Revista Colombiana de Matemáticas
Volumen 32 (1998), páginas 93–99

Wavelet transforms and singularities of L_2-functions in $\mathbb{R}^n$

Jaime Navarro

Universidad Autónoma Metropolitana, México D. F.

Abstract. For a function f in $L^2(\mathbb{R})$, a wavelet transform with respect to an admissible function is defined such that its singularities are precisely the points where f fails to be smooth.

Key words and phrases. Admissible functions, wavelet transform.

1991 Mathematics Subject Classification. 42A38, 44A05.

Introduction

In this paper a group structure on $\{(a,b) : a \in \mathbb{R}^+, b \in \mathbb{R}^n\}$ is used to define a wavelet transform of a function $f \in L^2(\mathbb{R}^n)$ with respect to an *admissible* function $h \in C_0^\infty(\mathbb{R}^n)$. For (a,b) in the group and letting $(U(a,b)h)(x) = \frac{1}{a^{n/2}} h(\frac{x-b}{a})$, a representation U of the group acting on the Hilbert space $L^2(\mathbb{R}^n)$ is defined.

By means of this representation, I. Daubechies [4] established the following resolution of the identity: for f, h in $L^2(\mathbb{R}^n)$, where h is radially symmetric (i.e., $h(x) = \eta(|x|)$, so that $h(x)$ depends only on $|x|$), we have

$$f = \frac{1}{C_h} \int_{\mathbb{R}^+} \int_{\mathbb{R}^n} \langle f, U(a,b)h \rangle \, U(a,b)h \, \frac{1}{a^{n+1}} \, db \, da, \tag{1}$$

where $\langle \ , \ \rangle$ is the inner product in $L^2(\mathbb{R}^n)$ and $C_h = \int_{\mathbb{R}^+} |\hat{\eta}(k)|^2 \frac{1}{k} dk < \infty$, $\hat{\eta}$ being the Fourier transform of η.

With the help of this resolution of the identity, and for (a,b) in the group, a wavelet transform $(L_h f)(a,b)$ of a function f in $L^2(\mathbb{R}^n)$ is defined with respect to an *admissible* function h in $L^2(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^+} |\hat{\eta}(k)|^2 \frac{1}{k} dk < \infty$, such that the singularities of $(L_h f)(a,b)$ are precisely the singularities of f.

Notations and definitions. *With G we denote the set $\{(a,b) : a \in \mathbb{R}^+, b \in \mathbb{R}^n\}$. In G we define $(a_1,b_1)\cdot(a_2,b_2) = (a_1a_2, a_1b_2+b_1)$. With this operation G becomes a group in which $(1,0)$ is the identity and $(a,b)^{-1} = (a^{-1}, -a^{-1}b)$. Moreover, G turns out to be a locally compact topological group with $d(a,b) = \frac{1}{a^{n+1}}da\,db$ and $d_1(a,b) = \frac{1}{a}da\,db$ as the left and right Haar measures, respectively.*

Definition 1. For h in $L^2(\mathbb{R}^n)$ and b in $\mathbb{R}^n$, the *traslation operator* T_b is $(T_bh)(x) = h(x-b)$, where $x \in \mathbb{R}^n$.

Definition 2. For h in $L^2(\mathbb{R}^n)$ and a in $\mathbb{R}^+$, the *dilation operator* J_a is $(J_ah)(x) = \frac{1}{a^{n/2}}h(\frac{x}{a})$, where $x \in \mathbb{R}^n$.

Definition 3. For h in $L^2(\mathbb{R}^n)$ and c in $\mathbb{R}^n$, the *rotation operator* E_c is $(E_ch)(x) = e^{2\pi i x\cdot c}h(x)$, where $x \in \mathbb{R}^n$.

