A note on the asymptotic stability in the whole of non-autonomous systems

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Abstract. In this paper we present some results on the global stability of the trivial solutions $x \equiv 0$ of the system $x' = f(t, x)$. Our main results are then applied to various systems.

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1. Introduction

Liapunov's principal theorems give sufficient conditions for the stability, asymptotic stability and instability of systems. In the last few years these stability concepts have been refined and further generalized in several directions, one of these being the asymptotic stability in the whole.

This paper is concerned with sufficient conditions guaranteeing that the trivial solution $x \equiv 0$ of the system

$$x' = f(t, x), \tag{1}$$

where the prime marks indicate differentiation with respect to t, is asymptotically stable in the whole.

Throughout this paper, the following notations will be used. With I we denote the interval $0 \le t < \infty$, and $\mathbb{R}^m$ will stand for Euclidean m-space; $\| \cdot \|$ will be an arbitrary norm in $\mathbb{R}^m$, and $S_r = \{x \in \mathbb{R}^m : \|x\| < r\}$.

A solution of (1) through a point (t_0, x_0) in $I \times \mathbb{R}^m$ will be denoted by $x(t; x_0, t_0)$, with $x(t_0; x_0, t_0) = x_0$.

With $C(\mathbb{R})$ and $CI(\mathbb{R})$ we respectively denote the families of continuous functions and increasing continuous functions defined on $\mathbb{R}$, and

$CS(\mathbb{R}) = \{h \in C(\mathbb{R}) : xh(x) > 0 \text{ for } x \neq 0\},$
$CC(\mathbb{R}) = CI(\mathbb{R}) \cap CS(\mathbb{R}),$
$CP_b^K(\mathbb{R}) = \{h \in C^k(\mathbb{R}) : h(x) \geq b > 0 \text{ for all } x\}, \ CP_b := CP_b^0.$

Finally, by $\mathcal{F}$ we denote the class of non decreasing continuous functions φ on I such $\varphi(u) > 0$ for all $u \in I$ and

$$\int_0^\infty \frac{du}{\varphi(\mu)} = \infty.$$

We consider system (1) for $f \in C(I \times S_r)$ and $f(t, 0) \equiv 0$.

With $V(t, x)$ we denote an arbitrary continuous scalar function defined on an open set $S \subset I \times \mathbb{R}^m$. In all what follows it is assumed that all these functions $V(t, x)$ have continuous partial derivatives with respect to all arguments. These functions will be called *Liapunov's functions*. Corresponding to $V(t, x)$, we define the function

$$V'_{(1)}(t, x) := \limsup_{h \to 0+} \frac{V(t + h, x + hf(t, x)) - V(t, x)}{h}, \tag{2}$$

called the *total derivative of $V(t, x)$ for system* (1). Under the above conditions,

$$V'_{(1)}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x).$$

We need the following definitions (cf. [3], [14]).

Definition 1. The solution $x(t) \equiv 0$ of (1) is *stable* if for any $\varepsilon > 0$ and any $t_0 \in I$ there exists a $\delta(t_0, \varepsilon) < 0$ such that if $x_0 \in S_{\delta(t_0, \varepsilon)}$ then $x(t; x_0, t_0) \in S_\varepsilon$ for all $t \geq t_0$.

Definition 2. *asymptotically stable* if it is *stable* and there exists a $\delta(t_0) > 0$ such that $\|x(t; x_0, t_0)\| \longrightarrow 0$ as $t \longrightarrow \infty$ for all $x_0 \in S_{\delta(t_0)}$.

Definition 3. The solution $x(t) \equiv 0$ of (1) is *asymptotically stable in the whole* if it is *stable* and every solution of (1) tends to zero as $t \longrightarrow \infty$.

Definition 4. The solution $x(t) \equiv 0$ of (1) is *quasi equiasymptotically stable in the whole* if for any $\alpha > 0$, any $\varepsilon > 0$ and any $t_0 \in I$, there exists $T(t_0, \varepsilon, \alpha) > 0$ such that if $x_0 \in S_\alpha$ then $x(t; x_0, t_0) \in S_\varepsilon$ for all $t \geq t_0 + T(t_0, \varepsilon, \alpha)$.

