Revista Colombiana de Matemáticas
Volumen 36 (2002), páginas 29–48

Schauder bases in an abstract setting

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Abstract. By using some generalized Riemann integrals instead of ordinary sums and multiplication systems of Banach spaces instead of Banach spaces, we establish some natural generalizations of the most basic facts on Schauder bases so that Hamel bases, and some other important unconditional bases, could also be included.

Key words and phrases. Defining nets for integration, multiplication systems of Banach spaces, generalized Riemann integrals and Schauder bases.

2000 Mathematics Subject Classification. Primary: 26A42. Secondary: 46B15.

Introduction

By using some generalized Riemann integrals [11] instead of ordinary sums and multiplication systems of Banach spaces [14] instead of Banach spaces, we shall establish some natural generalizations of the following basic facts on Schauder bases [4], [8].

The research of the second author has been supported by the grants OTKA T-030082 and FKFP 0310/1997 .

Definition 1. Let Z be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then a sequence $\mu = (\mu_n)$ in Z is called a Schauder basis for Z if for each $z \in Z$ there exists a unique sequence $\hat{z} = (\hat{z}_n)$ in $\mathbb{K}$ such that

$$z = \sum_{n=1}^{\infty} \hat{z}_n \mu_n.$$

Remark 1. If f and μ are sequences in $\mathbb{K}$ and Z, respectively, then we define

$$S_n(f, \mu) = \sum_{i=1}^{n} f_i \mu_i$$

for all $n \in \mathbb{N}$. Thus, $\sum_{n=1}^{\infty} f_n \mu_n = \lim_{n \to \infty} S_n(f, \mu)$ whenever this limit exists.

Theorem 1. *If μ is a Schauder basis for Z, and moreover*

$$\mathcal{L}_\mu = \big\{ f \in \mathbb{K}^{\mathbb{N}} : \quad \big(S_n(f, \mu)\big) \ \text{converges} \big\},$$

and

$$\big| f \big|_\mu = \sup_{n \in \mathbb{N}} \big| S_n(f, \mu) \big|$$

for all $f \in \mathcal{L}_\mu$, then $\mathcal{L}_\mu$ is a linear space over $\mathbb{K}$ and $|\ |_\mu$ is a complete norm on $\mathcal{L}_\mu$ such that the mapping $z \mapsto \hat{z}$ is a continuous linear injection of Z onto $\mathcal{L}_\mu$ such that $|z| \le |\hat{z}|_\mu$ for all $z \in Z$.

Definition 2. If μ is a Schauder basis for Z, then the number

$$C_\mu = \sup_{|z|=1} \big| \hat{z} \big|_\mu$$

is called the basis constant of μ. Moreover, for each $n \in \mathbb{N}$, the function $P_{\mu n}$ defined by

$$P_{\mu n}(z) = S_n(\hat{z}, \mu)$$

for all $z \in Z$ is called the nth μ-projection of Z.

Theorem 2. *If μ is a Schauder basis for Z, then $P_{\mu n}$ is a continuous linear map of Z into itself for all $n \in \mathbb{N}$ such that*

$$C_\mu = \sup_{n \in \mathbb{N}} \big\| P_{\mu n} \big\|$$

and

$$P_{\mu n} = P_{\mu n} \circ P_{\mu m} = P_{\mu m} \circ P_{\mu n}$$

for all $n, m \in \mathbb{N}$ with $n \le m$.

Remark 2. Note that

$$\|P_{\mu n}\| = \sup_{|z|=1} |P_{\mu n}(z)|$$

for all $n \in \mathbb{N}$.

Theorem 3. *If μ is a sequence in Z, then μ is a Schauder basis for Z if and only if the following three conditions hold :*

(1) $\mu_n \neq 0$ for all $n \in \mathbb{N}$;

(2) the linear hull of $\{\mu_n\}_{n=1}^{\infty}$ is dense in Z;

(3) there exists a nonnegative number C such that

$$|S_n(f,\mu)| \leq C|S_m(f,\mu)|$$

for all $n, m \in \mathbb{N}$, with $n \leq m$, and for all $f \in \mathbb{K}^{\mathbb{N}}$.

In order to keep this paper as self-contained as possible, the necessary prerequisites concerning the generalized Riemann integrals of [11] will be briefly laid out in the subsequent preparatory sections. However, a familiarity with some basic facts on nets [5] will be assumed.

1. Integration systems

Definition 1.1. An ordered pair $(\Omega, \mathcal{S})$ consisting of a set Ω and a family $\mathcal{S}$ of subsets of Ω will now be called a pre-measurable space.

Remark 1.2. The family $\mathcal{S}$ may usually be assumed to be a semi-ring or a ring in Ω [1].

However, for a preliminary consideration, the reader may assume that $\mathcal{S}$ is the family of all finite subsets of Ω.

Definition 1.3. If $(\Omega, \mathcal{S})$ is a pre-measurable space, then a family

$$\mathfrak{N} = \big((\sigma_\alpha, \tau_\alpha)\big)_{\alpha \in \Gamma},$$

where Γ is a directed set, $\sigma_\alpha = (\sigma_{\alpha i})_{i \in I_\alpha}$ and $\tau_\alpha = (\tau_{\alpha i})_{i \in I_\alpha}$ are finite families in $\mathcal{S}$ and Ω, respectively, will be called a defining net for integration over $(\Omega, \mathcal{S})$.

Remark 1.4. To define powerful defining nets for integration, we must usually assume that Ω is equipped with a generalized uniformity which is compatible, in a certain sense, with the family $\mathcal{S}$ [12].

However, for a preliminary consideration, the reader may assume that $\mathfrak{N}$ is one of the most important particular cases of the following simple defining net for integration which will actually define summation.

Example 1.5. Let Ω be a set and $\mathcal{S}$ be the family of all finite subsets of Ω. Suppose that $(A_\alpha)_{\alpha\in\Gamma}$ is a net in $\mathcal{S}$ and, for each $\alpha\in\Gamma$ define

$$\sigma_\alpha = (\{i\})_{i\in A_\alpha} \qquad \text{and} \qquad \tau_\alpha = (i)_{i\in A_\alpha}.$$

Then $\mathfrak{N} = \big((\sigma_\alpha, \tau_\alpha)\big)_{\alpha\in\Gamma}$ is a defining net for integration over $(\Omega, \mathcal{S})$.

Remark 1.6. Note that Γ may, in particular, be $\mathcal{S}$ directed by set inclusion. And A_α may, in particular, be α for all $\alpha\in\Gamma$.

Moreover, if in particular $\Omega=\mathbb{N}$ $(\Omega=\mathbb{Z})$ and $\Gamma=\mathbb{N}$, then we may naturally take $A_\alpha=\{i\}_{i=1}^{\alpha}$ $\big(A_\alpha=\{i\}_{i=-\alpha}^{\alpha}\big)$ for all $\alpha\in\Gamma$.

Definition 1.7. An ordered triple (X, Y, Z) of Banach spaces over $\mathbb{K}$, together with a bilinear map $(x, y)\mapsto xy$ from $X\times Y$ into Z such that

$$|xy| \le |x||y|$$

for all $x\in X$ and $y\in Y$, will be called a multiplication system with respect to the above bilinear map.

