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A remark on exponential dichotomies

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Abstract. A proof of the existence of an exponential dichotomy for the linear system $x'(t) = A(t)x(t)$ is given, based on the admissibility of the pair $(\mathcal{B}(\infty), \mathcal{B}_A(\infty))$, where $\mathcal{B}(\infty)$ is the space of continuous functions on the semi-axis $J = [0, \infty)$, values in $\mathbb{C}^n$ and having a limit as $t \to \infty$, and $\mathcal{B}_A(\infty)$ is the space of bounded functions f on J such that $A^{-1}f \in \mathcal{B}(\infty)$.

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1. **Introduction**

In this paper we consider the system of differential equations

$$x'(t) = A(t)x(t) + f(t),\ t \in J := [0, \infty), \tag{1}$$

where $A(t)$ is a continuous matrix function with complex entries. The function $f(t)$ belongs to a functional space we will define in the course of the paper.

Definition 1. Let $\mathcal{C}$ and $\mathcal{D}$ be function spaces. We say that the pair $(\mathcal{C},\ \mathcal{D})$ is admissible for equation (1) if for each f in the space $\mathcal{D}$ there exists a solution of (1) belonging to $\mathcal{C}$.

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Admissible pairs are important in the theory of differential equations (see [1], [2]), as they define the dichotomic behavior of the linear system

$$x'(t) = A(t)x(t). \tag{2}$$

Definition 2. We say that equation (2) has an exponential dichotomy on J, if there exist a fundamental matrix Φ of (2), a projection matrix P (i.e., $PP = P$) and positive constants K, α such that

$$\begin{aligned} |\Phi(t)P\Phi^{-1}(s)| &\leq Ke^{\alpha(s-t)}, \quad t \geq s \geq 0, \\ |\Phi(t)(I-P)\Phi^{-1}(s)| &\leq Ke^{\alpha(t-s)}, \quad s \geq t \geq 0. \end{aligned} \tag{3}$$

In this paper, we are concerned with the following classical result [1]:

Theorem A. *Equation (2) has an exponential dichotomy on J if for any bounded and continuous function $f(t)$ on J, equation (1) has at least one bounded solution.*

The aim of this paper is the characterization of exponential dichotomy by means of the admissibility of a pair of spaces of functions with limit at infinity.

2. Preliminaries

We will make use of the following spaces

$$\mathcal{B} := \{f : J \to \mathbb{C}^n : f \text{ is bounded and continuous}\},$$

$$\mathcal{B}(\infty) := \left\{f \in \mathcal{B} : \lim_{t\to\infty} f(t) \text{ exists}\right\}.$$

We call $\mathcal{B}(\infty)$ *the space of functions with limit at infinity.* These spaces, endowed with the norm $|f|_\infty = \sup\{|f(t)| : t \in J\}$, become Banach spaces. Furthermore, if $F : J \to \mathbb{C}^{n\times n}$ and $F(t)$ is invertible for each $t \in J$, we define

$$\mathcal{B}_F(\infty) := \left\{f \in \mathcal{B} : F^{-1}f \in \mathcal{B}(\infty)\right\}.$$

To this space we give the norm $|f|_F = |F^{-1}f|_\infty$. Provided that F is bounded on J, also $\mathcal{B}_F(\infty)$ is a Banach space. If equation (2) has an exponential dichotomy, then for any $f \in \mathcal{B}$, equation (1) has the following bounded solution:

$$x_f(t) = \int_0^t \Phi(t)P\Phi(s)f(s)\,ds - \int_t^\infty \Phi(t)(I-P)\Phi^{-1}(s)f(s)\,ds.$$

Let us introduce the following Green function:

$$G(t,\ s) = \begin{cases} \Phi(t)P\Phi(s), & t \geq s, \\ -\Phi(t)(I-P)\Phi^{-1}(s), & s > t. \end{cases} \tag{4}$$

By means of this function we can write the solution x_f in the form:

$$x_f(t) = \int_J G(t,\ s)f(s)ds. \tag{5}$$

If $A(t)$ is a bounded function, we will use the following identity:

$$\Phi(t)P\Phi^{-1}(0) - I = \int_J G(t,\ s)A(s)ds. \tag{6}$$

3. The main result

Theorem 1. *If the function $A(t)$ is bounded on J and the matrix $A(t)$ is invertible for each $t \in J$, then the following assertions are equivalent:*

(A) *The pair $(\mathcal{B},\ \mathcal{B})$ is admissible.*
(B) *The pair $(\mathcal{B}(\infty),\ \mathcal{B}_A(\infty))$ is admissible.*
(C) *Equation (2) has an exponential dichotomy on J.*