Definition 4. For (a,b) in G, define $U(a,b) = J_aT_b$. This family of operators is a representation of G acting on the Hilbert space $L^2(\mathbb{R}^n)$ by

$$(U(a,b)h)(x) = (J_aT_bh)(x) = \frac{1}{a^{\frac{n}{2}}}h\Big(\frac{x-b}{a}\Big). \tag{2}$$

Definition 5. A function h in $L^2(\mathbb{R}^n)$ is said to be admissible if

$$\int_G \big|\langle h, U(a,b)h\rangle\big|^2 d(a,b) < \infty. \tag{3}$$

Lemma 1. *A radially symmetric function h in $L^2(\mathbb{R}^n)$ is admissible if and only if*

$$C_h \equiv \int_{\mathbb{R}^+} |\hat{\eta}(k)|^2 \frac{1}{k} dk < \infty, \tag{4}$$

where $\hat{h}(y) = \hat{\eta}(|y|)$.

See the Appendix for the proof.

Definition 6. For a function f in $L^2(\mathbb{R}^n)$ and (a,b) in G, the *wavelet transform of f with respect to the admissible function h* in $L^2(\mathbb{R}^n)$ is defined as

$$(L_hf)(a,b) = \big\langle f, U(a,b)h\big\rangle. \tag{5}$$

We now state and prove the main result of this paper.

Theorem. *Suppose that h in $C_0^\infty(\mathbb{R}^n)$ is radially symetric, non-identically vanishing and such that $\int_{\mathbb{R}^n} h(x)\,dx = 0$. For f in $L^2(\mathbb{R}^n)$ and (a,b) in G, let $\mathcal{L}_\alpha(a,b) = a^{-1}a^{-\frac{n}{2}}D_b^\alpha(L_hf)(a,b)$. Then, for each multi-index α, $\mathcal{L}_\alpha$ is continuous at any point (a_1,b_1) in G. Furthermore, f is C^∞ in a neighborhood* *of $x = b_0$ if and only if for each multi-index α, $\lim_{(a,b)\to(0,b_1)} \mathcal{L}_\alpha(a,b)$ exists for each b_1 in a neighborhood of b_0.*

Proof. First we show that $\mathcal{L}_\alpha$ is continuous at (a_1, b_1) for $a_1 > 0$. Note that

$$(L_h f)(a,b) = \int_{\mathbb{R}^n} \frac{1}{a^{n2}} f(x)\overline{h\left(\frac{x-b}{a}\right)}dx = (f * (J_a\overline{h})^{\sim})(b)$$

where $\psi^{\sim}(x) = \psi(-x)$ and $*$ means convolution. Now, since $f \in L^2(\mathbb{R}^n)$ and $h \in C_0^\infty(\mathbb{R}^n)$, it follows that $f * (J_a\overline{h})^{\sim} \in C^\infty(\mathbb{R}^n)$ and $D_b^\alpha(f * (J_a\overline{h})^{\sim})(b) = (f * D_b^\alpha(J_a\overline{h})^{\sim})(b)$. Thus, $\mathcal{L}_\alpha(a,b) = a^{-1}a^{-\frac{n}{2}}\frac{(-1)^{|\alpha|}}{a^{|\alpha|}}(f * (J_a\overline{D^\alpha h})^{\sim})(b)$ is continuous at (a_1, b_1) for $a_1 > 0$.

Next we show that the smoothness of f implies the existence of the limit of $\mathcal{L}_\alpha(a,b)$ as $(a,b) \to (0,b_1)$. Suppose that f is C^∞ in a neighborhood of $x = b_0$ containing the closed ball $\overline{B_\Delta(b_0)}$, where $\Delta > 0$. Take b, b_1 in the open ball $B_{\frac{\Delta}{2}}(b_0)$. Note that if $L > 0$ is such that $supp\, h \subset B_L(0)$, then

$$(L_h f)(a,b) = \int_{B_L(0)} a^{\frac{n}{2}} f(b+ay)\overline{h(y)}\,dy.$$

Thus, for a such that $0 < a < \frac{\Delta}{2L}$,

$$D_b^\alpha(L_h f)(a,b) = \int_{B_L(0)} a^{\frac{n}{2}} D_b^\alpha f(b+ay)\overline{h(y)}\,dy.$$

