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On a class of variational-hemivariational inequalities

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Abstract. In this paper we consider a class of variational-hemivariational inequalities. We use the critical point theory for nonsmooth functionals due to Motreanu-Panagiotopoulos [9]. We derive nontrivial solutions using the mountain-pass theorem.

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1. Introduction

Our starting point is the paper of Motreanu-Panagiotopoulos [8] for hemivariational inequalities. Namely, the authors there want to answer the following question:

Find $u \in X$ and $\lambda \in \mathbb{R}$ satisfying the inequality

$$a(u,v) + \int_Z j^o(u,v)dx \geq \lambda(u,v) \text{ for all } v \in X$$

where $j : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function and $a(\cdot,\cdot)$ a continuous symmetric bilinear form.

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Our goal here is to have some existence results for such problems with the solution being at a closed, convex subset K of $W^{1,p}(Z)$ and in our case the differential operator is the p-Laplacian. Moreover, we seek for nontrivial solutions and for that purpose we use the mountain-pass theorem.

The problem under consideration is the following:

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^1-boundary Γ. Find $x \in W^{1,p}(Z)$ such that

$$\int_Z (\|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{R^N} dz + \int_Z F^o(z, x(z); y(z))dz \geq 0 \qquad (1)$$

for all $y \in K$. Here $K = \{x \in W^{1,p}(Z) : x(z) \geq 0\}$. Clearly, K is closed and convex on $W^{1,p}(Z)$ and finally $F : Z \times \mathbb{R} \to \mathbb{R}$ is the potential of some $f : Z \times \mathbb{R} \to \mathbb{R}$.

2. **Preliminaries**

Let X be a real Banach space and Y be a subset of X. A function $f : Y \to \mathbb{R}$ is said to satisfy a Lipschitz condition (on Y) provided that, for some nonnegative scalar K, one has

$$|f(y) - f(x)| \leq K\|y - x\|$$

for all points $x, y \in Y$. Let f be Lipschitz near a given point x, and let v be any other vector in X. The generalized directional derivative of f at x in the direction v, denoted by $f^o(x; v)$ is defined as follows:

$$f^o(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}$$

where y is a vector in X and t a positive scalar. If f is Lipschitz of rank K near x then the function $v \to f^o(x; v)$ is finite, positively homogeneous, subadditive and satisfies $|f^o(x; v)| \leq K\|v\|$. In addition f^o satisfies $f^o(x; -v) = (-f)^o(x; v)$. Now we are ready to introduce the generalized gradient which denoted by $\partial f(x)$ as follows:

$$\partial f(x) = \{w \in X^* : f^o(x; v) \geq \langle w, v\rangle \text{ for all } v \in X\}$$

Some basic properties of the generalized gradient of locally Lipschitz functionals are the following:

(a) $\partial f(x)$ is a nonempty, convex, weakly compact subset of X^* and $\|w\|_* \leq K$ for every w in $\partial f(x)$.

(b) For every v in X, one has

$$f^o(x; v) = \max\{\langle w, v\rangle : w \in d\partial f(x)\}.$$

If f_1, f_2 are locally Lipschitz functions then

$$\partial(f_1 + f_2) \subseteq \partial f_1 + \partial f_2.$$

Moreover, $(x, v) \to f^o(x; v)$ is upper semicontinuous and as function of v alone, is Lipschitz of rank K on X.

Let us mention the mean-value theorem of Lebourg.

Theorem 1 (Lebourg)**.** *Let x and y be points in X, and suppose that f is Lipschitz on an open set containing the line segment $[x, y]$. Then there exists a point $u \in (x, y)$ such that*

$$f(y) - f(x) \in \langle \partial f(u), y - x \rangle. \tag{2}$$

Let $R : X \to \mathbb{R} \cup \{\infty\}$ be such that $R = \Phi + \psi$ where $\Phi : X \to \mathbb{R}$ be a locally Lipschitz functional while $\psi : X \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous, convex but not defined everywhere functional.

