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Some results on the integral geometry of unions of independent families

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ABSTRACT. A notion of independence for families of varieties is presented, and some results of integral geometry are established in relation to their unions.

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1. Independent families

Let $\mathcal{F}_t$ and $\mathcal{F}_m$ be two families of varieties, p and s dimensional, placed in a space X_n and depending on parameters $A_1, A_2, \dots, A_t$ and $B_1, B_2, \dots, B_m$ respectively:

$$\mathcal{F}_t : F^\lambda(x_1, x_2, \dots, x_n, A_1, A_2, \dots, A_t) = 0, \quad \lambda = 1, 2, \dots, n-p,$$

and

$$\mathcal{F}_m : G^\mu(x_1, x_2, \dots, x_n, B_1, B_2, \dots, B_m) = 0, \quad \mu = 1, 2, \dots, n-s.$$

We denote such families by $\mathcal{F}_t(\underline{A})$ and $\mathcal{F}_m(\underline{B})$.

Definition 1.1. The families $\mathcal{F}_t(\underline{A})$ and $\mathcal{F}_m(\underline{B})$ are said to be independent if there exists no relation $\varphi(\underline{A}, \underline{B}) = 0$ between their parameter sets $\{\underline{A}\}$ and $\{\underline{B}\}$.

Let $\mathcal{F}_q$ be the family of systems of two independent families $\mathcal{F}_t(\underline{A})$ and $\mathcal{F}_m(\underline{B})$. The above definition means that the parameters $A_1, A_2, \ldots, A_t, B_1, B_2, \ldots, B_m$ are all essential in $\mathcal{F}_q(\underline{A}, \underline{B})$. Further, if the maximal group of invariance of $\mathcal{F}_q$ is the intersection of the maximal groups of invariance of $\mathcal{F}_t$ and $\mathcal{F}_m$, then $\mathcal{F}_q$ is also called the *independent union of* $\mathcal{F}_t$ *and* $\mathcal{F}_m$.

Hence, every group of invariance of an independent union of independent families is a group of invariance of both of them.

Theorem 1.2. *Let G_1, G_2 and G_3 be the maximal groups of invariance of the families of varieties $\mathcal{F}_q$, $\mathcal{F}_t$ and $\mathcal{F}_m$, respectively, where $\mathcal{F}_q = \mathcal{F}_t + \mathcal{F}_m$ is the family of systems of $\mathcal{F}_t$ and $\mathcal{F}_m$. If $\tau \in G_1$, then τ, or at least τ^2, belongs to $G_2 \cap G_3$.*

Proof. An element of $\mathcal{F}_q$ is a pair (V, W) where $V \in \mathcal{F}_t$, $W \in \mathcal{F}_m$. If $\tau \in G_1$ then $\tau(V, W) = (\tau(V), \tau(W)) = (V_1, W_1)$ with $V_1 \in \mathcal{F}_t$ and $W_1 \in \mathcal{F}_m$. Three cases may arise:

1. $\tau(V) = V_1$ and $\tau(W) = W_1$. This implies $\tau \in G_2$ and $\tau \in G_3$, so $\tau \in G_2 \cap G_3$.
2. $\tau(V) = W_1$, $\tau(W) = V_1$, and $\tau(V_1) \in \mathcal{F}_t$. This implies $\tau(W_1) \in \mathcal{F}_m$. Consequently τ changes the variety $V_1 \in \mathcal{F}_t$ into another variety of $\mathcal{F}_t$, and therefore $\tau \in G_2$. Analogously τ changes the variety $W_1 \in \mathcal{F}_m$ into a variety of $\mathcal{F}_m$, and so $\tau \in G_3$. Hence $\tau \in G_2 \cap G_3$.
3. $\tau(V) = W_1$, $\tau(W) = V_1$, and $\tau(V_1) \in \mathcal{F}_m$. This implies $\tau(W_1) \in \mathcal{F}_t$. Consequently τ^2 changes the variety $W \in \mathcal{F}_m$ into another variety of $\mathcal{F}_m$, so that $\tau^2 \in G_3$. Analogously, τ^2 changes the variety $V \in \mathcal{F}_t$ into a variety of $\mathcal{F}_t$, so $\tau^2 \in G_2$. Hence $\tau^2 \in G_2 \cap G_3$. ☑