Now, since f is C^∞ at the points in the region of integration, it follows from Taylor's formula that

$$D_b^\alpha f(b+ay) = D^\alpha f(b) + \int_0^1 \sum_{|\beta|=1} \frac{1}{\beta!} D_b^{\beta+\alpha} f(b+tay)a^{|\beta|}y^\beta dt,$$

for y in $B_L(0)$, so that

$$\begin{aligned}
\mathcal{L}_\alpha(a,b) =& a^{-1}a^{-\frac{n}{2}} D_b^\alpha(L_h f)(a,b) \\
=& a^{-1}\int_{B_L(0)} D^\alpha f(b)\overline{h(y)}dy \\
&+ \int_{B_L(0)}\int_0^1 \sum_{|\beta|=1} D_b^{\beta+\alpha} f(b+tay)y^\beta\overline{h(y)}dtdy.
\end{aligned}$$

Then, since $\int_{B_L(0)} h(y)dy = 0$ and $D^{\beta+\alpha}f$ is continuous near b_1, it follows that

$$\begin{aligned}
\lim_{(a,b)\to(0,b_1)} \mathcal{L}_\alpha(a,b) &= \int_{B_L(0)} \left(\sum_{|\beta|=1} D_b^{\beta+\alpha} f(b_1)\right) y^\beta\overline{h(y)}dy \\
&= \left(\sum_{|\beta|=1} D_b^{\beta+\alpha} f(b_1)\right)\int_{B_L(0)} \overline{h(y)}y^\beta dy.
\end{aligned} \tag{6}$$

Therefore, $\lim_{(a,b)\to(0,b_1)} \mathcal{L}_\alpha(a,b)$ exists for each b_1 in $B_{\frac{\Delta}{2}}(b_0)$.

Now we show that the existence of the limit implies the smoothness of f. Suppose that $L_\alpha(b_1) := \lim_{(a,b)\to(0,b_1)} \mathcal{L}_\alpha(a,b)$ exists for each b_1 in an open neighborhood containing the closed ball $\overline{B_R(b_0)}$, where $R > 0$.

For fixed x in the open ball $B_R(b_0)$, let

$$\mathcal{I}_\alpha(a,x,y) = \begin{cases} h(-y)\mathcal{L}_\alpha(a, x+ay) & \text{if } a > 0 \\ h(-y)L_\alpha(x) & \text{if } a = 0, \end{cases} \tag{7}$$

where $supp\, h \subset B_L(0)$, $L > 0$. Note that for such x, $\mathcal{I}_\alpha$ is well-defined for all a and y. Furthermore, for fixed y and $a \neq 0$, $\mathcal{I}_\alpha(a,x,y)$ is infinitely differentiable in the variable x, and we have the following three claims.

Claim 1. *$\mathcal{I}_\alpha$ is continuous at (a_1, x_1, y_1) for all a_1 in $\mathbb{R}^+$, x_1 in $\overline{B_R(b_0)}$ and y_1 in $\mathbb{R}^n$.*

In fact, if $a_1 \neq 0$, $\mathcal{I}_\alpha(a,x,y)$ is continuous at (a_1, x_1, y_1). Thus, we only need to consider the limit as $(a,x,y) \to (0, x_1, y_1)$. But

$$\begin{aligned} \lim_{(a,x,y)\to(0,x_1,y_1)} \mathcal{I}_\alpha(a,x,y) &= \lim_{(a,b)\to(0,x_1)} h(-y)\mathcal{L}_\alpha(a,b) \\ &= h(-y)\mathcal{L}_\alpha(0,x_1) = \mathcal{I}_\alpha(0,x_1,y_1). \end{aligned}$$

Then, $\mathcal{I}_\alpha$ is continuous at all (a_1, x_1, y_1) in $\mathbb{R}^+ \times \overline{B_R(b_0)} \times \mathbb{R}^n$. This proves Claim 1.