Remark 1.8. Multiplication systems of Banach spaces play an important role in advanced calculus [6, pp. 135, 372 and 455].

However, for a preliminary consideration, the reader may assume (X, Y, Z) $= (\mathbb{K}, Z, Z)$ with the usual multiplication by scalars.

Definition 1.9. An ordered triple $\big((\Omega,\mathcal{S}), \mathfrak{N}, (X, Y, Z)\big)$, consisting of a pre-measurable space $(\Omega, \mathcal{S})$, a defining net for integration

$$\mathfrak{N} = \Big(\big((\sigma_{\alpha i})_{i\in I_\alpha}, (\tau_{\alpha i})_{i\in I_\alpha}\big)\Big)_{\alpha\in\Gamma}$$

over $(\Omega, \mathcal{S})$ and a multiplication system (X, Y, Z) of Banach spaces over $\mathbb{K}$, will be called an integration system.

Remark 1.10. The above notations will be kept fixed throughout in the sequel. They contain all the fixed data necessary for our subsequent integration process.

2. Net integrals

Definition 2.1. A function f from Ω into X will be called an integrand and the family of all integrands will be denoted by $\mathcal{F}(\Omega, X)$. Moreover, a function μ from $\mathcal{S}$ into Y will be called an integrator. And the family of all integrators will be denoted by $\mathcal{M}(\mathcal{S}, Y)$.

Remark 2.2. Note that the families $\mathcal{F}(\Omega, X)$ and $\mathcal{M}(\mathcal{S}, Y)$ are vector spaces over $\mathbb{K}$ under the usual pointwise operations.

Definition 2.3. If $f \in \mathcal{F}(\Omega, X)$ $\mu \in \mathcal{M}(\mathcal{S}, Y)$ and

$$S_\alpha(f,\mu) = \sum_{i \in I_\alpha} f(\tau_{\alpha i})\mu(\sigma_{\alpha i})$$

for all $\alpha \in \Gamma$, then the limit

$$\int_\Omega f d\mu = \lim_{\alpha \in \Gamma} S_\alpha(f,\mu),$$

whenever it exists, will be called the $\mathfrak{N}$-integral of f with respect to μ. Moreover, if the above integral exists then we shall say that f is $\mathfrak{N}$-integrable with respect to μ and the family of all such functions f will be denoted by $\mathcal{L}_\mu(\Omega, X)$.

Remark 2.4. Note that under the notations of Example 1.5, we simply have

$$\int_\Omega f d\mu = \lim_{\alpha \in \Gamma} \sum_{i \in A_\alpha} f(i)\mu(\{i\})$$

for all $f \in \mathcal{F}(\Omega, X)$ and $\mu \in \mathcal{M}(\mathcal{S}, Y)$ with $f \in \mathcal{L}_\mu$.

Theorem 2.5. *If $f, g \in \mathcal{F}(\Omega, X)$ and $\mu, \nu \in \mathcal{M}(\mathcal{S}, Y)$ such that $f, g \in \mathcal{L}_\mu$ and $f \in \mathcal{L}_\nu$, and moreover $\lambda \in \mathbb{K}$, then*

(1) $\displaystyle\int_\Omega (f+g) d\mu = \int_\Omega f d\mu + \int_\Omega g d\mu;$

(2) $\displaystyle\int_\Omega f d(\mu+\nu) = \int_\Omega f d\mu + \int_\Omega g d\nu;$

(3) $\displaystyle\int_\Omega (\lambda f) d\mu = \lambda \int_\Omega f d\mu = \int_\Omega f d(\lambda\mu).$

Sketch of the proof. Note that the approximating sums $S_\alpha(f,\mu)$ are bilinear functions of f and μ. Therefore, by the continuity of the linear operations in Z, the above assertions are also true.

From Theorem 2.5, we can immediately get the following corollary.

Corollary 2.6. *If $\mu \in \mathcal{M}(\mathcal{S}, Y)$, then $\mathcal{L}_\mu(\Omega, X)$ is a linear subspace of $\mathcal{F}(\Omega, X)$.*

Moreover, in addition to Theorem 2.5, we can also easily establish the following remark.

Remark 2.7. If $f \in \mathcal{F}(\Omega, \mathbb{K})$ and $\mu \in \mathcal{M}(\mathcal{S}, Y)$ such that $f \in \mathcal{L}_\mu$, then

(1) $\displaystyle\int_\Omega (fx) d\mu = x \int_\Omega f d\mu = \int_\Omega f d(x\mu), \quad \text{for all } x \in X.$

Moreover, if $f \in \mathcal{F}(\Omega, X)$ and $\mu \in \mathcal{M}(\mathcal{S}, \mathbb{K})$ such that $f \in \mathcal{L}_\mu$, then

(2) $\displaystyle\int_\Omega (fy) d\mu = \left(\int_\Omega f d\mu\right) y = \int_\Omega f d(\mu y), \quad \text{for all } y \in Y.$

3. The supremum μ-norm

Definition 3.1. If $f \in \mathcal{F}(\Omega, X)$ and $\mu \in \mathcal{M}(\mathcal{S}, Y)$, then the extended real number

$$\big|f\big|_\mu = \sup_{\alpha \in \Gamma} \big|S_\alpha(f, \mu)\big|$$

will be called the supremum μ-norm of f with respect to the net $\mathfrak{N}$.

Theorem 3.2. *The above μ-norm $|\ |_\mu$ is an extended valued seminorm on $\mathcal{F}(\Omega, X)$ such that*

$$\left|\int_\Omega f d\mu\right| \le \big|f\big|_\mu$$

for all $f \in \mathcal{L}_\mu(\Omega, X)$.

Proof. By the corresponding definitions, it is clear that

$$\big|\lambda f\big|_\mu = \sup_\alpha \big|S_\alpha(\lambda f, \mu)\big| = \sup_\alpha \big|\lambda\big|\big|S_\alpha(f, \mu)\big| \le \big|\lambda\big|\big|f\big|_\mu$$

for all $\lambda \in \mathbb{K}$ and $f \in \mathcal{F}(\Omega, X)$. Hence, by writing $1/\lambda$ in place of λ, and λf in place of f, we can see that the corresponding equality is also true. Moreover, we can also easily see that

$$\begin{aligned}\big|f+g\big|_\mu &= \sup_\alpha \big|S_\alpha(f+g, \mu)\big| \\ &\le \sup_\alpha \Big(\big|S_\alpha(f, \mu)\big| + \big|S_\alpha(g, \mu)\big|\Big) \le \big|f\big|_\mu + \big|g\big|_\mu\end{aligned}$$

for all $f, g \in \mathcal{F}(\Omega, X)$. Therefore, $|\ |_\mu$ is an extended valued seminorm.

On the other hand, it is clear that

$$\left|\int_\Omega f d\mu\right| = \big|\lim_\alpha S_\alpha(f, \mu)\big| = \lim_\alpha \big|S_\alpha(f, \mu)\big| \le \big|f\big|_\mu$$

for all $f \in \mathcal{L}_\mu(\Omega, X)$.

