We show that G is not even superstable.

S.V. Ivanov gave a similar example, but with the following additional property: there is an m (in his example $m = 10^6$) such that, for every $b \notin B$ and every conjugacy class $C \neq \{1\}$ we have $(bB)^m = C^m = G$. This property is also satisfied by any simple bad group. For this reason his example is better than ours. His result was announced as an abstract in [I] but we do not know the actual proof of it. We presented our results for the first time in 1989 (see [Co1]) and we obtained them independently. We believe that our construction is simpler than Ivanov's and therefore we present it here.

2. $(*)$-groups

Definition. A $(*)$-*group* is a torsionfree group G with the following property:

$(*)$ For all nontrivial element $g \in G,\ C_G(g)$ is cyclic.

In the rest of the paper, if G is a $(*)$-group and $g \in G \setminus \{1\}$, we will call v_g one of the generators of the cyclic group $C_G(g)$.

Examples of $(*)$-groups:

(1) $\langle \mathbb{Z}, + \rangle$ is the only non trivial abelian $(*)$-group.
(2) Free groups are $(*)$-groups; cf [L-Sch].
(3) In Theorem 3 bellow we show that some HNN-extensions of $(*)$-groups are also $(*)$-groups.
(4) Subgroups of $(*)$-groups are $(*)$-groups.
(5) Free products of $(*)$-groups are $(*)$-groups. See Theorem 2 bellow.

Theorem 1. *Let G be a $(*)$-group, $g \in G - \{1\}$ and $v_g \in G$ so that $C_G(g) = \langle v_g \rangle$. Then:*

(1) $C_G(v_g) = \langle v_g \rangle$
(2) $C_G(g) = \langle v_g \rangle$ *is a maximal cyclic subgroup of G.*
(3) *For every $r \in \mathbb{Z}^*$, $C_G(g^r) = C_G(g)$.*
(4) $C_G\big(C_G(g)\big) = N_G\big(C_G(g)\big) = N_G\big(\langle g \rangle\big) = C_G(g)$.
(5) *For every $h \in G \smallsetminus \{1\}$: if $C_G(h) \cap C_G(g) \neq \langle 1 \rangle$, then $C_G(h) = C_G(g)$, and if $\left(\bigcup_{x\in G} C_G(h)^x\right) \cap \left(\bigcup_{x\in G} C_G(g)^x\right) \neq \langle 1 \rangle$, then $\bigcup_{x\in G} C_G(h)^x = \bigcup_{x\in G} C_G(g)^x$.*
(6) *The relation $a \leftrightarrow b$, "a commutes with b", is an equivalence relation in $G \smallsetminus \{1\}$.*
(7) *If $u^{-1}g^p u = g^q$, where $p \in \mathbb{Z}^*$, $q \in \mathbb{Z}$ and $u \in G$, then $p = q$.*
(8) *Let $x, h \in G$ such that $C_G(x) = \langle x \rangle$ and $h^m = x^p$, for $m, p \in \mathbb{Z}$, $m \neq 0$. Then m divides p and $h = x^{p/m}$.*
(9) *Every solvable subgroup of G is cyclic.*

Proof.

(1) There is a $p \in \mathbb{Z}$ such that $g = v_g^p$, because $g \in C_G(g) = \langle v_g \rangle$. Therefore
$$C_G(v_g) \subseteq C_G(g) = \langle v_g \rangle.$$
The other inclusion is clear.
(2) $\langle v_g \rangle$ is a maximal cyclic subgroup of G: if $\langle v_g \rangle \leq \langle u \rangle$, then u commutes with v_g, so
$$\langle u \rangle \leq C_G(v_g) \leq \langle v_g \rangle \leq \langle u \rangle.$$
(3) It is clear that $C_G(g) \leq C_G(g^r)$. But $C_G(g^r)$ is also a cyclic subgroup of G. By the maximality of $C_G(g)$, we have the equality.
(4) The following inclusions are clear
$$C_G(g) \leq C_G\big(C_G(g)\big) \leq N_G\big(C_G(g)\big) \leq N_G\big(\langle g \rangle\big).$$
We show that $N_G\big(\langle g \rangle\big) \leq C_G(g)$: Let $u \in N_G\big(\langle g \rangle\big)$, i.e., $g^u = g^{\pm 1}$. Then $u^2 \in C_G(g)$ i.e., $u^2 = v_g^q$ for some $q \in \mathbb{Z}$. If $u \neq 1$, i.e., $q \neq 0$, then we get $C_G(u^2) = C_G(u)$, so
$$u \in C_G(u) = C_G(u^2) = C_G(v_g^q) = C_G(v_g) = \langle v_g \rangle = C_G(g).$$
(5) Let $b \neq 1$ in $C_G(h) \cap C_G(g)$. By (2) $C_G(g)$ and $C_G(h)$ are maximal cyclic subgroups of G. Both are contained in $C_G(b)$, which is also cyclic. Hence
$$C_G(h) = C_G(b) = C_G(g).$$

The second part of the claim follows from here.