Claim 2. *$\mathcal{I}_\alpha$ is in $L^1(\mathbb{R}^+ \times \mathbb{R}^n)$ for fixed x in $B_R(b_0)$.*

In fact, for $a \neq 0$,

$$\mathcal{I}_\alpha(a,x,y) = h(-y)a^{-1}a^{-\frac{n}{2}} D_x^\alpha (L_h f)(a, x+ay).$$

Then

$$\begin{aligned} |\mathcal{I}_\alpha(a,x,y)| &= |h(-y)|\, a^{-\frac{2+n}{2}} \left| \frac{(-1)^{|\alpha|}}{a^{|\alpha|}} \langle f, T_{x+ay} D^\alpha J_a h \rangle \right| \\ &\le |h(-y)| a^{-\frac{2+n}{2}} a^{-|\alpha|} \|f\|_2 \|D^\alpha h\|_2. \end{aligned}$$

Now let

$$G_\alpha(a,y) = \begin{cases} |\mathcal{I}_\alpha(a,x,y)| & \text{if } 0 < a \le 1 \\ |h(-y)|\, a^{-\frac{2+n}{2} - |\alpha|}\, \|f\|_2\, \|D^\alpha h\|_2 & \text{if } a > 1. \end{cases} \tag{8}$$

Then $|\mathcal{I}_\alpha(a,x,y)| \le G_\alpha(a,y)$ for all (a,y) in $\mathbb{R}^+ \times \mathbb{R}^n$, and we can see that G_α is in $L^1(\mathbb{R}^+ \times \mathbb{R}^n)$ as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^+}\int_{\mathbb{R}^n} |G_\alpha(a,y)|dyda \\
&\qquad = \int_0^1 \int_{B_L(0)} |\mathcal{I}_\alpha(a,x,y)|dyda \\
&\qquad\quad + \int_1^\infty \int_{B_L(0)} |h(-y)|\, a^{-\frac{2+n}{2}-|\alpha|}\, \|f\|_2\, \|D^\alpha h\|_2 dyda \\
&\qquad = \int_0^1 \int_{B_L(0)} |\mathcal{I}_\alpha(a,x,y)|dyda \\
&\qquad\quad + \|f\|_2\, \|D^\alpha h\|_2 \left(\int_{B_L(0)} |h(-y)|dy\right)\left(\int_1^\infty a^{-\frac{2+n}{2}-|\alpha|}da\right).
\end{aligned}$$

Since $\mathcal{I}_\alpha(\cdot,x,\cdot)$ is continuous on $[0,1]\times\overline{B_L(0)}$ and $\int_1^\infty a^{-\frac{2+n}{2}-|\alpha|}da < \infty$, it follows that $G_\alpha \in L^1(\mathbb{R}^+ \times \mathbb{R}^n)$. Hence, $\mathcal{I}_\alpha(\cdot,x,\cdot) \in L^1(\mathbb{R}^+ \times \mathbb{R}^n)$. This proves Claim 2.

Claim 3. *For x in the open ball $B_R(b_0)$, let $w(x) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^n} \mathcal{I}_0(a,x,y)\,dyda$ and $I_\alpha(x) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^n} \mathcal{I}_\alpha(a,x,y)\,dyda$. Then $D^\alpha w(x) = I_\alpha(x)$ for any multi-index α.*

In fact, let x be in the open ball $B_R(b_0)$. By Claim 1, $\mathcal{I}_\alpha$ is continuous on $\mathbb{R}^+ \times \overline{B_R(b_0)} \times \mathbb{R}^n$, and by Claim 2, $|\mathcal{I}_\alpha(a,x,y)| \le S\, a^{-\frac{2+n}{2}-|\alpha|}\, \|f\|_2\, \|D^\alpha h\|_2$, for $a \ne 0$, where $S = \mathrm{Sup}\{|h(-y)| : y \in B_L(0)\}$. Thus,

$$\mathrm{Sup}\{|\mathcal{I}_\alpha(a,x,y)| : a \in \mathbb{R}^+,\ x \in B_R(b_0),\ y \in B_L(0)\}$$

exists.