Remark 3.3. Note that if $\mu \in \mathcal{M}(\mathcal{S}, Y)$, then we also have

$$|fx|_\mu = |x||f|_\mu$$

for all $f \in \mathcal{F}(\Omega, \mathbb{K})$ and $x \in X$.

Moreover, it is also worth noticing that if $f \in \mathcal{F}(\Omega, X)$, then the function $|\ |_f$ defined by

$$|\mu|_f = |f|_\mu$$

for all $\mu \in \mathcal{M}(\mathcal{S}, Y)$ is also an extended valued seminorm.

From Theorem 3.2, we can get at once the following corollary.

Corollary 3.4. *The family*

$$\mathcal{F}_\mu(\Omega, X) = \{ f \in \mathcal{F}(\Omega, X) : \quad |f|_\mu < +\infty \}$$

forms a closed linear subspace of the space $\mathcal{F}(\Omega, X)$.

In addition to this corollary, we can also prove the following result.

Theorem 3.5. *The family* $\mathcal{L}_\mu(\Omega, X)$ *forms a closed linear subspace of the space* $\mathcal{F}(\Omega, X)$.

Proof. Note that if (f_n) is a sequence in $\mathcal{L}_\mu(\Omega, X)$ and $f \in \mathcal{F}(\Omega, X)$, then we have

$$\begin{aligned}
&\overline{\lim_{(\alpha,\beta)}} \big| S_\alpha(f,\mu) - S_\beta(f,\mu) \big| \\
&\qquad \le \overline{\lim_{(\alpha,\beta)}} \Big(\big| S_\alpha(f,\mu) - S_\alpha(f_n,\mu) \big| + \big| S_\alpha(f_n,\mu) - S_\beta(f_n,\mu) \big| \\
&\qquad\qquad + \big| S_\beta(f_n,\mu) - S_\beta(f,\mu) \big| \Big) \\
&\qquad \le \overline{\lim_{(\alpha,\beta)}} \big| S_\alpha(f,\mu) - S_\alpha(f_n,\mu) \big| + \overline{\lim_{(\alpha,\beta)}} \big| S_\alpha(f_n,\mu) - S_\beta(f_n,\mu) \big| \\
&\qquad\qquad + \overline{\lim_{(\alpha,\beta)}} \big| S_\beta(f_n,\mu) - S_\beta(f,\mu) \big| \\
&\qquad = 2 \overline{\lim_{\alpha}} \big| S_\alpha(f_n,\mu) - S_\alpha(f,\mu) \big| \\
&\qquad = 2 \overline{\lim_{\alpha}} \big| S_\alpha(f_n - f,\mu) \big| \le 2 \big| f_n - f \big|_\mu
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence, if $\lim_n |f_n - f|_\mu = 0$, it follows that

$$\lim_{(\alpha,\beta)} \big| S_\alpha(f,\mu) - S_\beta(f,\mu) \big| = 0.$$

Therefore, $\big(S_\alpha(f,\mu)\big)$ is a Cauchy net in Z. And thus, by the completeness of Z, we have $f \in \mathcal{L}_\mu(\Omega, X)$.

Now, combining Theorem 3.5 and Corollary 3.4, we can also state

Corollary 3.6. *The family*

$$\mathcal{L}^*_\mu(\Omega, X) = \mathcal{L}_\mu(\Omega, X) \cap \mathcal{F}_\mu(\Omega, X)$$

forms a closed linear subspace of the space $\mathcal{F}(\Omega, X)$.

Remark 3.7. Note that if in particular $\Gamma = \mathbb{N}$ with its natural order, then we simply have $\mathcal{L}^*_\mu(\Omega, X) = \mathcal{L}_\mu(\Omega, X)$.

4. Admissible integrators

Definition 4.1. An integrator $\mu \in \mathcal{M}(\mathcal{S}, Y)$ will be called $\mathfrak{N}$-admissible if there exists a nonnegative $c \in \mathcal{F}(\Omega, \mathbb{R})$ such that

$$\big|f(t)\big| \leq c(t)\big|f\big|_\mu$$

for all $t \in \Omega$ and $f \in \mathcal{F}(\Omega, X)$.

Remark 4.2. If in addition to the notations of Example 1.5 for each $t \in \Omega$ there exist $\alpha_t, \beta_t \in \Gamma$ such that $\{t\} = A_{\alpha_t} \setminus A_{\beta_t}$, and moreover $|xy| = |x||y|$ for all $x \in X$ and $y \in Y$, then each $\mu \in \mathcal{M}(\mathcal{S}, Y)$, with $\mu(\{t\}) \neq 0$ for all $t \in \Omega$, is $\mathfrak{N}$-admissible.

In this case, we have

$$\begin{aligned}
\big|f(t)\big| &= \big|\mu(\{t\})\big|^{-1}\big|f(t)\mu(\{t\})\big| \\
&= \big|\mu(\{t\})\big|^{-1}\Bigg|\sum_{i\in A_{\alpha_t}} f(i)\mu(\{i\}) - \sum_{i\in A_{\beta_t}} f(i)\mu(\{i\})\Bigg| \\
&= \big|\mu(\{t\})\big|^{-1}\big|S_{\alpha_t}(f,\mu) - S_{\beta_t}(f,\mu)\big| \\
&\leq \big|\mu(\{t\})\big|^{-1}\big(\big|S_{\alpha_t}(f,\mu)\big| + \big|S_{\beta_t}(f,\mu)\big|\big) 2\big|\mu(\{t\})\big|^{-1}\big|f\big|_\mu
\end{aligned}$$

for all $t \in \Omega$ and $f \in \mathcal{F}(\Omega, X)$.

The importance of admissible integrators is apparent from the following theorem.

Theorem 4.3. *If $\mu \in \mathcal{M}(\mathcal{S}, Y)$ is an $\mathfrak{N}$-admissible, then the μ-norm $|\ |_\mu$ is a complete extended valued norm on $\mathcal{F}(\Omega, X)$.*

Proof. If $f \in \mathcal{F}(\Omega, X)$ such that $|f|_\mu = 0$, then by the above definition we have

$$|f(t)| \leq c(t)|f|_\mu = 0$$

for all $t \in \Omega$, and hence $f = 0$. Therefore, by Theorem 3.2, $|\ |_\mu$ is an extended valued norm.

On the other, if (f_n) is a Cauchy sequence in $\mathcal{F}(\Omega, X)$, then for each $\varepsilon > 0$ there exists an n_o such that

$$|f_n - f_m|_\mu < \varepsilon \quad \text{for all} \quad n, m \geq n_o.$$

Hence, by Definition 4.1, it follows that

$$|f_n(t) - f_m(t)| \leq |(f_n - f_m)(t)| \leq c(t)|f_n - f_m|_\mu \leq c(t)\varepsilon$$

for all $t \in \Omega$ and $n, m \geq n_o$. Therefore, $(f_n(t))$ is a Cauchy sequence in X for all $t \in \Omega$. Thus, by the completeness of X, we may define a function $f \in \mathcal{F}(\Omega, X)$ such that

$$f(t) = \lim_n f_n(t)$$

for all $t \in \Omega$. Now, since

$$\begin{aligned}\lim_n S_\alpha(f_n, \mu) &= \lim_n \sum_{i \in I_\alpha} f_n(\tau_{\alpha i})\mu(\sigma_{\alpha i}) \\ &= \sum_{i \in I_\alpha} f(\tau_{\alpha i})\mu(\sigma_{\alpha i}) = S_\alpha(f, \mu),\end{aligned}$$

we can also state that

$$\big|S_\alpha(f_n - f, \mu)\big| = \lim_m \big|S_\alpha(f_n - f_m, \mu)\big| \leq \big|f_n - f_m\big|_\mu < \varepsilon$$

for all $\alpha \in \Gamma$ and $n \geq n_o$, and hence $|f_n - f|_\mu \leq \varepsilon$ for all $n \geq n_o$. Therefore,

$$\lim_n |f_n - f|_\mu = 0.$$

From Theorem 4.3, by Corollaries 3.6 and 3.4, we can get at once the following corollary.

