# Zeta functions of singular curves over finite fields

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ABSTRACT. Let $X$ be a complete, geometrically irreducible, algebraic curve defined over a finite field $\mathbb{F}_q$ and let $\zeta(X,t)$ be its zeta function [Ser1]. If $X$ is a singular curve, two other zeta functions exist. The first is the Dirichlet series $Z(\mathrm{Ca}(X),t)$ associated to the effective Cartier divisors on $X$; the second is the Dirichlet series $Z(\mathrm{Div}(X),t)$ associated to the effective divisors on $X$. In this paper we generalize F. K. Schmidt's results on the rationality and functional equation of the zeta function $\zeta(X,t)$ of a non-singular curve to the functions $Z(\mathrm{Ca}(X),t)$ and $Z(\mathrm{Div}(X),t)$ by means of the singular Riemann-Roch theorem.

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In [G] Galkin introduced a zeta function for orders in global fields which coincides with Schmidt's zeta function in the case of a non-singular curve but satisfies a functional equation only in the Gorenstein case. Green [Gr] obtained a functional equation by slightly modifying the definition of the zeta function of Galkin. However, this new zeta function is not uniquely determined by the curve. Finally, Stöhr [Sto] defines the zeta function $Z(\mathrm{Div}(X),t)$ which coincides with the zeta function of Galkin in the Gorenstein case and always satisfies a functional equation ([Sto, p. 133]). In [Z] we studied the local factors of the zeta function $Z(\mathrm{Ca}(X),t)$ using an adelic approach a lá Tate. As a consequence we obtained the rationality and a global functional equation of

$Z(\mathrm{Ca}(X), t)$ ([Z, thm. 5.3]). In this paper we generalize F. K. Schmidt's results on the rationality and functional equation of the zeta function $\zeta(X, t)$ of a non-singular curve to the zeta functions $Z(\mathrm{Ca}(X), t)$ and $Z(\mathrm{Div}(X), t)$ using an approach based on the singular Riemann-Roch theorem. This allows us to find the residue at $s = 0$ of the zeta function $Z(\mathrm{Ca}(X), q^{-s})$ and $Z(\mathrm{Div}(X), t)$ (see thms. 2.1, 2.2, 3.1).

# 1. Preliminaries

In this section we present the basic facts about singular curves and zeta functions (cf. e.g. [Ros1], [Ros2], [Ser2], [Sto]).

Let $X$ be a complete, geometrically irreducible, algebraic curve defined over a field $k$. We denote by $K = k(X)$ the function field of $X$ over $k$, $g$ will stand for its arithmetic genus, $\tilde{g}$ for its geometric genus, $\tilde{X}$ will be the normalization of $X$ over $k$ (also named the non-singular model of $X$) and $\pi : \tilde{X} \longrightarrow X$ will denote the normalization map. The regular surjective map $\pi$ is birational. In particular, the function field of the smooth curve $\tilde{X}$ is $K/k$.

By a divisor of $X$ we mean a coherent fractional ideal sheaf or, equivalently, a formal product
$$\mathcal{A} = \prod_{P \in X} \mathcal{A}_P,$$
where for each point $P$ the $P$-component $\mathcal{A}_P$ (i.e. the stalk of $\mathcal{A}$ at $P$) is a non-zero fractional ideal of $\mathcal{O}_P$ and $\mathcal{A}_P = \mathcal{O}_P$ for all but finitely many points. Given two divisors $\mathcal{A}$ and $\mathcal{B}$, we define pointwise the product divisor $\mathcal{A}\mathcal{B}$ and the quotient divisor $\mathcal{A} : \mathcal{B}$ by
$$(\mathcal{A}\mathcal{B})_P := \mathcal{A}_P \mathcal{B}_P,$$
and
$$(\mathcal{A} : \mathcal{B})_P := \{ z \in K \mid z\mathcal{B}_P \subseteq \mathcal{A}_P \}.$$
We denote by $\mathrm{Div}(X)$ the set of divisors of $X$. A divisor $\mathcal{A}$ is called a locally principal or a Cartier divisor if each $\mathcal{A}_P$ is a principal fractional ideal. The Cartier divisors form a multiplicative group having the structure divisor
$$\mathcal{O} := \prod_{P \in X} \mathcal{O}_P,$$
as the identity. We denote by $\mathrm{Ca}(X)$ the group of the Cartier divisors. We define a partial order on $\mathrm{Div}(X)$ by
$$\mathcal{A} \leqslant \mathcal{B} \Longleftrightarrow \mathcal{A}_P \subseteq \mathcal{B}_P \quad \text{for all} \ \ P \in X.$$