Note that, by Claim 2, for x in $B_R(b_0)$, $\mathcal{I}_\alpha(a,x,y)$ is integrable and $D_x\mathcal{I}_\alpha(a,x,y)$ exists and is uniformly bounded for (a,y) in $\mathbb{R}^+ \times \mathbb{R}^n$. It follows that for each x in $B_R(b_0)$, $D_x\mathcal{I}_\alpha(a,x,y)$ is integrable and $D_x \int_{\mathbb{R}^+}\int_{\mathbb{R}^n} \mathcal{I}_\alpha(a,x,y)dyda = \int_{\mathbb{R}^+}\int_{\mathbb{R}^n} D_x\mathcal{I}_\alpha(a,x,y)dyda$. Thus, $D^\alpha w(x) = I_\alpha(x)$ for any multi-index α. This proves Claim 3.

Now, for $l > 0$ and any x, define

$$U_l(x) = \int_{\frac{1}{l}}^{l} \int_{\mathbb{R}^n} h(-y)a^{-1}a^{-\frac{n}{2}}(L_h f)(a, x+ay)\,dyda. \tag{9}$$

Then, by Claim 3, for every x in $B_R(b_0)$, $\lim_{l\to\infty} U_l(x) = w(x)$. That is, $U_l \to w$ pointwise on $B_R(b_0)$ as $l \to \infty$. On the other hand, by (1), $U_l \to C_h f$ weakly in $L^2(\mathbb{R}^+ \times \mathbb{R}^n)$ as $l \to \infty$. Then $f = C_h^{-1} w$ almost everywhere on $B_R(b_0)$, and because of Claim 3, f is C^∞ on $B_R(b_0)$. This completes the proof of our main theorem. ☑

Appendix

Proof of Lemma 1. Suppose that h in $L^2(\mathbb{R}^n)$ is admissible. Then

$$\int_G |\langle h, U(a,b)h\rangle|^2 d(a,b) < \infty,$$

and we have

$$\begin{aligned}
&\int_G \left|\langle h, U(a,b)h\rangle\right|^2 d(a,b) \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^+} \left|\langle \hat{h}, \widehat{J_a T_b h}\rangle\right|^2 \frac{1}{a^{n+1}} da\, db = \int_{\mathbb{R}^n}\int_{\mathbb{R}^+} \left|\langle \hat{h}, E_{-b} J_{\frac{1}{a}} \hat{h}\rangle\right|^2 \frac{1}{a^{n+1}} da\, db \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^+} \left|\int_{\mathbb{R}^n} \hat{h}(\xi)\overline{E_{-b} J_{\frac{1}{a}} \hat{h}(\xi)}\, d\xi\right|^2 \frac{1}{a^{n+1}} da\, db \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^+} \left|\int_{\mathbb{R}^n} \hat{h}(\xi)\overline{e^{-2\pi i b\cdot\xi}}\,\overline{J_{\frac{1}{a}} \hat{h}(\xi)}\, d\xi\right|^2 \frac{1}{a^{n+1}} da\, db \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^+} \left|\overline{\int_{\mathbb{R}^n} e^{-2\pi i b\cdot\xi}\overline{\hat{h}(\xi)} J_{\frac{1}{a}} \hat{h}(\xi)\, d\xi}\right|^2 \frac{1}{a^{n+1}} da\, db \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^+} \left|\int_{\mathbb{R}^n} e^{-2\pi i b\cdot\xi} \left(\overline{\hat{h}} J_{\frac{1}{a}} \hat{h}\right)(\xi)\, d\xi\right|^2 \frac{1}{a^{n+1}} da\, db \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^+} \left|\widehat{\left(\overline{\hat{h}} J_{\frac{1}{a}} \hat{h}\right)}(b)\right|^2 \frac{1}{a^{n+1}} da\, db = \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^n} \left|\widehat{(\overline{\hat{h}} J_{\frac{1}{a}} \hat{h})}(b)\right|^2 db\right) \frac{1}{a^{n+1}} da \\
&= \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^n} \left|(\overline{\hat{h}} J_{\frac{1}{a}} \hat{h})(y)\right|^2 dy\right) \frac{1}{a^{n+1}} da \\
&= \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^n} \left|\overline{\hat{h}(y)}\right|^2 \left|J_{\frac{1}{a}} \hat{h}(y)\right|^2 dy\right) \frac{1}{a^{n+1}} da \\
&= \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^n} \left|\hat{h}(y)\right|^2 \left|a^{\frac{n}{2}} \hat{h}(ay)\right|^2 dy\right) \frac{1}{a^{n+1}} da \\
&= \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^n} \left|\hat{h}(y)\right|^2 \left|\hat{h}(ay)\right|^2 dy\right) \frac{1}{a} da = \int_{\mathbb{R}^n} \left|\hat{h}(y)\right|^2 \left(\int_{\mathbb{R}^+} \left|\hat{h}(ay)\right|^2 \frac{1}{a} da\right) dy
\end{aligned}$$