Corollary 4.4. *If $\mu \in \mathcal{M}(\mathcal{S}, Y)$ is $\mathfrak{N}$-admissible, then $\mathcal{L}^*_\mu(\Omega, X)$ and $\mathcal{F}_\mu(\Omega, X)$ are Banach spaces.*

Remark 4.5. The supremum μ-norm $|\ |_\mu$ could throughout be replaced by the limit superior μ-norm

$$\big|f\big|^*_\mu = \overline{\lim_{\alpha \in \Gamma}} \big|S_\alpha(f, \mu)\big|.$$

However, since we have

$$\big|f\big|^*_\mu = \left|\int_\Omega f d\mu\right|$$

for all $f \in \mathcal{L}_\mu(\Omega, X)$, the limit superior μ-norm $|\ |^*_\mu$ cannot, in general, be an extended valued norm.

5. Generalized bases

Definition 5.1. An integrator $\mu \in \mathcal{M}(\mathcal{S}, Y)$ will be called an $\mathfrak{N}$-basis (resp. $\mathfrak{N}^*$-basis) for the multiplication system (X, Y, Z) if for each $z \in Z$ there exists a unique $\hat{z} \in \mathcal{L}_\mu(\Omega, X)$ $\big($resp. $\hat{z} \in \mathcal{L}^*_\mu(\Omega, X)\big)$ such that

$$z = \int_\Omega \hat{z} d\mu.$$

Remark 5.2. Note that if $\mathfrak{N}$ is as in Example 1.5 and (X,Y,Z) is as in Remark 1.8, then by Remark 2.4 the above definition already gives a substantial generalization of the notions of the Schauder and the Hamel bases of Z.

Simple applications of Definition 5.1 and Theorem 2.5 yield the following

Lemma 5.3. *An integrator* $\mu \in \mathcal{M}(\mathcal{S},Y)$ *is an* $\mathfrak{N}$*-basis* (*resp.* $\mathfrak{N}^*$*-basis*) *for* (X,Y,Z) *if and only if*

(1) *for each* $z \in Z$ *there exists* $f \in \mathcal{L}_\mu(\Omega,X)$ $\big($*resp.* $f \in \mathcal{L}^*_\mu(\Omega,X)\big)$ *such that* $z = \int_\Omega f d\mu$;

(2) $\int_\Omega f d\mu = 0$ *implies* $f = 0$ *for all* $f \in \mathcal{L}_\mu(\Omega,X)$ $\big($*resp.* $f \in \mathcal{L}^*_\mu(\Omega,X)\big)$.

Moreover, by using Theorems 2.5 and 3.2 we can also easily verify the following

Theorem 5.4. *If* μ *is an* $\mathfrak{N}$*-basis* (*resp.* $\mathfrak{N}^*$*-basis*) *for* (X,Y,Z) *then the mapping* $z \mapsto \hat{z}$ *is a linear injection of* Z *onto* $\mathcal{L}_\mu(\Omega,X)$ $\big($*resp.* $\mathcal{L}^*_\mu(\Omega,X)\big)$ *such that* $|z| \le |\hat{z}|_\mu$ *for all* $z \in Z$.

Sketch of the proof. To prove that $\hat{Z} = \mathcal{L}_\mu(\Omega,X)$, note that if $f \in \mathcal{L}_\mu(\Omega,X)$, then $z = \int_\Omega f d\mu$ is in Z. Therefore, we also have $z = \int_\Omega \hat{z} d\mu$. And hence, by the uniqueness property of $\hat{z}$, it follows that $f = \hat{z} \in \hat{Z}$.

Remark 5.5. In the sequel, an $\mathfrak{N}$-basis or $\mathfrak{N}^*$-basis μ will usually be said to have a property P if it has this property as an integrator.

Note that if μ is an admissible $\mathfrak{N}$-basis or $\mathfrak{N}^*$-basis for (X,Y,Z), then by Definition 4.1 we also have $|\hat{z}(t)| \le c(t)|\hat{z}|_\mu$ for all $z \in Z$ and $t \in \Omega$.

Moreover, in the latter particular case, we can also easily prove the next important

Theorem 5.6. *If* μ *is an admissible* $\mathfrak{N}^*$*-basis for* (X,Y,Z), *then there exists a nonnegative number* C *such that* $|\hat{z}|_\mu \le C|z|$ *for all* $z \in Z$.

Proof. In this case, by Corollary 4.4, $\mathcal{L}^*_\mu(\Omega,X)$ is also a Banach space. Moreover, by Theorem 5.4, the mapping $\hat{z} \mapsto z$ is a continuous linear injection of $\mathcal{L}^*_\mu(\Omega,X)$ onto Z. Therefore, by Banach's isomorphism theorem [3, p. 68], the inverse linear mapping $z \mapsto \hat{z}$ is also continuous. And thus, the assertion of the theorem is also true.

Remark 5.7. Note that if μ is as in Theorem 5.6, then by Remark 5.5 not only the 'Fourier-transform' $z \mapsto \hat{z}$, but also the 'coefficient functionals' $z \mapsto \hat{z}(t)$ are continuous.

Definition 5.8. If μ is a $\mathfrak{N}$-basis or an $\mathfrak{N}^*$-basis for (X,Y,Z), then the extended real number

$$C_\mu = \sup_{|z|=1} |\hat{z}|_\mu$$

will be called the basis constant of μ.

By Theorems 5.4 and 5.6, we evidently have the following

Theorem 5.9. *If μ is an admissible $\mathfrak{N}^*$-basis for (X, Y, Z), then $|\hat{z}|_\mu \leq C_\mu |z|$ for all $z \in Z$. Moreover $1 \leq C_\mu < +\infty$.*

Sketch of the proof. To prove that $1 \leq C_\mu$, note that $|z| \leq |\hat{z}|_\mu \leq C_\mu |z|$ for all $z \in Z$. Moreover, since $Z \neq \{0\}$, there exists a $z \in Z$ such that $|z| \neq 0$. Therefore, the required inequality is also true.

Definition 5.10. If μ is an $\mathfrak{N}$-basis or $\mathfrak{N}^*$-basis for (X, Y, Z), then for each $\alpha \in \Gamma$ the function $P_{\mu\alpha}$ defined by

$$P_{\mu\alpha}(z) = S_\alpha(\hat{z}, \mu)$$

for all $z \in Z$ will be called the αth μ-projection of Z.