A divisor $\mathcal{A}$ is called effective if $\mathcal{A} \geq \mathcal{O}$. For our purposes, it is more convenient to work with the above ordering than with the usual one.

The degree of a divisor is uniquely defined by the following properties:

(i) $\deg(\mathcal{O}) = 0$.

(ii) $\deg(\mathcal{B}) - \deg(\mathcal{A}) = \sum_{P \in X} \dim_k(\mathcal{B}_P/\mathcal{A}_P)$  whenever  $\mathcal{B} \geq \mathcal{A}$.

We observe that in general $\deg(\mathcal{AB}) \neq \deg(\mathcal{A}) \deg(\mathcal{B})$ (cf. [Ha, sect. 1]). However, if at least one of the divisors $\mathcal{A}$ or $\mathcal{B}$ is a Cartier divisor then the equality holds.

For each non-zero rational function $z \in K^*$, let $\mathrm{div}(z)$ be its principal divisor, i.e.,

$$\mathrm{div}(z) := \prod_{P \in X} z^{-1}\mathcal{O}_P.$$

We denote by $\mathrm{Prin}\,(X)$ the subgroup of principal divisors of $X$.

Let

$$L(\mathcal{A}) := \bigcap_{P \in X} \mathcal{A}_P = \{z \in K \mid \mathrm{div}(z)\mathcal{A} \geqq \mathcal{O}\}$$

be the $k-$vector space of global sections of $\mathcal{A}$ (also denoted by $H^0(X, \mathcal{A})$). We denote the dimension of the above $k-$vector space by $\ell(\mathcal{A})$ (also denoted by $h^0(X, \mathcal{A})$).

The Riemann-Roch theorem for function fields was generalized by Rosenlicht to curves with singularities (cf. e.g. [Ros1], [Sto]).

**Theorem 1.1.** *(Riemann-Roch theorem for singular curves). Each divisor $\mathcal{A}$ of $X$ satisfies*

$$\ell(\mathcal{A}) = \deg(\mathcal{A}) + 1 - g + \ell(\mathcal{C} : \mathcal{A}),$$

*where $\mathcal{C}$ denotes the canonical divisor of $X$.*

The local duality theorem also generalizes to singular curves (cf. e.g. [Sto, thm. 1.5])

**Theorem 1.2.** *(Local duality). Let $\mathcal{A}$, $\mathcal{B}$ be divisors of $X$ such that $\mathcal{A} \geqq \mathcal{B}$. Then for each point $P$ we have the $k$-isomorphism*

$$(\mathcal{C}_P : \mathcal{B}_P)/(\mathcal{C}_P : \mathcal{A}_P) \xrightarrow[\cong]{} Hom_k(\mathcal{A}_P/\mathcal{B}_P, k).$$

As a consequence of the theorem of local duality, we get the reciprocity formula (cf. e.g. [Sto, sect. 1.7]).