Definition 5. The solution $x(t) \equiv 0$ of (1) is *equiasymtotically stable in the whole* if it is *stable* and *quasi-equiasymptotically stable in the whole*.

The distinction between asymptotic stability in the large and asymptotic stability in the whole has often been obliterated by inaccurate translation of the Russian terminology.

Definition 6. A solution $x(t; x_0, t_0)$ of (1) is *bounded* if there exists a $\beta > 0$ such that $x(t; x_0, t_0) \in S_\beta$ for all $t \geq t_{0\prime}$, where β may depend on the solution.

Definition 7. The solutions of (1) are *equibounded* if for any $\alpha > 0$ and $t_0 \in I$ there exists $\beta(t_0, \alpha) > 0$ such that if $x_0 \in S_\alpha$ then $x(t; x_0, t_0) \in S_\beta(t_0, \alpha)$ for all $t \geq t_0$.

We now mention some theorems which will play an important role in the proofs of our main results.

Theorem A. *Suppose that there exists a Liapunov's function $V(t, x)$ defined on $I \times S_r$ satisfying the following conditions:*

   1. $V(t, 0) \equiv 0$.
   2. $a(\|x\|) \leq V(t, x)$, *where $a(t)$ is a positive definite function in $CI(\mathbb{R})$.*
   3. $V'_{(1)}(t, x) \leq 0$.

Then, the trivial solution of (1) is stable.

Theorem B. *Suppose that there exists a Liapunov's function $V(t, x)$ defined on $I \times \mathbb{R}^m$ which satisfies the following conditions:*
   1. $a(\|x\|) \leq V(t, x)$, *where $a(r) \in CC(\mathbb{R})$ and $a(r) \longrightarrow \infty$ as $r \longrightarrow \infty$.*
   2. $V'_{(1)}(t, x) \leq 0$.
Then, the solutions of (1) are equi-bounded.

For the proof of the above results, see Theorems 8.1 and 10.1 of [14].

Theorem C (Barbashin and Krasovskii). *If there exists a function $V(t, x)$ which is everywhere positive definite, radially unbounded, decreasing, and whose total derivative (2) for system (1) is negative definite, then solution $x(t) \equiv 0$ of (1) is asymptotically stable in the whole.*

For the proof, see [3, p. 248].

2. Main results

Theorem 1. *Suppose that there exists a continuous, positive definite function $a(r)$ such that $a(r) \longrightarrow \infty$ as $r \longrightarrow \infty$, and that the following conditions are fulfilled:*

   1. *There exists a Liapunov's function $V(t, x)$ such that $a(x) \leq V(t, x)$.*
   2. $V'_{(1)}(t, x) \leq -\lambda(t)V(t, x) - \mu(t)W(t, x)$, *where $\lambda, \mu \in CP_0(I)$ and $W(t, x)$ is a positive definite function.*

Then, the trivial solution $x \equiv 0$ of (1) is equi-asymptotically stable in the whole.

Proof. The conditions of Theorems A and B are satisfied, so that the solutions of (1) are stable and equibounded.

Let $x(t; t_0 \ x_0)$ be a unique solution of (1) satisfying $x_0 \in S_\varepsilon$ for ε sufficiently small.

Since μ is non-negative, we have

$$V'_{(1)}(t, x) \leq -\lambda(t)V(t, x).$$

From the above inequality and the comparison theorem we obtain that

$$V(t, x(t; t_0, x_0)) \leq V(t_0, x_0) \exp(-L(t)),$$

where

$$L(t) = \int_{t_0}^{t} \lambda(s) \, ds.$$

Now define

$$M(t_0, \varepsilon) = \max \{V(t_0, x) : \|x\| \leq \varepsilon\}$$

and

$$T(t_0, \tau, \varepsilon) = L^{-1} \left(1n \frac{M(t_0, \varepsilon)}{a(\tau)} \right).$$