(6) The relation " $\leftrightarrow$ " is clearly reflexive and symmetric. The transitivity follows easily from (5).

(7) By hypothesis $C_G(u^{-1}gu) \cap C_G(g) \neq 1$, so $C_G(g)^u = C_G(g)$. By (4) $u \in N_G\big(C_G(g)\big) = C_G(g)$. Hence $p = q$.

(8) We have

$$h \in C_G(h^m) = C_G(x^p) = C_G(x) = \langle x \rangle.$$

Then $h = x^q$, for some $q \in \mathbb{Z}$ and $x^p = h^m = x^{qm}$. Therefore $p = qm$, $q = p/m$ and $h = x^{p/m}$.

(9) Let $H \neq 1$ be a solvable subgroup of G. H contains a non trivial abelian normal subgroup $H^{(n)}$. Let $1 \neq h \in H^{(n)}$, then $H^{(n)} \leq C_G(h)$. Since $C_G(h)$ is cyclic, then $H^{(n)}$ is also cyclic. Let $H^{(n)} = \langle u \rangle$. Then

$$H \leq N_G\big(\langle u \rangle\big) = C_G(u),$$

and H is cyclic. ☑

Definition. Let G be a torsion-free group. $x \in G \setminus \{1\}$ is called *indecomposable* if $\langle x \rangle$ is a maximal cyclic subgroup of G.

Let G be a $(*)$-group and $x \in G \setminus \{1\}$. The set

$$C_x^G := \bigcup_{g \in G} C_G\,(x)^g$$

is called *the component of x in G*. x is called *atomic* if $C_G(x) = \langle x \rangle$.

Theorem 2. *Let G and H be $(*)$-groups. Then $G{*}H$, the free product of G and H, is also a $(*)$-group.*

Proof. Every $w \in G{*}H$, $w \neq 1$, can be written uniquely as a product $w_1 \cdots w_n$, where $w_1 \neq 1$, each w_i belongs G or H and consecutive factors w_i and w_{i+1} are not in the same group. This is called the *normal form* of w and we say that $w = w_1 \cdots w_n$ is *reduced.* The number $|w| = n$ is called the length of w.

We call $w = w_1 \cdots w_n$ *cyclicly reduced* if w_n and w_1 are in different factors or $n = 1$.

Let $w \in G * H$, $w \neq 1$. The proof will be complete once we prove by induction on $|w| = n$ the following claim:

Claim. *For every $r \in \mathbb{Z}^*$, $C_{G*H}(w^r) = C_{G*H}(w)$ is cyclic.*

If $|w| = 1$. then $w \in G$ or $w \in H$ and so $C_{G*H}(w^r)$ equals $C_G(w^r)$ or $C_H(w^r)$. Now we can apply Theorem 1 (3).

Suppose that $|w| \geq 2$. Without lost of generality, assume that w is cyclicly reduced (every element of $G * H$ is conjugated to one element which is cyclicly reduced). We may also assume, without lost of generality, that w is indecomposable; otherwise $w = v^r$ for some $v \in G * H$, v indecomposable, also cyclicly reduced and $|v| < |w|$. But, if the claim holds for v, it holds for $v^r = w$ as well. Therefore $w = w_1 \cdots w_m$ is cyclicly reduced and indecomposable. Let $u = u_1 \cdots u_n$ be in $C_{G*H}(w^r)$. We are done if we prove by induction on $|u| = n$ that

$$u \in \langle w \rangle \big(\leq C_{G*H}(w) \leq C_{G*H}(w^r) \big).$$

First we show that either $u = 1$ or $n \geq m$. Suppose, for a contradiction, that $u \neq 1$ and $n < m$. It follow by induction that $C_{G*H}(u)$ is cyclic. Let $C_{G*H}(u) = \langle v_u \rangle$.Then $u = v_u^q$ for some $q \in \mathbb{Z}^*$ and $|v_u| \leq |u|$. Therefore

$$C_{G*H}(v_u) = C_{G*H}(v_u^q) = C_{G*H}(u) = \langle v_u \rangle.$$

By hypothesis we have $w^r \in C_{G*H}(u) = \langle v_u \rangle$. Then $w^r = v_u^{\pm s}$ where $s \geq 0$. Hence

$$w \in C_{G*H}(w^r) = C_{G*H}(v_u^{\pm s}) = C_{G*H}(v_u) = \langle v_u \rangle.$$

Then $w = v_u^p$ for some $p \in \mathbb{Z}^*$. But w is indecomposable, so $|p| = 1$ and we have $|w| = |v_u| \leq |u| < |w|$, a contradiction.