Theorem 5.11. *If is an admissible $\mathfrak{N}^*$-basis for (X, Y, Z), then $P_{\mu\alpha}$ is a continuous linear map of Z into itself for all $\alpha \in \Gamma$ such that*

$$C_\mu = \sup_{\alpha \in \Gamma} \|P_{\mu\alpha}\|.$$

Sketch of the proof. To prove the latter equality, note that under the notation

$$\|P_{\mu\alpha}\| = \sup_{|z|=1} \big|P_{\mu\alpha}(z)\big|$$

we have

$$\begin{aligned} C_\mu &= \sup_{|z|=1} |\hat{z}|_\mu = \sup_{|z|=1} \sup_{\alpha \in \Gamma} \big|S_\alpha(\hat{z}, \mu)\big| \\ &= \sup_{|z|=1} \sup_{\alpha \in \Gamma} \big|P_{\mu\alpha}(z)\big| = \sup_{\alpha \in \Gamma} \sup_{|z|=1} \big|P_{\mu\alpha}(z)\big| = \sup_{\alpha \in \Gamma} \|P_{\mu\alpha}\|. \end{aligned}$$

Remark 5.12. Later we shall see that, under some natural conditions on μ and $\mathfrak{N}$, the μ-projections $P_{\mu\alpha}$ are also idempotent.

6. Regular integrators

Definition 6.1. An integrator $\mu \in \mathcal{M}(\mathcal{S}, Y)$ will be called finitely additive if

$$\mu\Big(\bigcup_{k \in K} A_k\Big) = \sum_{k \in K} \mu(A_k)$$

for any finite disjoint family $(A_k)_{k \in K}$ in $\mathcal{S}$ with $\bigcup_{k \in K} A_k \in \mathcal{S}$. And the family of all such integrators μ will be denoted by $\mathcal{M}_o(\mathcal{S}, Y)$.

Remark 6.2. Note that the family $\mathcal{M}_o(\mathcal{S}, Y)$ forms a linear subspace of $\mathcal{M}(\mathcal{S}, Y)$.

Definition 6.3. An integrator $\mu \in \mathcal{M}(\mathcal{S}, Y)$ will be called $\mathfrak{N}$-regular if

$$\mu(A) = \int_\Omega \chi_A \, d\mu$$

for all $A \in \mathcal{S}$. And the family of all such integrators μ will be denoted by $\mathcal{M}_{\mathfrak{N}}(\mathcal{S}, Y)$.

Remark 6.4. Note that by the corresponding definitions we have

$$\int_\Omega \chi_A \, d\mu = \lim_{\alpha \in \Gamma} \sum_{\tau_{\alpha i} \in A} \mu(\sigma_{\alpha i})$$

for all $A \in \mathcal{S}$ with $\chi_A \in \mathcal{L}_\mu$.

Simple applications of the above definitions and Theorem 2.5 give

Theorem 6.5. *The family $\mathcal{M}_{\mathfrak{N}}(\mathcal{S}, Y)$ forms a linear subspace of $\mathcal{M}_o(\mathcal{S}, Y)$.*

Sketch of the proof. Note that if $\mu \in \mathcal{M}_{\mathfrak{N}}(\mathcal{S}, Y)$ and $(A_k)_{k \in K}$ is as in Definition 6.1, then

$$\begin{aligned} \mu\Big(\bigcup_{k \in K} A_k\Big) &= \int_\Omega \chi_{\bigcup_{k \in K} A_k} \, d\mu \\ &= \int_\Omega \Big(\sum_{k \in K} \chi_{A_k}\Big) d\mu = \sum_{k \in K} \int_\Omega \chi_{A_k} \, d\mu = \sum_{k \in K} \mu(A_k). \end{aligned}$$

Therefore, $\mu \in \mathcal{M}_o(\mathcal{S}, Y)$ is also true.

Remark 6.6. In this respect, it is also worth mentioning that under the notations Example 1.5 the following assertions are equivalent:

(1) $\Omega = \underline{\lim}_{\alpha \in \Gamma} A_\alpha$; (2) $\mathcal{M}_{\mathfrak{N}}(\mathcal{S}, Y) = \mathcal{M}_o(\mathcal{S}, Y)$.

To prove the implication (2) $\Longrightarrow$ (1), note that if $y \in Y$ and $\mu(A) = \operatorname{card}(A) y$ for all $A \in \mathcal{S}$, then we have $\mu \in \mathcal{M}_o(\mathcal{S}, Y)$. Therefore, if the assertion (2) holds, then we also have $\mu \in \mathcal{M}_{\mathfrak{N}}(\mathcal{S}, Y)$. Hence, in particular, it follows that for each $t \in \Omega$ we have

$$y = \mu(\{t\}) = \int_\Omega \chi_{\{t\}} \, d\mu = \lim_\alpha \sum_{i \in A_\alpha} \chi_{\{t\}}(i) \mu(\{i\}) = \lim_\alpha \chi_{A_\alpha}(t) y.$$

Therefore, if $y \neq 0$, then there exists an $\alpha \in \Gamma$ such that for each $\beta \geq \alpha$ we have

$$\big|y - \chi_{A_\beta}(t) y\big| < \big|y\big|,$$

and hence $t \in A_\beta$. Consequently, $t \in \underline{\lim}_\alpha A_\alpha$, and thus the assertion (1) also holds.

The importance of regular integrators is apparent from the following theorem.

Theorem 6.7. *If* $\mu \in \mathcal{M}_{\mathfrak{N}}(\mathcal{S}, Y)$, *and moreover* $(A_k)_{k\in K}$ *and* $(x_k)_{k\in K}$ *are finite families in* $\mathcal{S}$ *and* X, *respectively, then*

$$\int_\Omega \left(\sum_{k\in K} \chi_{A_k} x_k \right) d\mu = \sum_{k\in K} x_k \mu(A_k).$$

Proof. By Theorem 2.5 and Remark 2.7, we evidently have

$$\begin{aligned}\sum_{k\in K} x_k \mu(A_k) &= \sum_{k\in K} x_k \int_\Omega \chi_{A_k} d\mu \\ &= \sum_{k\in K} \int_\Omega \chi_{A_k} x_k d\mu = \int_\Omega \left(\sum_{k\in K} \chi_{A_k} x_k \right) d\mu.\end{aligned}$$

Remark 6.8. To establish a certain converse to Theorem 6.7, note that if $\mu \in \mathcal{M}(\mathcal{S}, Y)$ such that $\int_\Omega \chi_A x d\mu = x\mu(A)$ for all $A \in \mathcal{S}$ and $x \in X$, and there exists a finite family $(x_k)_{k\in K}$ in X such that $|y| \leq \sum_{k\in K} |x_k y|$ for all $y \in Y$, then we can also state that μ is $\mathfrak{N}$-regular.

Definition 6.9. If $f \in \mathcal{F}(\Omega, X)$, then the function

$$f_\alpha = \sum_{i\in I_\alpha} \chi_{\sigma_{\alpha i}} f(\tau_{\alpha i})$$

will be called the αth $\mathfrak{N}$-trace of f.