Proof.
(A) $\Leftrightarrow$ (C). This follows from Theorem A. We observe that this equivalence holds without the requirements of invertibility of the matrices $A(t)$ or the boundedness of the function $A(t)$.
(A) $\Rightarrow$ (B). Let $f \in \mathcal{B}_A(\infty)$. Since $f \in \mathcal{B}_A$, formula (5) makes sense. Therefore x_f defines a solution of (1) belonging to $\mathcal{B}(\infty)$. We have to prove that $\lim_{t\to\infty} x_f$ exists. Using (6) we may write

$$x_f(t) = -A^{-1}(t)f(t) + \Phi(t)P\Phi^{-1}(0)A^{-1}(t)f(t) + I_1(t) + I_2(t), \tag{7}$$

where

$$I_1(t) := \int_0^t G(t,\ s)A(s)\left[A^{-1}(s)f(s) - A^{-1}(t)f(t)\right] ds,$$
$$I_2(t) := \int_t^\infty G(t,\ s)A(s)\left[A^{-1}(s)f(s) - A^{-1}(t)f(t)\right] ds.$$

Taking into account (3), we can estimate I_i, $i = 1,\ 2$. We have

$$\begin{aligned} |I_1(t)| &\le \int_0^t |\Phi(t)P\Phi^{-1}(s)||A(s)||A^{-1}(s)f(s) - A^{-1}(t)f(t)|\, ds \\ &\le \int_0^{t/2} \cdots + \int_{t/2}^t \cdots \\ &\le 2K|A|_\infty \alpha^{-1} e^{-\frac{\alpha}{2}t}|A^{-1}f|_\infty \\ &\quad + \alpha^{-1}K \sup_{s\in[t/2,\ t]} |A^{-1}(s)f(s) - A^{-1}(t)f(t)| \end{aligned} \tag{8}$$

and

$$\begin{aligned}|I_2(t)| &\le |A|_\infty \int_t^\infty e^{\alpha(t-s)}|A^{-1}(s)f(s) - A^{-1}(t)f(t)|ds \\ &\le 2\alpha^{-1}|A|_\infty K \sup_{s\in[t,\,\infty)} |A^{-1}(s)f(s) - A^{-1}(t)f(t)|.\end{aligned} \tag{9}$$

Since $f \in B_A(\infty)$, it is clear from (8) and (9) that $\lim_{t\to\infty} I_i(t) = 0$. From (7) we obtain that $\lim_{t\to\infty} x_f(t) = -(A^{-1}f)(\infty)$. Therefore, the function pair $(\mathcal{B}(\infty), \mathcal{B}_A(\infty))$ is admissible.

(B) $\Rightarrow$ (A). Let S be the subspace of $\mathbb{C}^n$ of values of initial conditions of solutions of equation (2) belonging to $\mathcal{B}(\infty)$, and let U be a supplementary subspace of S. We have the direct sum $\mathbb{C}^n = S \bigoplus U$. Then it is easy to prove that equation (1) has, for any $f \in \mathcal{B}_A(\infty)$, a unique solution, which we denote by $T(f)$, that belongs to $\mathcal{B}(\infty)$ and is such that the initial condition satisfies $T(f)(0) \in U$. It is also easy to verify that this correspondence is linear. Thus, we define this way a linear map $T:\ \mathcal{B}_A(\infty) \to \mathcal{B}(\infty)$ such that $T(f)$ satisfies (1) and $T(f)(0) \in U$. This map has a closed graph (the proof of this assertion is exactly the same as that of Proposition 3.4 in [1]). Therefore, it is bounded, i.e., there exists a constant M, such that

$$|T(f)| \le M|f|_A. \tag{10}$$

Let $f \in \mathcal{B}$ and for each $n = 1, 2, ...$, let $\theta_n(t)$ be a continuous function such that $|\theta_n|_\infty = 1$, $\theta_n(t) = 1$ if $t \in [0, n]$ and $\theta_n(t) = 0$ if $t \ge n + 1$. Let $\{f_n\}$ be the sequence in $\mathcal{B}_A$ defined by

$$f_n(t) = \theta_n(t)f(t). \tag{11}$$

For each function f_n, we consider the solution $x_n = T(f_n)$ of the equation

$$x'(t) = A(t)x(t) + f_n(t). \tag{12}$$

According to (10), for any index n we have

$$|x_n|_\infty \le M|f_n|_A \le M|f|_A. \tag{13}$$

From (12) and (13) we obtain that the sequences $\{x_n\}$ and $\{x'_n\}$ are bounded on any compact subinterval of J. From the Ascoli-Arzelá theorem, there exists then a subsequence $\{x_n^1\}$ of $\{x_n\}$ uniformly convergent on $[0, 1]$ to a continuous function u_1 on the interval $[0, 1]$. By the same argument, there exists a subsequence $\{x_n^2\}$ of $\{x_n^1\}$ converging uniformly on the interval $[0, 2]$ to a continuous function u_2 such that $u_1 = u_2$ on $[0, 1]$. Carrying out this process iteratively, we obtain, for any natural number N, a subsequence $\{x_n^N\}$ of $\{x_n^{N-1}\}$, converging uniformly to a continuous function u_N on the interval $[0, N]$, and such that $u_N = u_{N-1}$ on $[0, N-1]$. Defining $u(t) = u_N(t)$ if $t \in [0, N]$, we obtain that the diagonal sequence $\{x_n^n\}$ converges uniformly to u on each compact subinterval of J. From (13), we obtain that $u \in \mathcal{B}$. From (11) and (12) it follows that u satisfies $u' = A(t)u + f$. This means that u is a solution of (1) in the space $\mathcal{B}$. ☑

References

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