A point x in X is said to be a critical point of R if $x \in D(\psi)$ and if it satisfies the inequality

$$\Phi^o(x; y - x) + \psi(y) - \psi(x) \geq 0 \text{ for every } y \in X. \tag{3}$$

Definition 1. We say that $R : X \to \mathbb{R} \cup \{\infty\}$ with $R = \Phi + \psi$ satisfies H_1 is Φ is locally Lipschitz and ψ proper, convex and lower semicontinuous.

Let us now state the formulation of our (PS) condition.

(PS) If $\{x_n\}$ is a sequence such that $R(x_n) \to c$ and

$$\Phi^o(x_n; y - x_n) + \psi(y) - \psi(x_n) \geq -\varepsilon_n \|y - x_n\| \text{ for every } y \in X. \tag{4}$$

where $\varepsilon_n \to 0$, then $\{x_n\}$ has a convergent subsequence.

The following theorem is a mountain-pass theorem for functionals which satisfies condition H_1 and (PS) (see Motreanu-Panagiotopoulos [9], Cor. 3.2).

Theorem 2. *If $f : X \to \mathbb{R}$ satisfies H_1 and (PS) on the reflexive Banach space X and the hypotheses*

(i) there exist positive constants ρ and a such that

$$f(u) \geq a \text{ for all } x \in X \text{ with } \|x\| = \rho;$$

(ii) $f(0) = 0$ and there a point $e \in X$ such that

$$\|e\| > \rho \text{ and } f(e) \leq 0,$$

then there exists a critical value $c \geq a$ of f determined by

$$c = \inf_{g \in G} \max_{t \in [0,1]} f(g(t))$$

where

$$G = \{g \in C([0, 1], X) : g(0) = 0, g(1) = e\}$$

In what follows we will use the well-known inequality

$$\sum_{j=1}^{N}(a_j(\eta)-a_j(\eta^{'}))(\eta_j-\eta_j^{'})\geq C|\eta-\eta^{'}|^p, \tag{5}$$

for $\eta,\eta^{'}\in R^N$, with $a_j(\eta)=|\eta|^{p-2}\eta_j$.

3. **Hemivariational inequalities with constraints**

Let $f:Z\times\mathbb{R}\to\mathbb{R}$. Then we introduce the following functions:

$$f_1(z,x)=\liminf_{x^{'}\to x} f(z,x^{'}),\ \ f_2(z,x)=\limsup_{x^{'}\to x} f(z,x^{'}).$$

In this section we state and prove an existence result for a variational-hemivariational inequality. So our hypotheses on the data are:

$\mathbf{H(f)}$: $f_1,f_2:Z\times\mathbb{R}\to\mathbb{R}$ is N-measurable (i.e. if $x(z)$ is measurable then so is $f_{1,2}(z,x(z))$);

(i) for almost all $z\in Z$ and all $x\in\mathbb{R}$, $|f(z,x)|\leq a(z)+c|x|^{\theta-1}$ with $a\in L^\infty(Z), c>0, 1\leq\theta<p$;

(ii) uniformly for almost all $z\in Z$ we have that $\frac{f_{1,2}(z,x)}{|x|^{\theta-2}x}\to f_+(z)$ as $x\to\infty$ where $f_+\in L^1(Z), f_+\geq 0$ with strict inequality on a set of positive Lebesgue measure.

(iii) Uniformly for almost all $z\in Z$ we have that

$$\limsup_{x\to 0}\frac{pF(z,x)}{|x|^p}\leq h(z),$$

with $h\in L^\infty(Z)$ and $h(z)\leq 0$ with strict inequality on a set of positive measure. Here, by $F(z,x)$ we denote the integral of f, that is $F(z,x)=\int_o^x f(z,r)dr$.