Now, as an immediate consequence of Theorem 6.7, we can also state

Corollary 6.10. *If* $\mu \in \mathcal{M}_{\mathfrak{N}}(\mathcal{S}, Y)$, *then*

$$S_\alpha(f, \mu) = \int_\Omega f_\alpha d\mu$$

for all $\alpha \in \Gamma$ *and* $f \in \mathcal{F}(\Omega, X)$.

7. Normal integrators

Definition 7.1. An integrator $\mu \in \mathcal{M}(\mathcal{S}, Y)$ will be called $\mathcal{S}$-finite if

$$|\chi_A|_\mu < +\infty$$

for all $A \in \mathcal{S}$. And the family of all such integrators μ will be denoted by $\mathcal{M}^*(\mathcal{S}, Y)$.

Remark 7.2. Note that by the corresponding definitions we have

$$\big|\chi_{A}\big|_{\mu} = \sup_{\alpha\in\Gamma} \Bigg| \sum_{\tau_{\alpha i}\in A} \mu(\sigma_{\alpha i}) \Bigg|$$

for all $A \in \mathcal{S}$.

Theorem 7.3. *The family $\mathcal{M}^{*}(\mathcal{S}, Y)$ forms a linear subspace of $\mathcal{M}(\mathcal{S}, Y)$.*

Sketch of the proof. Recall that, by Remark 3.3, the function $|\ |_{A}$ defined by

$$|\mu|_{A} = |\mu|_{\chi_{A}}$$

for all $\mu \in \mathcal{M}(\mathcal{S}, Y)$ is an extended valued seminorm for every $A \in \mathcal{S}$.

Definition 7.4. An integrand $f \in \mathcal{F}(\Omega, X)$ will be called $\mathcal{S}$-simple if

$$f = \sum_{k\in K} \chi_{A_k} x_k$$

for some finite families $(A_k)_{k\in K}$ and $(x_k)_{k\in K}$ in $\mathcal{S}$ and X, respectively. And the family of all such integrands f will be denoted by $\mathcal{F}_{\mathcal{S}}(\Omega, X)$.

Remark 7.5. Note that the family $\mathcal{F}_{\mathcal{S}}(\Omega, X)$ is a linear subspace of $\mathcal{F}(\Omega, X)$.

The importance of $\mathcal{S}$-finite integrators is apparent from the following theorem.

Theorem 7.6. *If $\mu \in \mathcal{M}(\mathcal{S}, Y)$, then the following assertions are equivalent:*

(1) $\mu \in \mathcal{M}^{*}(\mathcal{S}, Y)$; *(2)* $\mathcal{F}_{\mathcal{S}}(\Omega, X) \subset \mathcal{F}_{\mu}(\Omega, X)$.

Sketch of the proof. Recall that, by Remark 3.3, we have

$$|\chi_{A} x|_{\mu} = |x||\chi_{A}|_{\mu}$$

for all $A \in \mathcal{S}$ and $x \in X$.

Therefore, if $(A_k)_{k\in K}$ and $(x_k)_{k\in K}$ are finite families in $\mathcal{S}$ and X, respectively, and the assertion (1) holds, then by Theorem 3.2 we also have

$$\Bigg|\sum_{k\in K} \chi_{A_k} x_k\Bigg|_{\mu} \leq \sum_{k\in K} \big|\chi_{A_k} x_k\big|_{\mu} = \sum_{k\in K} |x_k|\big|\chi_{A_k}\big|_{\mu} < +\infty.$$

Consequently, the function $\sum_{k\in K} \chi_{A_k} x_k$ is in $\mathcal{F}_{\mu}(\Omega, X)$, and thus the assertion (2) also holds.

Definition 7.7. An integrator $\mu \in \mathcal{M}(\mathcal{S}, Y)$ will be called $\mathfrak{N}$-normal if it is $\mathfrak{N}$-regular and $\mathcal{S}$-finite. And the family of all such integrators μ will be denoted by $\mathcal{M}^*_{\mathfrak{N}}(\mathcal{S}, Y)$.

Remark 7.8. Note that thus we have

$$\mathcal{M}^*_{\mathfrak{N}}(\mathcal{S}, Y) = \mathcal{M}_{\mathfrak{N}}(\mathcal{S}, Y) \cap \mathcal{M}^*(\mathcal{S}, Y).$$

Therefore, $\mathcal{M}^*_{\mathfrak{N}}(\mathcal{S}, Y)$ is also a linear subspace of $\mathcal{M}(\mathcal{S}, Y)$.

Now, as a useful consequence of Theorems 6.7 and 7.6, we can also state

Theorem 7.9. *If μ is a regular $\mathfrak{N}$-basis or a normal $\mathfrak{N}^*$-basis for (X, Y, Z), and moreover $(A_k)_{k\in K}$ and $(x_k)_{k\in K}$ are finite families in $\mathcal{S}$ and X, respectively, then*

$$\Big(\sum_{k\in K} x_k \mu(A_k)\Big)^{\wedge} = \sum_{k\in K} \chi_{A_k} x_k.$$

Proof. If $\mu \in \mathcal{M}_{\mathfrak{N}}(\mathcal{S}, Y)$, then by Theorem 6.7 we have

$$\sum_{k\in K} x_k \mu(A_k) = \int_\Omega \Big(\sum_{k\in K} \chi_{A_k} x_k\Big) d\mu.$$

While, if $\mu \in \mathcal{M}^*_{\mathfrak{N}}(\mathcal{S}, Y)$, then in addition to the above equality, by Theorem 7.6, we also have

$$\sum_{k\in K} \chi_{A_k} x_k \in \mathcal{L}^*_\mu(\Omega, X).$$

Therefore, by Definition 5.1, the required assertion is also true.

Corollary 7.10. *If μ is a regular $\mathfrak{N}$-basis or a normal $\mathfrak{N}^*$-basis for (X, Y, Z), then*

$$f_\alpha = S_\alpha(f, \mu)^{\wedge}$$

for all $\alpha \in \Gamma$ and $f \in \mathcal{F}(\Omega, X)$.

Proof. By the corresponding definitions and Theorem 7.9, we evidently have

$$f_\alpha = \sum_{i\in I_\alpha} \chi_{\sigma_{\alpha i}} f(\tau_{\alpha i}) = \Big(\sum_{i\in I_\alpha} f(\tau_{\alpha i})\mu(\sigma_{\alpha i})\Big)^{\wedge} = S_\alpha(f, \mu)^{\wedge}$$

for all $\alpha \in \Gamma$ and $f \in \mathcal{F}(\Omega, X)$.

8. Stable defining nets

Definition 8.1. The defining net $\mathfrak{N}$ will be called lower stable if for each $\alpha \in \Gamma$ and $i \in I_\alpha$ there exists a unique $j \in I_\alpha$ such that $\tau_{\alpha i} \in \sigma_{\alpha j}$, and for this j we have $\tau_{\alpha i} = \tau_{\alpha j}$.

Moreover, the defining net $\mathfrak{N}$ will be called upper stable if for each $\alpha \in \Gamma$ and $i \in I_\alpha$ there exists a unique $j \in I_\alpha$ such that $\tau_{\alpha j} \in \sigma_{\alpha i}$, and for this j we have $\sigma_{\alpha i} = \sigma_{\alpha j}$.