**Corollary 1.3.** (Reciprocity formula). For each divisor $\mathcal{A}$,

$$\mathcal{C} : (\mathcal{C} : \mathcal{A}) = \mathcal{A}.$$

From now on, we understand point to mean closed point. Let $P$ be a point of $X$ and $\mathcal{O}_P$ be the local ring of $X$ at $P$. Let $Q_1, Q_2, \ldots, Q_d$ be the points of $\tilde{X}$ lying over $P$, i.e., $\pi^{-1}(P) = \{Q_1, \ldots, Q_d\}$, and let $\mathcal{O}_{Q_1}, \ldots, \mathcal{O}_{Q_d}$ be the

corresponding local rings at these points. Since the function fields of $X$ and $\widetilde{X}$ are the same and $\widetilde{X}$ is a non-singular curve, the local rings $\mathcal{O}_{Q_1}, \ldots, \mathcal{O}_{Q_d}$ are valuation rings of $K/k$ over $\mathcal{O}_P$. The integral closure of $\mathcal{O}_P$ in $K$ is $\widetilde{\mathcal{O}}_P = \bigcap_{Q \in \pi^{-1}(P)} \mathcal{O}_Q$.

The degree of singularity of $X$ at $P$ is defined as

$$\delta_P = \dim_k(\widetilde{\mathcal{O}}_P / \mathcal{O}_P).$$

By Theorem. 1 in [Ros1], $\delta_P < \infty$. The total degree of singularity of $X$ is defined as

$$\delta = \sum_{P \in X} \delta_P.$$

The degree of singularity $\delta_P$ remains invariant under completion. The total degree of singularity $\delta$ remains invariant under separable constant extensions (cf. [Ros1, thm. 12]).

We recall that the genus formula of a complete, geometrically irreducible, algebraic curve $X$ is (see [H])

$$g = \tilde{g} + \sum_{P \in X} \delta_P,$$

where $\tilde{g}$ is the geometric genus of $X$.

The conductor ideal $\mathcal{F}_P$ of $\widetilde{\mathcal{O}}_P$ in $\mathcal{O}_P$ is defined as

$$\mathcal{F}_P = \{x \in K \mid x\widetilde{\mathcal{O}}_P \subseteq \mathcal{O}_P\}.$$

This ideal is the largest common ideal of $\widetilde{\mathcal{O}}_P$ and $\mathcal{O}_P$. Furthermore, since $\delta_P < \infty$, then $\mathcal{F}_P \neq 0$. On the other hand, $\widetilde{\mathcal{O}}_P$ is a Dedekind domain with a finite number of maximal ideals. Thus, $\mathcal{F}_P$ is an $\widetilde{\mathcal{O}}_P$-principal ideal. The degree of the conductor ideal $\mathcal{F}_P$ is defined as

$$\deg \mathcal{F}_P := \dim_k(\widetilde{\mathcal{O}}_P / \mathcal{F}_P).$$

The degree of $\mathcal{F}_P$ is also invariant under completions and under separable constant extensions.

We say that a local ring $\mathcal{O}_P$ is Gorenstein if $\deg \mathcal{F}_P = 2\delta_P$. An algebraic curve is called a Gorenstein curve if all its local rings are Gorenstein.

The dualizing sheaf $\omega_X$ is locally free of rank 1 if and only if $X$ is a Gorenstein curve (cf. [A-K, chap. VIII, prop. 1.16]). We denote by $\mathcal{C}$ the canonical divisor of a complete, geometrically irreducible, algebraic curve $X$. Then $\mathcal{C}_P$ is a principal ideal if and only if $\omega_{X,P}$ is free of rank 1. Summarizing, we have the following result of Rosenlicht.