Theorem 3. *If **H(f)** holds then problem (1) has a nontrivial solution $x\in K$.*

Proof. Let $\Phi:W^{1,p}(Z)\to\mathbb{R}$ and $\psi:W^{1,p}(Z)\to\mathbb{R}\cup\{\infty\}$ be defined such that

$$\Phi(x)=-\int_Z F(z,x(z))dz \text{ and } \psi(x)=\frac{1}{p}\|Dx\|_p^p+I_K(x).$$

In the definition of $\Phi(\cdot)$, $F(z,x)=\int_o^x f(z,r)dr$ and I_K is the indicator function of $K=\{x\in W^{1,p}(Z):x(z)\geq 0 \text{ a.e. on } Z\}$. It is easy to see that K is closed, convex and thus I_K is convex and lower semicontinuous.

Set $R=\Phi+\psi$. Recall that Φ is locally Lipschitz and ψ is lower semicontinuous, proper and convex.

Claim 1 $R(\cdot)$ satisfies the (PS)-condition.

Let $\{x_n\}_{n\geq 1}\subseteq W^{1,p}(Z)$ such that $R(x_n)\to c$ when $n\to\infty$ and

$$\Phi^o(x_n;x-x_n)+\psi(x)-\psi(x_n)\geq -\varepsilon_n\|x-x_n\|$$

with $\varepsilon_n \to 0$. Note that $\{x_n\} \in K$ because $|R(x_n)| \leq M$. In the above inequality choose $x = x_n + \delta x_n$ and then divide with δ. Also,

$$\frac{1}{p}\|Dx_n\|_p^p - \frac{1}{p}\|Dx_n + \delta Dx_n\| = \frac{1}{p}\|Dx_n\|_p^p(1 - (1+\delta)^p).$$

So if we divide this with δ and let $\delta \to 0$ we have that is equal with $-\|Dx_n\|_p^p$. Finally there exists $v_n(z) \in [-f_1(z, x_n(z)), -f_2(z, x_n(z))]$ such that $\langle v_n, x_n \rangle = \Phi^o(x_n; x_n)$. So, it follows that

$$\int_Z -v_n x_n(z)dz - \|Dx_n\|_p^p \geq -\varepsilon_n \|x_n\|.$$

Suppose that $\{x_n\} \subseteq W^{1,p}(Z)$ was unbounded. Then (at least for a subsequence), we may assume that $\|x_n\| \to \infty$. Let $y_n = \frac{x_n}{\|x_n\|}, n \geq 1$. By passing to a subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(Z), y_n \to y \text{ in } L^p(Z), y_n(z) \to y(z) \text{ a.e. on} Z \text{ as } n \to \infty$$

and $|y_n(z)| \leq k(z)$ a.e. on Z with $k \in L^p(Z)$.

Recall that from the choice of the sequence $\{x_n\}$ we have $|R(x_n)| \leq M_1$ for some $M_1 > 0$ and all $n \geq 1$, thus

$$\frac{1}{p}\|Dx_n\|_p^p - \int_Z F(z, x_n(z))dz \leq M_1,$$

(since $I_K \geq 0$). Dividing by $\|x_n\|^p$ we obtain

$$\frac{1}{p}\|Dy_n\|_p^p - \int_Z \frac{F(z, x_n(z))}{\|x_n\|^p}dz \leq \frac{M_1}{\|x_n\|^p}. \tag{6}$$

But we have

$$\begin{aligned}\left|\int_Z \frac{F(z, x_n(z))}{\|x_n\|^p}dz\right| &\leq \frac{1}{\|x_n\|^p}\int_Z \int_0^{|x_n(z)|} |f(z,r)|drdz \\ &\leq \frac{1}{\|x_n\|^p}(\|\alpha\|_\infty \|x_n\| + \frac{c}{\theta}\|x_n\|^\theta) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

So by passing to the limit as $n \to \infty$ in (6), we obtain

$$\lim_{n\to\infty} \frac{1}{p}\|Dy_n\|_p^p = 0$$

from which it follows $\|Dy\|_p = 0$ (recall that $Dy_n \xrightarrow{w} Dy$ in $L^p(Z, R^N)$ as $n \to \infty$) and consequently, $y = \xi \in R$.

Note that $y_n \to \xi$ in $W^{1,p}(Z)$ and since $\|y_n\| = 1, n \geq 1$ we infer that $\xi \neq 0$. We deduce that $|x_n(z)| \to +\infty$ a.e. on Z as $n \to \infty$.