Remark 8.2. Note that if in particular $\tau_{\alpha i} \in \sigma_{\alpha i}$ for all $\alpha \in \Gamma$ and $i \in I_\alpha$, and the family $(\sigma_{\alpha i})_{i \in I_\alpha}$ is disjoint for all $\alpha \in \Gamma$, then the defining net $\mathfrak{N}$ is already both lower and upper stable.

Definition 8.3. The defining net $\mathfrak{N}$ will be called lower superstable if for each $\alpha, \beta \in \Gamma$, with $\alpha \leq \beta$, and for each $i \in I_\alpha$ there exists a unique $j \in I_\beta$ such that $\tau_{\alpha i} \in \sigma_{\beta j}$, and for this j we have $\tau_{\alpha i} = \tau_{\beta j}$.

Moreover, the defining net $\mathfrak{N}$ will be called upper superstable if for each $\alpha, \beta \in \Gamma$, with $\alpha \leq \beta$, and for each $i \in I_\alpha$ there exists a unique $j \in I_\beta$ such that $\tau_{\beta j} \in \sigma_{\alpha i}$, and for this j we have $\sigma_{\alpha i} = \sigma_{\beta j}$.

Remark 8.4. Note that the defining net $\mathfrak{N}$ given in Example 1.5 is lower or upper superstable if and only if the net $(A_\alpha)_{\alpha \in \Gamma}$ is nondecreasing.

The appropriateness of the above definitions is apparent from the following theorem.

Theorem 8.5. *If $\mu \in \mathcal{M}(\mathcal{S}, Y)$ and the defining net $\mathfrak{N}$ is lower or upper stable, then*

$$S_\alpha(f, \mu) = S_\alpha(f_\alpha, \mu)$$

for all $\alpha \in \Gamma$ and $f \in \mathcal{F}(\Omega, X)$.

Moreover, if the defining net $\mathfrak{N}$ is lower, resp. upper superstable, then

$$S_\alpha(f, \mu) = S_\alpha(f_\beta, \mu), \qquad \text{resp.} \qquad S_\alpha(f, \mu) = S_\beta(f_\alpha, \mu)$$

for all $\alpha, \beta \in \Gamma$, with $\alpha \leq \beta$, and for all $f \in \mathcal{F}(\Omega, X)$.

Proof. If $\alpha, \beta \in \Gamma$ are such that for each $i \in I_\alpha$ there exists a unique $j \in I_\beta$ such that $\tau_{\alpha i} \in \sigma_{\beta j}$, and for this j we have $\tau_{\alpha i} = \tau_{\beta j}$, then it is clear that

$$\begin{aligned} S_\alpha(f_\beta, \mu) &= \sum_{i \in I_\alpha} f_\beta(\tau_{\alpha i}) \mu(\sigma_{\alpha i}) \\ &= \sum_{i \in I_\alpha} \Big(\sum_{j \in I_\beta} \chi_{\sigma_{\beta j}}(\tau_{\alpha i}) f(\tau_{\beta j}) \Big) \mu(\sigma_{\alpha i}) \\ &= \sum_{i \in I_\alpha} \Big(\sum_{\tau_{\alpha i} \in \sigma_{\beta j}} f(\tau_{\beta j}) \Big) \mu(\sigma_{\alpha i}) \\ &= \sum_{i \in I_\alpha} f(\tau_{\alpha i}) \mu(\sigma_{\alpha i}) = S_\alpha(f, \mu) \end{aligned}$$

for all $f \in \mathcal{F}(\Omega, X)$.

While if $\alpha, \beta \in \Gamma$ such that for each $i \in I_\alpha$ there exists a unique $j \in I_\beta$ such that $\tau_{\beta j} \in \sigma_{\alpha i}$, and for this j we have $\sigma_{\alpha i} = \sigma_{\beta j}$, then it is clear that

$$\begin{aligned} S_\beta(f_\alpha, \mu) &= \sum_{j \in I_\beta} f_\alpha(\tau_{\beta j})\mu(\sigma_{\beta j}) \\ &= \sum_{j \in I_\beta} \Big(\sum_{i \in I_\alpha} \chi_{\sigma_{\alpha i}}(\tau_{\beta j}) f(\tau_{\alpha i}) \Big) \mu(\sigma_{\beta j}) \\ &= \sum_{i \in I_\alpha} f(\tau_{\alpha i}) \Big(\sum_{j \in I_\beta} \chi_{\sigma_{\alpha i}}(\tau_{\beta j}) \mu(\sigma_{\beta j}) \Big) \\ &= \sum_{i \in I_\alpha} f(\tau_{\alpha i}) \Big(\sum_{\tau_{\beta j} \in \sigma_{\alpha i}} \mu(\sigma_{\beta j}) \Big) \\ &= \sum_{i \in I_\alpha} f(\tau_{\alpha i}) \mu(\sigma_{\alpha i}) = S_\alpha(f, \mu) \end{aligned}$$

for all $f \in \mathcal{F}(\Omega, X)$.

Corollary 8.6. *If μ is a regular $\mathfrak{N}$-basis or a normal $\mathfrak{N}^*$-basis for (X, Y, Z) and the defining net $\mathfrak{N}$ is lower or upper stable, then*

$$S_\alpha(f, \mu) = P_{\mu\alpha}\big(S_\alpha(f, \mu)\big)$$

for all $\alpha \in \Gamma$ and $f \in \mathcal{F}(\Omega, X)$. Moreover, if the defining net $\mathfrak{N}$ is lower, resp. upper superstable, then

$$S_\alpha(f, \mu) = P_{\mu\alpha}\big(S_\beta(f, \mu)\big), \qquad \text{resp.} \qquad S_\alpha(f, \mu) = P_{\mu\beta}\big(S_\alpha(f, \mu)\big)$$

for all $\alpha \in \Gamma$, with $\alpha \leq \beta$, and for all $f \in \mathcal{F}(\Omega, X)$.

Sketch of the proof. If the defining net $\mathfrak{N}$ is, for instance, lower superstable, by Theorem 8.5 and Corollary 7.10, we have

$$S_\alpha(f, \mu) = S_\alpha(f_\beta, \mu) = S_\alpha\big(S_\beta(f, \mu)^\wedge, \mu\big) = P_{\mu\alpha}\big(S_\beta(f, \mu)\big)$$

for all $\alpha \in \Gamma$, with $\alpha \leq \beta$, and for all $f \in \mathcal{F}(\Omega, X)$.

Corollary 8.7. *If μ is a regular $\mathfrak{N}$-basis or a normal $\mathfrak{N}^*$-basis for (X, Y, Z) and the defining net $\mathfrak{N}$ is lower or upper stable, then*

$$P_{\mu\alpha} = P_{\mu\alpha} \circ P_{\mu\alpha}$$

for all $\alpha \in \Gamma$. Moreover, if the defining net $\mathfrak{N}$ is lower, resp. upper superstable, then

$$P_{\mu\alpha} = P_{\mu\alpha} \circ P_{\mu\beta}, \qquad \text{resp.} \qquad P_{\mu\alpha} = P_{\mu\beta} \circ P_{\mu\alpha}$$

for all $\alpha \in \Gamma$ *with* $\alpha \leq \beta$.