**Theorem 1.4.** *Let $X$ be a complete, geometrically irreducible, algebraic curve defined over $k$. The curve $X$ is Gorenstein if and only if its canonical divisor is a Cartier divisor.*

We denote by $\mathrm{Ca}^0(X)$ the subgroup of degree zero divisors of $\mathrm{Ca}(X)$, by $\mathrm{Pic}\,(X)$ the quotient group $\mathrm{Ca}(X)/\mathrm{Prin}\,(X)$ and by $\mathrm{Pic}^{\,0}(X)$ the quotient group

$$\mathrm{Ca}^0(X)/\mathrm{Prin}\,(X).$$

Let $\bar{k}$ be an algebraic closure of $k$. $\mathrm{Pic}^{\,0}(X\otimes_k \bar{k})$ is the *generalized Jacobian* of $X\otimes_k \bar{k}$ (see [Ser2], [Ros2]).If $k$ is a finite field $\mathrm{Pic}^{\,0}(\tilde{X})$ is the group of rational points of $\mathrm{Pic}^{\,0}(X\otimes_k \bar{k})$. The generalized Jacobian can be enlarged to a projective algebraic scheme, called the *compactified Jacobian*, which parametrizes the equivalence classes of zero degree divisors of $X\otimes_k \bar{k}$. The rational points of the compactified Jacobian correspond bijectively to equivalence classes of zero degree divisors of $X$.

## 2. Rationality and functional equation of $Z(\mathrm{Ca}(X),t)$

In this section we present a generalization of results of F. K. Schmidt on the rationality and on the functional equation of $\zeta(X,t)$ to the function $Z(\mathrm{Ca}(X),t)$ (cf. [Sti, chap. V]). The zeta function $Z(\mathrm{Ca}(X),t)$ is defined as follows:

$$Z(\mathrm{Ca}(X),t) := \sum_{\mathcal{A}\geq\mathcal{O}} t^{\deg(\mathcal{A})},$$

where $\mathcal{A}$ runs through all Cartier divisors on $X$ and $t = q^{-s}$, $s \in \mathbb{C}, Re(s) > 0$. We shall see later on that $Z(\mathrm{Ca}(X),t)$ converges analytically and uniformly on the semiplane $Re(s) > 0$. This zeta function decomposes formally into an Euler product

$$Z(\mathrm{Ca}(X),t) = \prod_{P\in X} Z(\mathcal{O}_P,t) = \prod_{P\in X}\left(\sum_{I\supseteq\mathcal{O}_P} t^{\dim_k(I/\mathcal{O}_P)}\right), \qquad (2.1)$$

and where $I$ runs through all fractional principal ideals of $\mathcal{O}_P$, such that $I \supseteqq \mathcal{O}_P$. By comparing product (2.1) with $\zeta(\tilde{X},t) = Z(\mathrm{Ca}(\tilde{X}),t)$, we see that it converges on the semiplane $Re(s) > 0$. On the other side, the correspondence $I \longrightarrow I^{-1}$ is a bijection between the principal fractional ideals containing $\mathcal{O}_P$ and the principal ideals contained in $\mathcal{O}_P$. Furthermore, this correspondence preserves degrees, therefore, we have that

$$Z(\mathcal{O}_P,t) = \sum_{I\subseteq\mathcal{O}_P} t^{\dim_k(\mathcal{O}_P/I)}.$$

We observe that if $P$ is a smooth point of $X$, then $Z(\mathcal{O}_P,t) = (1 - t^{\deg(P)})^{-1}$, where $\deg(P)$ is the cardinality of the residue field of $P$.

**Theorem 2.1.** *Let $X$ be a complete, geometrically irreducible algebraic curve over a finite field $k = \mathbb{F}_q$. Then the zeta function $Z(\mathrm{Ca}(X), t)$ is a rational function. More precisely,*

$$Z(\mathrm{Ca}(X), t) = \frac{L(\mathrm{Ca}(X), t)}{(1 - t)(1 - qt)},$$

*where $L(\mathrm{Ca}(X), t) \in \mathbb{Z}[t]$ is a polynomial of degree at most $2g$, and $L(\mathrm{Ca}(X), 1) = \#\mathrm{Pic}^0(X)$, i.e. $L(\mathrm{Ca}(X), 1)$ is the number of rational points of the generalized Jacobian of $X \bigotimes_k \bar{k}$.*

*Proof.*
In order to generalize the proof of F.K. Schmidt we need to prove the following three claims.