From the choice of the sequence $\{x_n\} \subseteq W^{1,p}(Z)$, we have

$$\int_Z -v_n(z)x_n(z)dz - \|Dx_n\|_p^p \geq -\varepsilon_n \|x_n\| \tag{7}$$

and

$$\|Dx_n\|_p^p - p\int_Z F(z, x_n(z))dz \geq -pM_1. \tag{8}$$

Adding (7) and (8), we obtain

$$\int_Z (-v_n(z))x_n(z) - pF(z, x_n(z)))dz \geq -pM_1 - \varepsilon_n\|x_n\|.$$

Dividing this inequality by $\|x_n\|^\theta$ we have

$$\int_Z \frac{-v_n(z)}{\|x_n\|^{\theta-1}} y_n(z)dz - \int_Z \frac{pF(z, x_n(z))}{\|x_n\|^\theta} dz \geq -\frac{1}{\|x_n\|^\theta} pM_1 - \frac{\varepsilon_n}{\|x_n\|^{\theta-1}} \tag{9}$$

Note that

$$\int_Z \frac{-v_n(z)}{\|x_n\|^{\theta-1}} y_n(z)dz = \int_Z \frac{-v_n(z)}{|x_n(z)|^{\theta-2}x_n(z)} |y_n(z)|^\theta dz \to |\xi|^\theta \int_Z f_+(z)dz$$

as $n \to \infty$.

Also by virtue of hypothesis $\mathbf{H(f)}$ (ii), given $z \in Z \setminus N, |N| = 0$ ($|C|$ denotes the Lebesgue measure of a measurable set $C \subseteq Z$) and $\varepsilon > 0$, we can find $M_\varepsilon > 0$ such that for all $|r| \geq M_\varepsilon$ we have $|f_+(z) - \frac{f_{1,2}(z,r)}{|r|^{\theta-2}r}| \leq \varepsilon$. Then, if $x_n(z) \to +\infty$, we have

$$\begin{aligned}
\frac{1}{|x_n(z)|^\theta} F(z, x_n(z))dz &\geq \frac{1}{|x_n(z)|^\theta} F(z, M_\varepsilon)dz \\
&\quad + \frac{1}{|x_n(z)|^\theta} \int_{M_\varepsilon}^{x_n(z)} (f_+(z)|r|^{\theta-2}r - \varepsilon|r|^{\theta-2}r)dr \\
&= \frac{1}{|x_n(z)|^\theta}\eta(z) + \frac{|x_n(z)|^\theta - M_\varepsilon^\theta}{\theta|x_n(z)|^\theta}(f_+(z) - \varepsilon)
\end{aligned}$$

for some $\eta \in L^1(Z)$. It follows that

$$\liminf_{n\to\infty} \frac{F(z, x_n(z))}{|x_n(z)|^\theta} \geq \frac{1}{\theta}(f_+(z) - \varepsilon) \tag{10}$$

Similarly we obtain that

$$\limsup_{n\to\infty} \frac{F(z, x_n(z))}{|x_n(z)|^\theta} \leq \frac{1}{\theta}(f_+(z) + \varepsilon) \tag{11}$$

From (10) and (11) and since $\varepsilon > 0$ and $z \in Z \setminus N$ were arbitrary, we infer that

$$\frac{F(z, x_n(z))}{|x_n(z)|^\theta} \to \frac{1}{\theta} f_+(z) \text{ a.e. on } Z \text{ as } n \to \infty$$

whence

$$\begin{aligned}
\int_Z \frac{F(z, x_n(z))}{\|x_n\|^\theta} dz &= \int_Z \frac{F(z, x_n(z))}{|x_n(z)|^\theta} \frac{|x_n(z)|^\theta}{||x_n||^\theta} dz \\
&= \int_Z \frac{F(z, x_n(z))}{|x_n(z)|^\theta} |y_n(z)|^\theta dz \to \xi^\theta \int_Z \frac{1}{\theta} f_+(z) \text{ as } n \to \infty
\end{aligned} \tag{12}$$