Observe that since $L(t)$ is strictly increasing then L^{-1} exists. Then we have, for $t > t_0 + T(t_0, \tau, \varepsilon)$,

$$V(t, x(t; t_0, x_0)) < M(t_0, \varepsilon) \frac{a(\tau)}{M(t_0, \varepsilon)} = a(\tau).$$

Since $a(r)$ is increasing, and by condition 1 we have

$$\|x(t; t_0, x_0)\| < \tau \text{ for } t > t_0 + T(t_0, \tau, \varepsilon),$$

which implies the equi-asymptotic stability in the whole of the trivial solution, the theorem is proved.   ☑

Theorem 2. *Suppose that the functions a, b defined on S_r and $W(t, x)$ defined on $I \times \mathbb{R}^m$ are positive definite and that the following two conditions hold:*

1. $a(\|x\|) \leq V(t, x) \leq b(\|x\|), a(r) \longrightarrow \infty$ as $r \longrightarrow \infty$.
2. $V'_{(1)}(t, x) \leq -\lambda(t)\phi(V(t, x)) - \mu(t))W(t, x)$, where $\phi \in CI(\mathbb{R}^+)$ and is positive definite.

Then the trivial solution $x \equiv 0$ of the system (1) is asymptotically stable in the whole.

Proof. If

$$U(t,x) := \int_0^{V(t,x)} du/\psi(u),$$

where $\psi \in \mathcal{F}(\mathbb{R}^+)$, then $U(t,x)$ satisfies the conditions of the Barbashin-Krasovski theorem (Theorem C above), and the proof of the theorem is complete. ☑

Remark 1. Observe the advantage of resorting to the inequalities for $V'_{(1)}(t,x)$ obtained in Theorems 1 and 2, not only for applications (Theorems 3 and 4 below), but also to render account of previous results (see [2], [3], [9] and [14]).

3. **Applications and related results**

The damped linear oscillator of one degree of freedom is described by the second order differential equation

$$x'' + h(t)x' + m^2 x = 0, t \in I, \tag{3}$$

where $m > 0$ is a constant, the "damping" coefficient $h : I \longrightarrow I$ being measurable and locally integrable.

It is an old problem to find conditions on h guaranteeing the asymptotic stability of the equilibrium $x = x' = 0$, which means that for every solution of (3),

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0 \tag{4}$$

holds. It is known (see e.g., [5], [13]) that the condition

$$\int_0^\infty h(t)dt = \infty \tag{5}$$

is necessary for asymptotic stability. It is also known that if $h(t) \equiv h_0 > 0$, where h_0 is constant, then the equilibrium is asymptotically stable. Results in [1], [4], [7], [8], [10] [11], [13] show that the condition

$$h \in CP_b(I) \tag{6}$$

is not necessary for asymptotic stability: it can be essentially weakened. Hale in [4] proved that if $h(t) = 2 + e^t$,the equilibrium of (2) with $k \equiv 1$ is not asymptotically stable.

It is therefore natural to pose the problem of when the null solution of (3) is asymptotically stable in the large, or, in other words, when each solution curve of (3) approaches 0 as $t \longrightarrow \infty$. This problem is of paramount importance for applications of stability theory.

A. Castro and R. Alonso [2] considered the special case

$$x'' + h(t)x' + x = 0, \tag{7}$$

of equation (3), under condition (6) with $b = 2a$ (a sufficiently small, i.e., $a \ll \frac{1}{2}$) and $h \in C^1(I)$. Letting $y = x'$, we obtain the system

$$\begin{aligned} y &= x' \\ y' &= -x - h(t)y, \end{aligned} \tag{8}$$

defined on $I \times S_r$, with $r > 0$. Further they required that the condition

$$ah'(t) + 2h(t) \le 4a$$

be fulfilled, and applying Theorem 4 of [2] obtained results on the asymptotic stability of the trivial solution of (8) (and consequently of (7)). In the next theorem we shall prove that condition (6) with $b = 2a, h \in C(I)$ and

$$h < \frac{\sqrt{1 - a^3} + 1}{a}, \tag{9}$$

are sufficient for the asymptotic stability in the whole of the trivial solution of (8).