Remark 1.3. Clearly the maximal group of invariance G_1 of $\mathcal{F}_q = \mathcal{F}_t + \mathcal{F}_m$ always contains the intersection $G_2 \cap G_3$ of the maximal groups of invariance of $\mathcal{F}_t$ and $\mathcal{F}_m$. Therefore, in the cases 1 and 2 of the above theorem, we have $G_1 = G_2 \cap G_3$, so that, if $\mathcal{F}_t$ and $\mathcal{F}_m$ are independent, $\mathcal{F}_q$ is their independent union.

2. Systems of independent families

Let $\mathcal{F}_q(\underline{A}, \underline{B}) = \mathcal{F}_t(\underline{A}) + \mathcal{F}_m(\underline{B})$, $(q = t + m)$ be the family of systems of two independent families. Let us suppose that $\mathcal{F}_t$ and $\mathcal{F}_m$ are both measurable with respect to the maximal group of invariance G_r of $\mathcal{F}_q(\underline{A}, \underline{B})$, $r \geq q$, with respective densities

$$d\psi_t = \Phi_1(A_1, A_2, \ldots, A_t) dA_1 \wedge dA_2 \wedge \cdots \wedge dA_t, \tag{1}$$

and

$$d\psi_m = \Phi_2(B_1, B_2, \ldots, B_m) dB_1 \wedge dB_2 \wedge \cdots \wedge dB_m. \tag{2}$$

The group H_r associated to G_r in the parameter space X_q is measurable if and only if there exists a single non-trivial solution $\Phi = \Phi(C_1, C_2, \dots, C_q)$ of the Deltheil system

$$\sum_{i=1}^{q} \frac{\partial(\xi_h^i \Phi)}{\partial C_i} = 0, \quad h = 1, 2, \dots, r. \tag{3}$$

Here ξ_h^i are the coefficients of the infinitesimal transformations of the group H_r and $C_1, C_2, \dots, C_q$, the essential parameters of $\mathcal{F}_q$ ([2], [7]).

Since $\mathcal{F}_t$ and $\mathcal{F}_m$ are independent families, the Deltheil system (3) may be written in the form

$$\sum_{k=1}^{t} \frac{\partial(\xi_h^k \Phi)}{\partial A_k} + \sum_{j=1}^{m} \frac{\partial(\xi_h^{j+t} \Phi)}{\partial B_j} = 0, \quad h = 1, 2, \dots, r. \tag{4}$$

The coefficients of the infinitesimal transformations ([7]) are

$$\xi_h^i = \left(\frac{\partial C_i'}{\partial \alpha_h}\right)\bigg|_0,$$

where $\{C_1', C_2', \dots, C_q'\}$ is a group isomorphic to G_r and $\alpha_1, \alpha_2, \dots, \alpha_r$ are the parameters of G_r. The independence of $\mathcal{F}_t$ and $\mathcal{F}_m$ also implies that

$$\{C_1', C_2', \dots, C_q'\} = \{A_1', A_2', \dots, A_t', B_1', B_2', \dots, B_m'\},$$

where $\{A_1', A_2', \dots, A_t'\}$ is the contribution of $\mathcal{F}_t$ and $\{B_1', B_2', \dots, B_m'\}$ that of $\mathcal{F}_m$. That is, $\{A_1', A_2', \dots, A_t'\}$ and $\{B_1', B_2', \dots, B_m'\}$ jointly determine a group isomorphic to G_r and are associated to the families $\mathcal{F}_t$ and $\mathcal{F}_m$, respectively. Consequently, we have

$$\xi_h^k = \left(\frac{\partial A_k'}{\partial \alpha_h}\right)\bigg|_0, \quad k = 1, 2, \dots, t,$$
$$\xi_h^{j+t} = \left(\frac{\partial B_j'}{\partial \alpha_h}\right)\bigg|_0, \quad j = 1, 2, \dots, m.$$