Thus by passing to the limit in (9), we obtain

$$(1-\frac{p}{\theta})\xi^\theta \int_Z f_+(z) \geq 0,$$

a contradiction to hypothesis **H(f)** (ii) (recall $p > \theta$). If $x_n(z) \to -\infty$, with similar arguments as above we show that

$$\int_Z \frac{F(z, x_n(z))}{\|x_n\|^\theta} dz \to \xi^\theta \int_Z \frac{1}{\theta} f_+(z) \text{ as } n \to \infty.$$

Therefore, it follows that $\{x_n\} \subseteq W^{1,p}(Z)$ is bounded. Hence we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}(Z), x_n \to x$ in $L^p(Z), x_n(z) \to x(z)$ a.e. on Z as $n \to \infty$ and $|x_n(z)| \leq k(z)$ a.e. on Z with $k \in L^p(Z)$. Note that K is closed and convex so it is weakly closed; thus $x \in K$.

So we have

$$-\varepsilon_n \|x - x_n\| \leq \langle Ax_n, x - x_n \rangle - \int_Z v_n(z)(x - x_n(z))dz$$

with $v_n(z) \in [f_1(z, x_n(z)), f_2(z, x_n(z))]$ and $A : W^{1,p}(Z) \to W^{1,p}(Z)^*$ such that $\langle Ax, y \rangle = \int_Z (\|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{R^N} dz$. But $x_n \xrightarrow{w} x$ in $W^{1,p}(Z)$, so $x_n \to x$ in $L^p(Z)$ and $x_n(z) \to x(z)$ a.e. on Z by virtue of the compact embedding $W^{1,p}(Z) \subseteq L^p(Z)$. Then we have that $\limsup \langle Ax_n, x_n - x \rangle = 0$ (note that v_n is bounded). By virtue of the inequality (5) we have that $Dx_n \to Dx$ in $L^p(Z)$. So we have $x_n \to x$ in $W^{1,p}(Z)$. The claim is proved.

Now let $W^{1,p}(Z) = X_1 \oplus X_2$ with $X_1 = \mathbb{R}$ and $X_2 = \{y \in W^{1,p}(Z) : \int_Z y(z)dz = 0\}$. For every $\xi \geq 0$ we have

$$R(\xi) = \Phi(\xi) + I_K(\xi) = -\int_Z F(z, \xi)dz.$$

By virtue of hypothesis $\mathbf{H(f)}_2$ (ii) we conclude that $R(\xi) \to -\infty$ as $\xi \to \infty$. On the other hand for $y \in X_2$, we have

$$\begin{aligned} R(y) &\geq \frac{1}{p}\|Dy\|_p^p - \int_Z F(z, y(z))dz \quad (\text{since } I_K(y) \geq 0) \\ &\geq \frac{1}{p}\|Dy\|_p^p - c_2\|y\|_p - c_3\|y\|_p^\theta \end{aligned}$$

for some $c_2, c_3 > 0$ (since $\theta < p$, see $\mathbf{H(f)}_3$ (i))

From the Poincare-Wirtinger inequality we know that $\|Dy\|_p$ is an equivalent norm on X_2. So we have

$$R(y) \geq \frac{1}{p}\|Dy\|_p^p - c_4\|Dy\|_p - c_5\|Dy\|_p^\theta$$

for some $c_4, c_5 > 0$. So, $R(\cdot)$ is coercive on X_2 (recall $\theta < p$) hence, bounded below on X_2.