Since h is radiallly symmetric, so is $\hat{h}$. Then

$$\begin{aligned}
\int_G |\langle h, U(a,b)h\rangle|^2 d(a,b) &= \int_{\mathbb{R}^n} |\hat{h}(y)|^2 \left(\int_{\mathbb{R}^+} |\hat{\eta}(a|y|)|^2 \frac{1}{a} da\right) dy \\
&= \int_{\mathbb{R}^n} |\hat{h}(y)|^2 \left(\int_{\mathbb{R}^+} |\hat{\eta}(k)|^2 \frac{1}{k} dk\right) dy \\
&= \left(\int_{\mathbb{R}^n} |\hat{h}(y)|^2 dy\right) C_h,
\end{aligned}$$

where $C_h = \int_{\mathbb{R}+} |\hat{\eta}(k)|^2 \frac{1}{k} dk < \infty$.

By working backwards, it is proved that if $C_h = \int_{\mathbb{R}+} |\hat{\eta}(k)|^2 \frac{1}{k} dk < \infty$ then h is admissible. This completes the proof of Lemma 1. ☑

References

[1] A. O. Barut and R. Raczka, *Theory of Group Representations and Applications*, World Scientific, Singapore, 1986.

[2] C. K. Chui, *An Introduction to Wavelets*, Academic Press, New York, 1992.

[3] J. M. Combes, A. Grossmann, and Ph Tchamitchian, *Wavelets-Time Frequency Methods and Phase Space*, Springer-Verlag, 1989.

[4] I. Daubechies, *Ten Lectures on Wavelets*, Siam, Philadelphia, 1992.

[5] G. B. Folland, *Introduction to Partial Differential Equations*, Princeton University Press, New Jersey, 1976.

[6] A. Grossmann and J. Morlet, *Decomposition of Hardy functions into square integrable wavelets of constant shape*, Siam J. Math. Anal. **15** (1984), 723–736.

[7] A. Grossmann, J. Morlet, and T. Paul, *Transforms associated to square integrable group representations, I. General results*, J. Math. Phys. **26** (1985), 2473–2479.

[8] ______, *Transforms associated to square integrable group representations, II. Examples*, Ann, Inst. H. Poincaré **45** (1986), 293–309.

[9] C. E. Heil and D. F. Walnut, *Continuous and discrete wavelet transforms*, Siam Review **31** no. 4 (1982), 628–666.

Universidad Autónoma Metropolitana
Unidad Azcapotzalco
División de Ciencias Básicas
Apdo. Postal 16-306
México D.F. 02000
México