*Claim 1.*
$$\#\mathrm{Pic}^0(X) < \infty. \tag{2.2}$$

*Claim 2. There exists a Cartier divisor of degree 1.*

*Claim 3. For any integer $d$, the number of divisor classes in $\mathrm{Pic}(X)$ of degree $d$ is independent of $d$ and is equal to the cardinality of $\mathrm{Pic}^0(X)$.*

After this, we can follow the argument of F.K. Schmidt as in [Sti, chap. V]. It is important to note that in this argument it is irrelevant to know whether a canonical divisor is a Cartier divisor or not. We denote by $[\mathcal{A}]$ the linear equivalence class of a divisor $\mathcal{A}$ and by $\pi^*(\mathcal{A})$ the pullback of $\mathcal{A}$.

To establish claim 1, we first observe that $\mathrm{Pic}^0(\tilde{X})$ is a finite group (more precisely, the group of divisor classes of degree zero of $K$(cf. [Sti, chap. V]), so, it is sufficient to show that the kernel of the morphism

$$\pi_0^* : \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0(\tilde{X})$$
$$\pi_0^*([\mathcal{A}]) \longmapsto [\pi^*(\mathcal{A})],$$

is a finite group.
We note that $[\mathcal{A}] \in \ker(\pi_0^*)$ if and only if $\pi^*(\mathcal{A})$ is a principal divisor of $\tilde{X}$, i.e.

$$\pi^*(\mathcal{A})_Q = z^{-1}\mathcal{O}_Q = \mathcal{A}_P \mathcal{O}_Q, \tag{2.3}$$

for some $z \in K^*$ and every point $Q$ of $\widetilde{X}$ lying over $P$. Therefore

$$\mathcal{F}_P \subseteqq z\mathcal{A}_P \subseteqq \tilde{\mathcal{O}}_P.$$

Since $\deg \mathcal{F}_P < \infty$, the above relation implies that the kernel of $\pi_0^*$ contains only a finite number of linear equivalence classes. The equivalence class of an effective Cartier divisor $\mathcal{A}$ contains $\frac{q^{\ell(\mathcal{A})-1}}{q-1}$ linearly equivalent Cartier divisors. Therefore the kernel of $\pi_0^*$ is a finite group.

The second claim can be reduced to the non-singular case using the fact that the morphism $\pi^* : \mathrm{Ca}(X) \longrightarrow \mathrm{Ca}(\tilde{X})$ is surjective and preserves degrees. The preservation of the degree of a Cartier divisor under $\pi^*$ follows from the fact that $\dim_k \mathcal{O}_P/h\mathcal{O}_P = \dim_k \tilde{\mathcal{O}}_P/h\tilde{\mathcal{O}}_P$, for every non-zero rational function $h$ of $X$. Thus, there exists a Cartier divisor of degree 1.

We observe that claim 1 and claim 2 imply that the number of effective Cartier divisors with a given degree is finite. Using the same argument as in the non-singular case, we prove that $Z(\mathrm{Ca}(X), t)$ converges absolutely and uniformly on the semiplane $Re(s) > 0$ (cf. [Sti, chap. V, prop. 1.6]).

The last claim follows from claim 2 and the fact that $\deg(\mathcal{A}\mathcal{B}) = \deg(\mathcal{A}) + \deg(\mathcal{B})$, for any two Cartier divisors $\mathcal{A}$ and $\mathcal{B}$ (see [Ha, sect. 1]). $\quad\checkmark$

**Corollary 2.2.** *The quotient $\frac{Z(\mathcal{O}_P, t)}{Z(\tilde{\mathcal{O}}_P, t)}$ is a rational function of $t$.*