Assume now that $u \neq 1$. We have $n \geq m$ and since $u \in C_{G*H}(w^r)$, the following equality holds:

$$u_1 \cdots u_{n-m} \cdots u_n w_1 \cdots w_m \cdots w_1 \cdots w_m u_n^{-1} \cdots u_1^{-1} = w^r.$$

The word on the left side must be reducible for the words to have the same lenght. This can only happen if $u_n w_1$ or $w_m u_n^{-1}$ is reducible. It is clear that only one of them is reducible because w is cyclicly reduced. Therefore there are two cases, but we will consider only the case in which $w_m u_n^{-1}$ is reducible.

Let $i \leq m-1$ be maximal such that $w_{m-i} \cdots w_m u_n^{-1} \cdots u_{n-i}^{-1} =: b_i$ is an element of G or H. One can easily see that $i = m-1$; otherwise $w_{m-i-1} b_i u_{n-i-1}^{-1}$ would be reduced and so would be the word

$$u w^{r-1} w_1 \cdots w_{m-i-1} b_i u_{n-i-1}^{-1} \cdots u_1^{-1} = u w^r u^{-1} = w^r.$$

By a length argument we get $m = |w| > |u| = n$, a contradiction.

Let $\hat{u} = u_1 \cdots u_{n-m} b_{m-1}^{-1}$. Then $u = \hat{u} w$ and so $w^r = \hat{u} w^r \hat{u}^{-1}$. Since $|\hat{u}| < |u|$, by induction we get that $\hat{u} \in \langle w \rangle$. Therefore $u \in \langle w \rangle$. ☑

Now we make a brief introduction to HNN-extensions. Let G be a group with two isomorphic subgroups A and B. Let $\varphi : A \longrightarrow B$ be an isomorphism. The *HNN-extension* of G with respect to A, B and φ is the group

$$G^* = \big\langle G, t;\ t^{-1} a t = \varphi(a),\ a \in A \big\rangle.$$

Each element $w \in G^*$ can be written in the form $w = w_0 t^{\epsilon_1} w_1 \cdots t^{\epsilon_n} w_n$, where the w_i's are in G and $\epsilon_i = \pm 1$. This representation of w is called reduced if it does not contain the substring $t^{-1} w_i t$ with $w_i \in A$ or $t w_i t^{-1}$ with $w_i \in B$.

If we choose a system of right coset representatives for A and B in G, then we get for each element $w \in G^*$ a *normal form* $w = w_0 t^{\epsilon_1} \cdots t^{\epsilon_n} w_n$ $(n \geq 0)$ with the properties:

(a) w_0 is an element of G.
(b) If $\epsilon_i = -1$, then w_i is one of the representatives for the cosets of A in G.
(c) If $\epsilon_i = 1$, then w_i is one of the representatives for the cosets of B in G.
(d) $t^\epsilon 1 t^{-\epsilon}$ is not a substring.

$|w| = n$ denotes the length of w. $w = w_0 t^{\epsilon_1} \cdots t^{\epsilon_n} w_n$ $(n \geq 0)$ is called *cyclicly reduced* if every cyclic permutation of w is reduced.

Theorem 3.

(i) *Let G be a $(*)$-group, $x, z \in G \setminus \{1\}$ be atomic elements with different components. Then the HNN-extension $G^* = \langle G, t;\ t^{-1} x t = z \rangle$ is also a $(*)$-group and for every $u \in G \setminus \{1\}$, $C_{G^*}(u) = C_G(u)$.*
(ii) *If $y \in G$ is a G-component other than that of x or that of z, i.e. $C_y^G \neq C_x^G$ and $C_y^G \neq C_z^G$, then $C_y^{G^*} \neq C_x^{G^*}$ and $C_y^{G^*} \neq C_z^{G^*}$.*

Proof. We use the following conventions: $y_1 := x$ and $y_{-1} := z$. With this notation we have that $t^{-\delta} y_\delta^m t^\delta = y_{-\delta}^m$, where $\delta = \pm 1$ and $m \in \mathbb{Z}$. If $w \in G$, $\epsilon = \pm 1$ and $\delta = \pm 1$, then $t^\epsilon w t^\delta$ is reducible if and only if $\epsilon = -\delta$ and $w = y_\delta^m$ for some $m \in \mathbb{Z}$. We use the standard theory of HNN-extensions which can be found in, say, [L-Sch].