Since $\mathcal{F}_t$ is measurable with respect to G_r then by (1) we obtain

$$\sum_{k=1}^{t} \frac{\partial\big(\xi_h^k \Phi_1(A_1, A_2, \dots, A_t)\big)}{\partial A_k} = 0, \quad h = 1, 2, \dots, r, \tag{5}$$

and since also $\mathcal{F}_m$ is measurable with respect to G_r, it follows from (2) that

$$\sum_{j=1}^{m} \frac{\partial\big(\xi_h^{j+t} \Phi_2(B_1, B_2, \dots, B_m)\big)}{\partial B_j} = 0, \quad h = 1, 2, \dots, r. \tag{6}$$

Now take $\Phi = \Phi_1\Phi_2$ and replace in (3). We get

$$\begin{aligned}\sum_{i=1}^{q} \frac{\partial(\xi_h^i \Phi)}{\partial C_i} &= \sum_{k=1}^{t} \frac{\partial(\xi_h^k \Phi)}{\partial A_k} + \sum_{j=1}^{m} \frac{\partial(\xi_h^{j+t} \Phi)}{\partial B_j} \\ &= \Phi_2(B_1, B_2, \dots, B_m) \sum_{k=1}^{t} \frac{\partial\big(\xi_h^k \Phi_1(A_1, A_2, \dots, A_t)\big)}{\partial A_k} \\ &\quad + \Phi_1(A_1, A_2, \dots, A_t) \sum_{j=1}^{m} \frac{\partial\big(\xi_h^{j+t} \Phi_2(B_1, B_2, \dots, B_m)\big)}{\partial B_j} = 0.\end{aligned} \tag{7}$$

The above argument shows that the Deltheil system associated to the group H_r, which is isomorphic to the maximal group of invariance G_r of the family of systems of two independent families, both measurable with respect to G_r, always has at least a non-trivial solution. Furthermore, (7) ensures the following result.

Theorem 2.1. *If $\mathcal{F}_q = \mathcal{F}_t + \mathcal{F}_m$ is the family of systems of two independent families of varieties $\mathcal{F}_t$, $\mathcal{F}_m$ $(q = t + m)$, both measurable with respect to the maximal group of invariance G_r of $\mathcal{F}_q$ $(r \geq q)$, the densities being $d\psi_t$ and $d\psi_m$, then $\mathcal{F}_q$ assumes, with respect to G_r, the density $d\psi_q = d\psi_t \wedge d\psi_m$.*

Corollary 2.2. *If the group H_r isomorphic to G_r in the parameter space X_q of $\mathcal{F}_q$ is transitive, then $\mathcal{F}_q$ is measurable, and its density is $d\psi_q = d\psi_t \wedge d\psi_m$.*

Remark 2.3. The assumption that $\mathcal{F}_t$ and $\mathcal{F}_m$ are both measurable with respect to the maximal group of invariance G_r of $\mathcal{F}_q$ implies that G_r is a group of invariance of both these families. Consequently, by Theorem 1.2, $\mathcal{F}_q$ is the independent union of $\mathcal{F}_t$ and $\mathcal{F}_m$.

More in general, the theorem holds whenever $\mathcal{F}_t$ and $\mathcal{F}_m$ have a density with respect to a same group of invariance.