*Proof.* By taking a partial resolution of singularities of $X$, we may assume that $P$ is the only singular point of $X$. Then

$$\frac{Z(\mathrm{Ca}(X), t)}{Z(\mathrm{Ca}(\tilde{X}), t)} = \prod_{Q \in \pi^{-1}(P)} (1 - t^{\deg(Q)})\ Z(\mathcal{O}_P, t) = \frac{Z(\mathcal{O}_P, t)}{Z(\tilde{\mathcal{O}}_P, t)}.$$

The result thus follows from the previous theorem. $\quad\checkmark$

**Theorem 2.3.** *Let $X$ be a complete, geometrically irreducible, algebraic curve defined over a finite field $k = \mathbb{F}_q$. Then the zeta function $Z(\mathrm{Ca}(X), t)$ satisfies the functional equation*

$$Z(\mathrm{Ca}(X), t) = q^{g-1} t^{2g-2} Z\left(\mathrm{Ca}(X), \frac{1}{qt}\right) \tag{2.3}$$

*if and only if $X$ is a Gorenstein curve.*

*Proof.* If $X$ is a Gorenstein curve, the argument of Schmidt for the non-singular case, the reciprocity (Corollary 1.2), and the observations made in the proof of Lemma 2.1 imply the functional equation (2.3) (see [Sti, chap. V, prop. 1.13]). Conversely, if the zeta function $Z(\mathrm{Ca}(X), t)$ satisfies the functional equation (2.3) and $g \geq 1$, then the Riemann-Roch theorem and (2.2) imply that

$$Z(\mathrm{Ca}(X), t) = Z_1(X, t) + Z_2(X, t)$$

with

$$Z_1(X, t) = \frac{1}{q-1} \sum_{0 \leq \deg[\mathcal{A}] \leq 2g-2} q^{\ell(\mathcal{A})} t^{\deg(\mathcal{A})} = \frac{1}{q-1} \sum_{j=0}^{2g-2} a_j t^j \tag{2.4}$$

and

$$Z_2(X,t) = \frac{1}{q-1} \sum_{\deg[\mathcal{A}] \geq 2g-1} q^{\ell(\mathcal{A})} t^{\deg(\mathcal{A})} - \frac{1}{q-1} \sum_{\deg[\mathcal{A}] \geq 0} t^{\deg(\mathcal{A})}$$

$$= \frac{1}{q-1} \left( \#\text{Pic}^{\,0}(X) q^g t^{2g-1} \frac{1}{1-qt} - \frac{\#\text{Pic}^{\,0}(X)}{1-t} \right). \qquad (2.5)$$

By hypothesis, $Z(\text{Ca}(X), t)$ satisfies the functional equation $(2.3)$. A direct verification shows that the function $Z_2(X,t)$ also satisfies the functional equation $(2.3)$. Therefore, $Z_1(X,t)$ satisfies the functional equation $(2.3)$. This implies that

$$a_{2g-2-j} = a_j q^{g-1-j}, \qquad j = 0, 1, \ldots, 2g-2.$$

On the other hand, $a_0 = 1$. Thus, $a_{2g-2} = q^{g-1}$. Since $g \geq 1$, a divisor class $[\mathcal{C}]$ with $\deg(\mathcal{C}) = 2g-2$ and $\ell(\mathcal{C}) = g$ must appear in the sum in $(2.4)$. These properties characterize the canonical class. Hence, by theorem 1.4, $X$ is Gorenstein. In the case $g = 0$, the genus formula implies $\delta = 0$, so that $X$ is a non-singular curve. Thus, in this case, also $X$ is Gorenstein.    ☑

As a consequence of the functional equation $(2.3)$, the degree of the polynomial $L(\text{Ca}(X), t)$ is $2g$.