(i) That G^* is torsionfree follows from the general theory of HNN-extensions. We show that G^* has also the property $(*)$. We shall be done when we prove by induction on $|w|$ the following claim:

Claim A. *For every $w \in G^* \setminus \{1\}$, $C_{G^*}(w)$ is cyclic (equal to $\langle v_w \rangle$) and $C_{G^*}(w^r) = C_{G^*}(w)$ for every $r \in \mathbb{N}^*$.*

We consider two cases:

Case I. Suppose $w \in G \setminus \{1\}$. It is enough to proof that $N_{G^*}\big(\langle w \rangle\big) = N_G\big(\langle w \rangle\big)$, because

$$N_G\big(\langle w \rangle\big) = C_G(w) \subseteq C_{G^*}(w) \subseteq N_{G^*}\big(\langle w \rangle\big).$$

Let $u = u_0 t^{\delta_1} \cdots t^{\delta_n} u_n \in N_{G^*}\big(\langle w \rangle\big)$ be reduced in normal form. We want to show that $n = 0$. Suppose that $n \geq 1$. We can suppose that $u_n = 1$ and $w = y_{-\delta_n}^p$ for some $p \in \mathbb{Z}$ since $u w u^{-1} = w^r$ for $r \in \mathbb{Z}$, i.e.,

$$u_0 t^{\delta_1} \cdots t^{\delta_n} u_n w u_n^{-1} t^{-\delta_n} \cdots t^{-\delta_1} u_0^{-1} w^{-r} = 1$$

which implies that $t^{\delta_n} u_n w u_n t^{-\delta_n}$ is reducible, i.e., $w = u_n^{-1} y^p_{-\delta_n} u_n$ for some $p \in \mathbb{Z}$. Let $u' = u_n u u_n^{-1} = u_n u_0 t^{\delta_1} \cdots t^{\delta_n}$ and $w' = u_n w u_n^{-1} = y^p_{-\delta_n}$. Then $u' w {u'}^{-1} = w^r$, i.e., $u' \in N_{G^*}\big(\langle w' \rangle\big)$.

If $n = 1$, then $u_0 t^{\delta_1} y^p_{-\delta_1} t^{-\delta_1} u_o^{-1} = y^{pr}_{-\delta_1}$ i.e, $u_0 y^p_{\delta_1} u_o^{-1} = y^{pr}_{-\delta_1}$. This implies that

$$\Big(\bigcup_{g \in G} \langle z \rangle^g\Big) \cap \Big(\bigcup_{g \in G} \langle x \rangle^g\Big) \neq \langle 1 \rangle.$$

But this contradicts our assumption that x and z have different components.

If $n \geq 2$, then $t^{\delta_{n-1}} u_{n-1} y^p_{\delta_n} u_{n-1}^{-1} t^{-\delta_{n-1}}$ is reducible, i.e., $u_{n-1} y^p_{\delta_n} u_{n-1}^{-1} = y^q_{-\delta_{n-1}}$ for some $q \in \mathbb{Z}$. By Theorem 1 ((5) and (7)), we get that $\delta_n = -\delta_{n-1}$, $q = p$ and $u_{n-1} = y^s_{\delta_n}$, $s \in \mathbb{Z}$.

Then the word u would be reducible at $t^{\delta_{n-1}} u_{n-1} t^{\delta_n} = t^{-\delta_n} y^s_{\delta_n} t^{\delta_n}$ which is a contradiction. Whence $N_{G^*}\big(\langle w \rangle\big) = N_G\big(\langle w \rangle\big)$ and $C_{G^*}(w) = C_G(w)$.

Before we consider the case $|w| \geq 1$, we need two lemmas. The first one is a result of the theory of HNN-extensions.

Lemma 4. *Let $G^* = \big\langle G, t;\ t^{-1} A t = B \big\rangle$ be a HNN-extension of G. Let $v \in G^*$. Then, there are words a and b in G^* such that b is cyclicly reduced, $v = aba^{-1}$ and aba^{-1} is reduced.*

Proof. We prove this by induction on $|v|$. The cases $|v| = 0$ and $|v| = 1$ are trivial.

Suppose $|v| \geq 2$ and let $v = v_0 t^{\epsilon_1} v_1 \cdots t^{\epsilon_n} v_n$ be reduced. If v is cyclicly reduced we are done; so we can assume that the word $\alpha := t^{\epsilon_n} v_n v_1 t^{\epsilon_1}$ belongs to G and $v = v_0 t^{\epsilon_1} \tilde{v} \big(v_0 t^{\epsilon_1}\big)^{-1}$, where

$$\tilde{v} = \begin{cases} v_1 t^{\epsilon_2} v_2 \cdots t^{\epsilon_{n-1}} v_{n-1} \alpha, & \text{if } n \geq 3 \\ v_1 \alpha, & \text{if } n = 2. \end{cases}$$

In the second case take $a = v_0 t^{\epsilon_1}$, $b = v_1 \alpha$. In the first case we can apply induction to $\tilde{v} = v_1 t^{\epsilon_2} v_2 \cdots t^{\epsilon_{n-1}} v_{n-1} \alpha$ and then $\tilde{v} = \tilde{a} b \tilde{a}^{-1}$ is reduced for some b cyclicly reduced.