We consider the functions

$$\begin{aligned} H(t) &= \exp\left(-\int_0^t h(s)\,ds\right), \\ R(x,y) &= \frac{x^2 + y^2}{2} \\ V(t,x,y) &= (H(t) + 2)R(x,y) + axy. \end{aligned}$$

Taking $a(x,y) = R(x,y), b(x,y) = 3R(x,y), \lambda(t) \equiv a, \mu(t) = (H(t)+1)h(t)$ and $c(x,y) = 2R(x,y)$, all assumptions of Theorem 2 with $\phi(u) = u$ are satisfied. Thus, we obtain the following result.

Theorem 3. *Assume that $h \in C(I)$ and that conditions (5) and (9) hold. Then, the equilibrium of (3) is asymptotically stable in the whole.*

Remark 2. In the case of asymptotic stability, our conditions(6) and (9) and Theorem 3 are consistent with results in [2], [6], [7], [8], [9], [11], [13] and [14].

We now consider the system

$$\begin{aligned} x' &= \alpha(y) - \beta(y)f(x) \\ y' &= -a(t)g(x) \end{aligned} \tag{10}$$

where α, β, f and g are continuous real valued functions and $a(t)$ is a positive continuously differentiable function on I. We define

$$G(x) = \int_0^x g(r)\, dr, \quad A(y) = \int_0^y \alpha(s)\, ds, \tag{11}$$

and assume that the conditions

(a) $\alpha \in CC(\mathbb{R})$,
(b) $\beta \in CP_b(\mathbb{R})$,
(c) $f, g \in CS(\mathbb{R})$,
(d) $a \in CP_0^1(\mathbb{R})$

hold. In [12] we prove that under these assumptions all solutions of (10) are continuable toward the future.

If $\alpha(y) = y, \beta(y) \equiv 1, a(t) \equiv 1$, then system (10) is the well known Liénard's equation.

Theorem 4. *Under conditions (a)–(d) above, suppose in addition that*

1. $G(x) \le bf(x)g(x)$, *for all* x,
2. $G(\overset{+}{-}\infty) = \pm\infty$,
3. $\frac{a'(t)}{a(t)} \ge 1$ *for all* t,

where G is as in (11). Then the trivial solution of (10) is equi-asymptotically stable in the whole.

Proof. Let $a(x, y) = \beta(x)$, $b(x, y) = \frac{A(y)}{a(0)} + \beta(x)$, $V(t, x, y) = A(y) + a(t)G(x)$ and $b(t) = \frac{a(0)}{a(t)}$. Let $W(t, x, y) = (b(t) + 1)V(t, x, y)$. We observe that

$$W'_{(10)}(t, x, y) \le -W(t, x, y) - b(t)V(t, x, y)$$

where

$$\begin{aligned}
W'_{(10)}(t, x, y) = &- \frac{a(0)}{a(t)}b(t)V(t, x, y) \\
&+ (b(t) + 1)(\alpha(y)y' + a'(t)b(x) + a(t)g(x)x')
\end{aligned}$$

(see [9], [12] and [14] for more details) satisfies the requirements of Theorem 1 with $\lambda(t) \equiv 1$ and $\mu(t) = b(t)$. Thus, we obtain the desired result. ☑

Remark 3. In the special case $\alpha(y) = y$, $\beta(y) \equiv 1, f(x) = x$, $a(t) \equiv 1$ and $g(x) = x$, of (10), conditions 1, 2 and 3 are easily verified, and system (10) becomes $x'' + x' + x = 0$, which satisfies the Routh-Hurwitz criterion for asymptotic stability in the whole.

Remark 4. From condition 3 of Theorem 4, we have that $a'(t) > 0$, for all $t \in I$. Thus, this theorem gives the author previous result [9, Theorem 2] on asymptotic stability in the whole.

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