**Corollary 2.4.** *The zeta function $Z(\mathcal{O}_P, t)$ satisfies the functional equation*

$$\frac{Z(\mathcal{O}_P, t)}{Z(\widetilde{\mathcal{O}}_P, t)} = q^{\delta_P} t^{2\delta_P} \frac{Z\left(\mathcal{O}_P, \frac{1}{qt}\right)}{Z\left(\widetilde{\mathcal{O}}_P, \frac{1}{qt}\right)}$$

*if and only if $\mathcal{O}_P$ is Gorenstein.*

## 3. Rationality and functional equation of $Z(\text{Div}(X), t)$

In this section, we study the rationality and functional equation of the zeta function $Z(\text{Div}(X), t)$ associated to the set of divisors $\text{Div}(X)$. This zeta function is defined as follows:

$$Z(\text{Div}(X), t) = \sum_{\mathcal{A} \geq \mathcal{O}} t^{\deg(\mathcal{A})},$$

where $\mathcal{A}$ runs through all effective divisors on $X$ and $t = q^{-s}$, $s \in \mathbb{C}, Re(s) > 0$. This zeta function decomposes formally into the following Euler product

$$Z(\text{Div}(X), t) = \prod_{P \in X} Z(\mathcal{O}_P, t) = \prod_{P \in X} \left( \sum_{I \supseteq \mathcal{O}_P} t^{\dim_k(I/\mathcal{O}_P)} \right),$$

where $I$ runs trough all $\mathcal{O}_P$-fractional ideals containing $\mathcal{O}$.

**Theorem 3.1.** *Let $X$ be a complete, geometrically irreducible algebraic curve over a finite field $k = \mathbb{F}_q$. Then the zeta function $Z(\mathrm{Ca}(X), t)$ is a rational function. More precisely,*

$$Z(\mathrm{Ca}(X), t) = \frac{L(\mathrm{Div}(X), t)}{(1 - t)(1 - qt)},$$

*where $L(\mathrm{Div}(X), t) \in \mathbb{Z}[t]$ is a polynomial of degree at most $2g$, and $L(\mathrm{Ca}(X), 1) = \#Cl^0(X)$, the number of classes of divisors of degree zero, i.e. $L(\mathrm{Div}(X), 1)$ is the number of rational points of the compactified Jacobian of $X \bigotimes_k \bar{k}$.*

*Proof.* The proof is similar to the proof of theorem 2.1. In order to prove that

$$\#Cl^0(X) = \mathrm{Div}^0(X)/Prin(X) < \infty,$$

we observe that $\mathrm{Pic}^{\,0}(X)$ acts on $Cl^0(X)$ by multiplication and the quotient set is isomorphic to

$$\left\{ \prod_{P \in X_{sing}} \mathcal{A}_P \mid \mathcal{A}_P \text{ is an ideal in } \mathcal{O}_P \text{ and } \mathcal{F}_P \subseteq \mathcal{A}_P \subseteq \widetilde{\mathcal{O}}_P \right\}. \qquad (3.1)$$

Since $\dim_k(\widetilde{\mathcal{O}}_P/\mathcal{F}_P) < \infty$ and $\#\mathrm{Pic}^{\,0}(X) < \infty$, we conclude from (3.1) that $\#Cl^0(X) < \infty$. Now we can follow the proof of theorem 2.1. $\quad \checkmark$

The zeta function $Z(\mathrm{Div}(X), t)$ satisfies the functional equation (2.3). The proof of this fact follows from the singular Riemann-Roch theorem and the reciprocity, by a similar reasoning to that of Schmidt (see [Sti, chap. V]). Stöhr defined $Z(\mathrm{Div}(X), t)$ and proved that $Z(\mathrm{Div}(X), t)$ satisfies the functional equation (2.4) (cf. [Sto, p. 133]).
The local factors of $Z(\mathrm{Div}(X), t)$ are rational functions (as in Corollary 2.2) and satisfy a functional equation (as in Corollary 2.4). The functional equation (2.3) implies that the degree of the polynomial $L(\mathrm{Div}(X), t)$ is $2g$.

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