Take $a = v_0 t^{\epsilon_1} \tilde{a}$. Then, by a length argument, $v = aba^{-1}$ is reduced, . ☑

Remark. *Let v, a and b be as in previous lemma. It is clear that for every $r \in \mathbb{N}$, $v^r = ab^r a^{-1}$ and $|v^r| = r|b| + 2|a|$. Therefore*

$$|v^{r+1}| \geq |v^r| \quad \text{and} \quad |v^{r+1}| = |v^r| \text{ if and only if } |b| = 0.$$

Lemma 5. *Let $w \in G^*$ as in Theorem 3. Then there is a $v_w \in G^*$, indecomposable, such that $w = v_w^r$ for some $r \in \mathbb{N}$. If w is cyclicly reduced, so is v_w.*

Proof. First we prove the statement for w cyclicly reduced by induction on $|w|$. If $|w| = 0$, then $C_{G^*}(w) = C_G(w) = \langle v_w \rangle$, where v_w is an atomic element of G. Then $w = v_w^r$ for some $r \in \mathbb{N}$, and v_w is indecomposable, since it is indecomposable in G.

If $|w| \geq 1$ and w is decomposable, let $w = v^r$ ($r \in \mathbb{N}$, $r \geq 2$). Then v is cyclicly reduced; otherwise there are words a and b in G^* such that b is cyclicly reduced, $|a| > 1$, $v = aba^{-1}$ and aba^{-1} is reduced. But then $w = ab^r a^{-1}$ is not cyclicly reduced, a contradiction. Therefore $|w| = r|v|$ and $|v| < |w|$. The required statement follows by induction.

Now if $v \in G^*$ is arbitrary, v is conjugate to a cyclicly reduced element w: $v = uwu^{-1}$ with w cyclicly reduced. By the previous argument we find $v_w \in G^*$ indecomposable such that $w = v_w^r$ for some $r \in \mathbb{N}$. It follows that $v = (uv_w u^{-1})^r$ and one can easily verifies that $uv_w u^{-1}$ is also indecomposable. ☑

Now we continue the proof of Theorem 3.

Case II. $k := |w| \geq 1$. By Lemma 4, it is clear that we can assume, without lost of generality, that w is cyclicly reduced. We can also suppose that w is indecomposable; otherwise $w = v_w^r$, where $r \geq 2$ and v_w is cyclicly reduced and indecomposable. Since $|v_w| < |w|$, it follows by induction that $C_{G^*}(v_w)$ is cyclic and $C_{G^*}(v_w^q) = C_{G^*}(v_w)$ for every $q \in \mathbb{N}^*$. Then $C_{G^*}(w) = C_{G^*}(v_w^r) = C_{G^*}(v_w)$ is cyclic and for every $s \in \mathbb{N}^*$ we have

$$C_{G^*}(w^s) = C_{G^*}(v_w^{rs}) = C_{G^*}(v_w) = C_{G^*}(w).$$

Let $w = w_0 t^{\delta_1} w_1 \cdots t^{\delta_k}$ be cyclicly reduced and indecomposable. Let $u = u_0 t^{\epsilon_1} u_1 \cdots t^{\epsilon_n} u_n$ be a reduced element of $C_{G^*}(w^r)$ ($r \geq 1$). We show by induction on $|u| = n$ that $u \in \langle w \rangle$. First we prove that $u = 1$ or $n \geq k$. arguing for a contradiction, suppose that $u \neq 1$ and $n < k$. It follows by induction that $C_{G^*}(u)$ is cyclic. Let $C_{G^*}(u) = \langle v_u \rangle$. Then $u = v_u^q$ for some $q \in \mathbb{Z}^*$ and $|v_u| \leq |u|$. It follows that

$$C_{G^*}(v_u) = C_{G^*}(v_u^q) = C_{G^*}(u) = \langle v_u \rangle.$$

By hypothesis, $w^r \in C_{G^*}(u) = \langle v_u \rangle$. Then $w^r = v_u^{\pm s}$ ($s \geq 1$) and

$$w \in C_{G^*}(w^r) = C_{G^*}(v_u^{\pm s}) = C_{G^*}(u) = \langle v_u \rangle.$$

Then $w = v_u^{\pm p}$ ($p \geq 1$). Since w is indecomposable, $p = 1$ and $|w| = |v_u| \leq |u| < |w|$, a contradiction.

So we can suppose that $u \neq 1$ and $n \geq k$. Since $u \in C_{G^*}(w^r)$ we have that $uw^r u^{-1} = w^r$. More explicitly:

$$\begin{aligned} &u_0 t^{\epsilon_1} u_1 \cdots t^{\epsilon_{n-k}} u_{n-k} t^{\epsilon_{n-k+1}} \cdots t^{\epsilon_{n-1}} u_{n-1} t^{\epsilon_n} u_n w_0 t^{\delta_1} \cdots \\ &\cdots t^{\delta_k} \cdots w_0 t^{\delta_1} w_1 \cdots t^{\delta_k} u_n^{-1} t^{-\epsilon_n} \cdots t^{-\epsilon_1} u_0^{-1} = w^r. \end{aligned}$$

We prove that the word from the left either at $t^{\epsilon_n} u_n w_0 t^{\delta_1}$ or at $t^{\delta_k} u_n^{-1} t^{-\epsilon_n}$ is reducible, but not at both positions. Since w is cyclicly reduced, $|w^r| = r|w|$.