In order to use the mountain-pass theorem it remains to show that there exists $\rho > 0$ such that for $\|x\| = \rho$ we have that $R(x) \geq a > 0$. In fact, we will show that for every sequence $\{x_n\} \subseteq W^{1,p}(Z)$ with $\|x_n\| = \rho_n \downarrow 0$ we have that $R(x_n) > 0$. Indeed, suppose not. Then there exists some sequence $\{x_n\}$ such that $R(x_n) \leq 0$. Thus, we have

$$\frac{1}{p}\|Dx_n\|_p^p \leq \int_Z F(z, x_n(z))dz.$$

Recall that $I_K \geq 0$. Divide this inequality with $\|x_n\|^p$. Let $y_n(z) = \frac{x_n(z)}{\|x_n\|}$. Then we have

$$\|Dy_n\|_p^p \leq \int_Z p\frac{F(z, x_n(z))}{\|x_n\|^p}dz.$$

From **H(f)** (iii) we have that for almost all $z \in Z$ for any $\varepsilon > 0$ we can find $\delta > 0$ such that for $|x| \leq \delta$ we have

$$pF(z, x) \leq (h(z) + \varepsilon)|x|^p.$$

On the other hand, for almost all $z \in Z$ and all $|x| \geq \delta$ we have

$$p|F(z, x)| \leq c_1|x| + c_2|x|^\theta + c_3 \leq c_1|x|^p + c_2|x|^{p^*} + c_4.$$

Thus we can always find $\gamma > 0$ such that $p|F(z, x)| \leq (h(z) + \varepsilon)|x|^p + \gamma|x|^{p^*}$ for all $x \in R$. Indeed, choose $\gamma \geq c_2 + \frac{c_4}{|\delta|^{p^*}} + |h(z) + \varepsilon - c_1|\ |\delta|^{p-p^*}$. Therefore, we obtain

$$\begin{aligned}\|Dy_n\|_p^p &\leq \int_Z (h(z) + \varepsilon)|y_n(z)|^p dz + \gamma \int_Z \frac{|x_n(z)|^{p^*}}{\|x_n\|^p}dz \\ &\leq \int_Z (h(z) + \varepsilon)|y_n(z)|^p dz + \gamma_1\|x_n\|^{p^*-p}.\end{aligned} \tag{13}$$

Here we have used the fact that $W^{1,p}(Z)$ embeds continuously in $L^{p^*}(Z)$. So we obtain

$$0 \leq \|Dy_n\|_p^p \leq \varepsilon\|y_n\|_p^p + \gamma_1\|x_n\|^{p^*-p} \text{ recall that } h(z) \leq 0.$$

Therefore in the limit we have that $\|Dy_n\|_p \to 0$. Recall that $y_n \to y$ weakly in $W^{1,p}(Z)$. So $\|Dy\|_p \leq \liminf \|Dy_n\|_p \leq \limsup \|Dy_n\|_p \to 0$. So $\|Dy\|_p = 0$, thus $y = \xi \in \mathbb{R}$. Note that $Dy_n \to Dy$ weakly in $L^p(Z)$ and $\|Dy_n\|_p \to \|Dy\|_p$ so $y_n \to y$ in $W^{1,p}(Z)$. Since $\|y_n\| = 1$ we have that $\|y\| = 1$ so $\xi \neq 0$. Suppose that $\xi > 0$. Going back to (13) we have

$$0 \leq \int_Z (h(z) + \varepsilon)y_n^p(z)dz + \gamma_1\|x_n\|^{p^*-p}.$$

In the limit we have

$$0 \leq \int_Z (h(z) + \varepsilon)\xi^p dz \leq \varepsilon\xi^p|Z| \text{ recall that } h(z) \leq 0.$$

Thus we obtain that $\int_Z h(z)\xi^p dz = 0$. But this is a contradiction. The same holds when $\xi < 0$. So the claim is proved. Now, by mountain pass theorem we have that there exists $x \in W^{1,p}(Z)$ such that

$$\Phi^o(x; y - x) + \psi(y) - \psi(x) \geq 0$$

for all $y \in W^{1,p}(Z)$. Choose $y = x + tv$ with $v \in K$. Dividing by $t > 0$ we have in the limit

$$\int_Z F^o(z, x(z); v(z))dz + \langle Ax, v\rangle \geq \Phi^o(x; v) + \langle Ax, v\rangle \geq 0$$

for all $v \in K$. ☑

Remark 1. Note that if $K = W^{1,p}(Z)$ then from above we have that $-Ax \in \partial\Phi(x)$ and the subdifferential is in the sense of Clarke.

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