Sketch of the proof. If the defining net $\mathfrak{N}$ is, for instance, lower superstable, by the corresponding definitions and Corollary 8.6, we have

$$P_{\mu\alpha}(z) = \S_{\alpha}(\hat{z}, \mu) = P_{\mu\alpha}\big(S_{\beta}(\hat{z}, \mu)\big) = P_{\mu\alpha}\big(P_{\mu\beta}(z)\big)$$

for all $\alpha \in \Gamma$, with $\alpha \leq \beta$, and for all $z \in Z$.

Remark 8.8. Note that if $\mu \in \mathcal{M}(\mathcal{S}, Y)$ and the defining net $\mathfrak{N}$ is upper superstable, then by Theorem 8.5 we also have

$$S_{\alpha}(f, \mu) = \int_{\Omega} f_{\alpha} d\mu$$

for all $\alpha \in \Gamma$ and $f \in \mathcal{F}(\Omega, X)$.

9. Characterization of admissible normal $\mathfrak{N}^*$-bases

The importance of superstable defining nets is apparent from the following two theorems which give a natural generalization of Theorem 3.

Theorem 9.1. *If* μ *is an admissible normal* $\mathfrak{N}^*$*-basis for* (X, Y, Z) *and the defining net* $\mathfrak{N}$ *is lower superstable, then the following two assertions hold:*

(1) *The set* $\big\{S_{\alpha}(f, \mu) : \ \alpha \in \Gamma, \ f \in \mathcal{L}^*_{\mu}(\Omega, X)\big\}$ *is dense in* Z;
(2) $\big|S_{\alpha}(f, \mu)\big| \leq C_{\mu}\big|S_{\beta}(f, \mu)\big|$ *for all* $\alpha \in \Gamma$, *with* $\alpha \leq \beta$, *and for all* $f \in \mathcal{F}(\Omega, X)$.

Proof. By Definitions 5.1 and 2.3, it is clear that the assertion (1) holds. Moreover, by using Corollary 8.6 and Theorem 5.11, we can easily see that

$$\big|S_{\alpha}(f, \mu)\big| = \big|P_{\mu\alpha}\big(S_{\beta}(f, \mu)\big)\big| \leq \big\|P_{\mu\alpha}\big\|\big|S_{\beta}(f, \mu)\big| \leq C_{\mu}\big|S_{\beta}(f, \mu)\big|$$

for all $\alpha \in \Gamma$, with $\alpha \leq \beta$, and for all $f \in \mathcal{F}(\Omega, X)$.

Remark 9.2. By the above theorem, an admissible normal $\mathfrak{N}^*$-basis μ for (X, Y, Z) may be called monotone if $C_{\mu} = 1$.

Theorem 9.3. *If* $\mu \in \mathcal{M}^*_{\mathfrak{N}}(\mathcal{S}, Y)$ *is* $\mathfrak{N}$*-admissible and the defining net* $\mathfrak{N}$ *is upper superstable, then* μ *is an admissible normal* $\mathfrak{N}^*$*-basis for* (X, Y, Z) *if the following two conditions hold:*

(1) *The set* $\big\{S_{\alpha}(f, \mu) : \ \alpha \in \Gamma, \ f \in \mathcal{F}(\Omega, X)\big\}$ *is dense in* Z;
(2) *There exists a nonnegative number* C *such that*

$$\big|S_{\alpha}(f, \mu)\big| \leq C\big|S_{\beta}(f, \mu)\big|$$

for all $\alpha \in \Gamma$, *with* $\alpha \leq \beta$, *and for all* $f \in \mathcal{L}^*_{\mu}(\Omega, X)$.

Proof. If $z \in Z$, then by condition (1) there exist sequences (α_n) and (f_n) in Γ and $\mathcal{F}(\Omega, X)$ respectively, such that

$$z = \lim_n S_{\alpha_n}(f_n, \mu).$$

Since the integrator μ is $\mathfrak{N}$-normal, by theorems 6.7 and 7.6, we have

$$f_{n\alpha_n} = (f_n)_{\alpha_n} \in \mathcal{L}^*_\mu(\Omega, X)$$

for all $n \in \mathbb{N}$. Moreover, if $m, n \in \mathbb{N}$, then by condition (2) and Theorem 8.5, it is clear that

$$\begin{aligned} \left|S_\alpha(f_{m\alpha_m} - f_{n\alpha_n},\ \mu)\right| &\le C\left|S_\beta(f_{m\alpha_m} - f_{n\alpha_n},\ \mu)\right| \\ &= C\left|S_\beta(f_{m\alpha_m}, \mu) - S_\beta(f_{n\alpha_n}, \mu)\right| \\ &= C\left|S_{\alpha_m}(f_m, \mu) - S_{\alpha_n}(f_n, \mu)\right| \end{aligned}$$

for all $\alpha, \beta \in \Gamma$ with $\alpha_m \le \beta$ and $\alpha_n \le \beta$. Hence, it follows that

$$\left|f_{m\alpha_m} - f_{n\alpha_n}\right|_\mu \le C\left|S_{\alpha_m}(f_m, \mu) - S_{\alpha_n}(f_n, \mu)\right|.$$

Therefore, $(f_{n\alpha_n})$ is a a Cauchy sequence in $\mathcal{L}^*_\mu(\mathcal{S}, Y)$. Thus, by Corollary 4.4, there exists an $f \in \mathcal{L}^*_\mu(\mathcal{S}, Y)$ such that

$$\lim_n \left|f_{n\alpha_n} - f\right|_\mu = 0.$$

Hence, by Corollary 6.10 and Theorem 3.2, it is clear that

$$z = \lim_n S_{\alpha_n}(f_n, \mu) = \lim_n \int_\Omega f_{n\alpha_n} d\mu = \int_\Omega f d\mu.$$

Now, by Lemma 5.3, it remains to show only that if $h \in \mathcal{L}^*_\mu(\Omega, X)$ is such that $\int_\Omega h d\mu = 0$, then $h = 0$. For this, note that by condition (2) we have

$$\left|S_\alpha(h, \mu)\right| \le C\left|S_\beta(h, \mu)\right|$$

for all $\alpha \in \Gamma$, with $\alpha \le \beta$. Therefore,

$$\left|S_\alpha(h, \mu)\right| \le C \lim_\beta \left|S_\beta(h, \mu)\right| = \left|\int_\Omega h d\mu\right| = 0,$$

and hence $S_\alpha(h, \mu) = 0$ for all $\alpha \in \Gamma$. Thus, in particular, we have $|h|_\mu = 0$. Hence, since the integrator μ is now $\mathfrak{N}$-admissible, it is clear that $h = 0$.

Now, as an immediate consequence of Theorems 9.1 and 9.3, we can also state

Corollary 9.4. *If $\mu \in \mathcal{M}^*_{\mathfrak{N}}(\mathcal{S}, Y)$ is $\mathfrak{N}$-admissible and the defining net $\mathfrak{N}$ is both lower and upper superstable, then the conditions (1) and (2) of Theorem 9.3 imply the assertions (1) and (2) of Theorem 9.1.*

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(Recibido en julio de 2001)

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