Then it is clear that the word on the left of the equality above is reducible and the only two positions where this is possible are the ones mentioned above. Suppose it is reducible at both positions. Then $t^{\epsilon_n}u_n w_0 t^{\delta_1}$ is reducible, i.e. $\epsilon_n = -\delta_1$ and $w_0 = u_n^{-1}y_{\delta_1}^p$ for some $p \in \mathbb{Z}^*$ and $t^{\delta_k}u_n^{-1}t^{-\epsilon_n}$ is also reducible, i.e., $\delta_k = \epsilon_n = -\delta_1$ and $u_n^{-1} = y_{\delta_1}^q$ for some $q \in \mathbb{Z}^*$.

If $k = 1$, then $\delta_k = -\delta_1$ is already a contradiction. If $k > 1$, then w would not be cyclicly reduced because

$$t^{\delta_k}w_0t^{\delta_1} = t^{-\delta_1}u_n^{-1}y_{\delta_1}^p t^{\delta_1} = t^{-\delta_1}y_{\delta_1}^{p+q}t^{\delta_1} = y_{-\delta_1}^{p+q}.$$

This is also a contradiction.

To finish the proof we show that in both cases we have $u \in \langle w\rangle$.

Subcase IIa: The word on the left of the equality above is reducible at $t^{\delta_k}u_n^{-1}t^{-\epsilon_n}$.

Let $i \le k-1$ maximal so that

$$t^{\delta_{k-i}}w_{k-i}\cdot \ldots \cdot t^{\delta_{k-1}}w_{k-1}t^{\delta_k}u_n^{-1}t^{-\epsilon_n}\cdot \ldots \cdot u_{n-i}^{-1}t^{-\epsilon_{n-i}} = y_{\epsilon_{n-i}}^p$$

for some $p \in \mathbb{Z}^*$. Let

$$\tilde{w} = t^{\delta_{k-i}}w_{k-i}\cdots t^{\delta_{k-1}}w_{k-1}t^{\delta_k}, \quad \bar{w} = w_0t^{\delta_1}t^{\delta_{k-i-1}}w_{k-i-1}$$

and

$$\bar{u} = u_0t^{\epsilon_1}u_1\cdots t^{\epsilon_{n-i-1}}u_{n-i-1}, \quad \tilde{u} = t^{\epsilon_{n-i}}u_{n-i}\cdots t^{\epsilon_n}u_n.$$

Then $\tilde{u} = y_{\epsilon_{n-i}}^{-p}\tilde{w}$. We show now that $i = k-1$, i.e., $\bar{w} = w_0$. Suppose, for a contradiction, that $|\bar{w}| \ge 1$. Then $\bar{w}y_{\epsilon_{n-1}}^p\bar{u}^{-1}$ is not reducible. But $w^r = uw^ru^{-1} = uw^{r-1}\bar{w}y_{\epsilon_{n-i}}^p\bar{u}^{-1}$ and $uw^{r-1}\bar{w}$ is also reduced. It follows that

$$r|w| = |u| + (r-1)|w| + |\bar{w}| + |\bar{u}|,$$

i.e., $|w| = |u| + |\bar{w}| + |\bar{u}| > |u|$, a contradiction. We then have that $w = w_0\tilde{w}$ and $u = \bar{u}y_{\epsilon_{n-i}}^{-p}w_0^{-1}w$. Let $\hat{u} = \bar{u}y_{\epsilon_{n-i}}^{-p}w_0^{-1}$. Then $u = \hat{u}w$ and

$$w^r = uw^ru^{-1} = \hat{u}ww^rw^{-1}\hat{u}^{-1} = \hat{u}w^r\hat{u}^{-1}.$$

Since $|\hat{u}| < |u|$, it follows by induction that $\hat{u} \in \langle w\rangle$. Therefore $u \in \langle w\rangle$.

Subcase IIb: The word on the left of the equality above is reducible at $t^{\epsilon_n}u_nw_0t^{\delta_1}$.

Let $j \le k-1$ maximal, so that

$$t^{\epsilon_{n-j}}u_{n-j}\cdot\cdots t^{\epsilon_{n-1}}u_{n-1}t^{\epsilon_n}u_nw_0t^{\delta_1}w_1t^{\delta_2}\cdot\cdots w_jt^{\delta_{j+1}} = y_{-\delta_{j+1}}^q$$

for some $q \in \mathbb{Z}^*$. Let

$$\bar{w} = w_0t^{\delta_1}\cdot\cdots w_jt^{\delta_{j+1}}, \quad \tilde{w} = w_{j+1}t^{\delta_{j+2}}\cdot\cdots t^{\delta_k}$$

and

$$\bar{u} = u_0 t^{\epsilon_1} u_1 \cdots t^{\epsilon_{n-j-1}} u_{n-j-1}, \quad \tilde{u} = t^{\epsilon_{n-j}} u_{n-j} \cdot \cdots \cdot t^{\epsilon_n} u_n.$$

Then $\tilde{u} = y^q_{-\delta_{j+1}} \bar{w}^{-1}$. We show that $j = k-1$, i.e., $\tilde{w} = 1$. Suppose, for a contradiction, that $|\tilde{w}| \geq 1$, i.e. $\bar{u} y^q_{-\delta_{j+1}} \tilde{w}$ is reduced. Then

$$w^r = uw^r u^{-1} = \bar{u}\tilde{u}\bar{w}\tilde{w}w^{r-1}u^{-1} = \bar{u} y^q_{-\delta_{j+1}} \tilde{w} w^{r-1} u^{-1}$$

and $\tilde{w}w^{r-1}u^{-1}$ is reduced. It follows that

$$r|w| = |\bar{u}| + |\tilde{w}| + (r-1)|w| + |u|,$$

i.e., $|w| = |\bar{u}| + |\tilde{w}| + |u| > |u|$, a contradiction. Then $\bar{w} = w$ and $\tilde{u} = y^q_{-\delta_{j+1}} w^{-1}$.

Let $\hat{u} = \bar{u} y^q_{-\delta_{j+1}}$. Then

$$w^r = uw^r u^{-1} = \hat{u} w^{-1} w^r w \hat{u}^{-1} = \hat{u} w^r \hat{u}^{-1}.$$

Since $|\hat{u}| < |u|$, it follows by induction that $\hat{u} \in \langle w \rangle$. Then $u = \hat{u} w^{-1} \in \langle w \rangle$.

(ii) Arguing for contradiction, suppose that $C^{G^*}_y = C^{G^*}_x$. We may assume, without lost of generality, that y is atomic. Then there is a reduced word $u = u_0 t^{\epsilon_1} u_1 \cdot \ldots \cdot t^{\epsilon_k} u_k$ $(k \geq 1)$ in G^*, so that $u^{-1}xu = y$.

If $k = 1$, then $y = u_1^{-1} t^{-\epsilon_1} u_0^{-1} x u_0 t^{\epsilon_1} u_1$ is reducible. So $u_0^{-1} x u_0$ is in $\langle x \rangle$ or in $\langle z \rangle$. But $C^{G^*}_x \neq C^{G^*}_z$. Then $u_0^{-1} x u_0 \in \langle x \rangle$. It follows that

$$u_0^{-1} x u_0 = x, \; u_0 = x^{r_0} \quad (r_0 \in \mathbb{Z}) \text{ and } \epsilon_1 = 1.$$

Then $u_1^{-1} z u_1 = y$, a contradiction. Then we can assume that $k \geq 2$. So $t^{-\epsilon_2} u_1^{-1} t^{-\epsilon_1} u_0^{-1} x u_0 t^{\epsilon_1} u_1 t^{\epsilon_2}$ is reducible. It follows that

$$u_0 = x^{r_0} \quad (r_0 \in \mathbb{Z}), \; \epsilon_1 = 1, \; u_1 = z^{r_1} \quad (r_1 \in \mathbb{Z}), \text{ and } \epsilon_2 = -1.$$

Then $t^{\epsilon_1} u_1 t^{\epsilon_2}$ would be reducible, a contradiction. ☑

3. Pseudo-bad groups

Now we use Theorem 3 in order to construct a pseudo-bad group.

Definition. *A Chain of $(*)$-groups $\left(G_i\right)_{i \in I}$ is called a $(*)$-chain, if for every $i, j \in I$, if $i \leq j$, then $G_i \subseteq G_j$, and for every $i \in I$, every $u \in G_i$ and every $j \geq i$, $C_{G_j}(u) = C_{G_i}(u)$.*

Theorem 6. *Let $\left(G_i\right)_{i\in I}$ be a $(*)$-chain. Then $G=\bigcup_{i\in I} G_i$ is a $(*)$-chain and for every $i\in I$ and every $u\in G_i$, $C_G(u)=C_{G_i}(u)$.*

Proof. Let $u\in G_i$. It is enough to prove that $C_G(u)\subseteq C_{G_i}(u)$. Let $v\in C_G(u)$, then $v\in G_j$ for some $j\geq i$. Whence $v\in C_{G_j}(u)=C_{G_i}(u)$. ☑

Corollary 7. *Every $(*)$-group G can be embedded in a $(*)$-group G', so that $G\setminus\{1\}$ is totally contained in one component of G' and for every $u\in G\setminus\{1\}$, $C_{G'}(u)=C_G(u)$.*

Proof. Let $\left(x_\alpha\right)_{\alpha<\beta}$ be an ordering of a representatives system for the components of G consisting only of atomic elements. Let

$$G'=\left\langle G,\left(t_\alpha\right)_{\alpha<\beta-\{0\}};\ t_\alpha^{-1}x_0t_\alpha=x_\alpha\right\rangle$$

$G'=\bigcup_{\alpha<\beta} G_\alpha$, where $G_0:=G$, $G_{\alpha+1}:=\left\langle G_\alpha,t_{\alpha+1};\ t_{\alpha+1}^{-1}x_0t_{\alpha+1}=x_{\alpha+1}\right\rangle$ and $G_\lambda=\bigcup_{\alpha<\lambda} G_\alpha$ for some λ a limit ordinal. From Theorem 3 and Theorem 6 it follows that $\left(G_\alpha\right)_{\alpha<\beta}$ is a $(*)$-chain and so the corollary follows. ☑

The next corollary follows likewise.

Corollary 8. *Every $(*)$-group G can be embedded in a $(*)$-group $\tilde{G}$ with only one component, i.e., $\tilde{G}=\bigcup_{g\in\tilde{G}} C_{\tilde{G}}(x)^g$ for some fixed x.*

Proof. Applying successively Corollary 7, we get a $(*)$-chain

$$G=G^{(0)}\leq G'\leq G^{(2)}\leq\cdots,$$

where for every n, $G^{(n)}\setminus\{1\}$ is contained in one component of $G^{(n+1)}$. $\tilde{G}=\bigcup_{n<\omega} G^{(n)}$ is a $(*)$-group with only one component. ☑

Let $G=F(x,z)$ be the free group over $\{x,z\}$. G is a $(*)$-group. Let $\tilde{G}$ be as in Corollary 8 and $B=\langle x\rangle$.

Theorem 10. *The group $\tilde{G}$ of Corollary 8 is a pseudo-bad group. Moreover, the maximal solvable subgroups of $\tilde{G}$ are B and its conjugates.*

Proof. By construction $\tilde{G}$ is a $(*)$-group with only one component. By Theorem 1 ((4), (9) and (2)), $N_G(B)=B$ and the maximal solvable subgroups of $\tilde{G}$ are B and its conjugates. Then, all we have to prove is that $\tilde{G}$ is simple. Let $N\neq\{1\}$ be a normal subgroup of $\tilde{G}$; N contains an $a\in B\setminus\{1\}$. Without lost of generality, we may assume that $a=x^r$ for some $r\in\mathbb{N}^*$, r minimal such that $x^r\in N$. It is clear that

$$N=\bigcup_{g\in\tilde{G}}\langle x^r\rangle^g.$$

Then $z^r \in N$ and $x^r z^r \in N$, i.e., $x^r z^r = \left(u^r\right)^n$ for some atomic element u and some $n \in \mathbb{Z}$. Then $u \in F(x,z)$. Since $x^r z^r$ is indecomposable, $r \cdot |n| = 1$ and so $r = 1$. We have then $N = \tilde{G}$. ☑

In fact, we have a family of pseudo-bad groups, one for each $(*)$-group G we start with. $\tilde{G}$ looks like a minimal simple bad group, but it is not of finite Morley rank since $\tilde{G}$ contains the definable subgroup $B \cong \mathbb{Z}$. We even have the following result.

Theorem 11. *Let G be a non trivial $(*)$-group (e.g., $G = F(x,z)$), and let $\tilde{G}$ be like in Corollary 8. Then $\tilde{G}$ is not superstable.*

Proof. We prove the following claim:

Claim. *For every $b_1, \cdots, b_n \in \tilde{G}$, there exists a $g \in \tilde{G}$, such that $b_1 g, \cdots, b_n g$ are not squares in $\tilde{G}$.*

$\tilde{G}$ is by construction a union of $(*)$-groups G_α's. Let α_0 be such that $b_1, \cdots, b_n \in G_{\alpha_0}$. Then $G_{\alpha_0+1} = \left\langle G_{\alpha_0}, t_{\alpha_0+1};\ t_{\alpha_0+1}^{-1} x_0 t_{\alpha_0+1} = x_{\alpha_0+1} \right\rangle$, and it is clear that $b_1 t_{\alpha_0+1}, \cdots, b_n t_{\alpha_0+1}$ are not squares in G_{α_0+1}, neither in $\tilde{G}$; otherwise $b_i t_{\alpha_0+1} = u^2$ for some $u \in \tilde{G} \setminus G_{\alpha_0+1}$. Then $u \in C_{\tilde{G}}(u^2) \subseteq G_{\alpha_0+1}$, a contradiction.

By the claim and a lemma from [Po 2], it follows that if $\tilde{G}$ is in fact stable; then if a is a generic element over $\tilde{G}$, a is not a square. So a^2 is not generic over $\tilde{G}$. But a is algebraic over a^2 because a is the only square root of a^2. Then $\tilde{G}$ is not superstable since for every super-stable group the following holds: if a generic element is algebraic over an element b, then b itself is generic. ☑

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