Example 2.4. In the projective space $\mathbb{P}_3$, let $\mathcal{F}_9$ be the family of non-degenerate quadrics having elliptic points ($\Delta < 0$, where Δ is the determinant of the quadrics), and let $\mathcal{F}_6$ be the family of pairs plane–point, with the point out of the plane:

$$\begin{aligned}\mathcal{F}_9 : x^2 &+ (A^2 + C)y^2 + (B + D)z^2 + 2Axy + 2Bxz \\ &+ 2(AB + E)yz + 2Lx + 2My + 2Nz + P = 0,\end{aligned}$$

$$\mathcal{F}_6 : \begin{cases} A_1x + A_2y + A_3z + 1 = 0, \\ x = \alpha_1, \\ y = \alpha_2, \\ z = \alpha_3, \qquad \left(\sum_{i=1}^{3} A_i\alpha_i + 1 \neq 0\right). \end{cases}$$

These families are measurable ([3], [5]), with respective densities

$$d\psi_9 = |\Delta|^{-\frac{5}{2}} dA \wedge dB \wedge dC \wedge dD \wedge dE \wedge dL \wedge dM \wedge dN \wedge dP,$$

and

$$d\psi_6 = (A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 + 1)^{-4} dA_1 \wedge dA_2 \wedge dA_3 \wedge d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3.$$

The maximal group of invariance of the family of systems $\mathcal{F}_{15} = \mathcal{F}_9 + \mathcal{F}_6$ is the projective group G_{15}, with respect to which both $\mathcal{F}_9$ and $\mathcal{F}_6$ are measurable. Then $\mathcal{F}_{15}$ is the independent union of $\mathcal{F}_9$ and $\mathcal{F}_6$, and so, by Theorem 2.1, it assumes the density $d\psi_{15} = d\psi_9 \wedge d\psi_6$ with respect to G_{15}. Here $r = q = 15$, so that if the determinant $\overline{\Delta} = |\xi_h^i| \neq 0$, then Corollary 2.2 holds. Note that $\overline{\Delta}$ is a polynomial; hence, it is non-trivial as soon as we can find a point $Q \in X_{15}$ whose coordinates specify a variety $V \in \mathcal{F}_{15}$ such that $\overline{\Delta}(Q) \neq 0$. If we take the point

$$\begin{aligned} Q &= (A, B, C, D, E, L, M, N, P, A_1, A_2, A_3, \alpha_1, \alpha_2, \alpha_3) \\ &= (1, 1, 2, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1) \end{aligned}$$

then $\Delta = -2$ and $A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 + 1 = 4$. Therefore, the variety V associated to Q belongs to $\mathcal{F}_{15}$. Moreover, we have $\overline{\Delta}(Q) = -3072 \neq 0$. Hence, we can apply Corollary 2.2, to deduce that $\mathcal{F}_{15}$ is measurable, with density

$$d\psi_{15} = |\Delta|^{-\frac{5}{2}} \left(\sum_{i=1}^{3} (A_i\alpha_i + 1) \right)^{-4} dA \wedge \cdots \wedge dP \wedge dA_1 \wedge \cdots \wedge d\alpha_3.$$

In the same way we can handle the independent union $\overline{\mathcal{F}}_{15} = \overline{\mathcal{F}}_9 + \mathcal{F}_6$ of the measurable families [3]

$$\begin{aligned} \overline{\mathcal{F}}_9 : x^2 + (A^2 + C)y^2 + (B + D)z^2 + 2Axy + 2Bxz \\ + 2(AB + E)yz + 2Lx + 2My + 2Nz + P = 0 \end{aligned}$$

consisting of non-degenerate quadrics having hyperbolic points ($C(DC - E^2) \leq 0$ and $\Delta > 0$, where Δ is the determinant of the quadrics) and the previous $\mathcal{F}_6$. Here we can take the point

$$\begin{aligned} Q &= (A, B, C, D, E, L, M, N, P, A_1, A_2, A_3, \alpha_1, \alpha_2, \alpha_3) \\ &= (1, 1, 2, 0, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1). \end{aligned}$$

With this choice we have $\Delta = 1$ and $A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 + 1 = 4$. Therefore the variety V associated to Q belongs to $\overline{\mathcal{F}}_{15}$. Moreover $\overline{\Delta}(Q) = -384 \neq 0$ and hence, by [3] and Corollary 2.2, $\overline{\mathcal{F}}_{15}$ is measurable, with density

$$d\overline{\psi}_{15} = |\Delta|^{-\frac{5}{2}} \left(\sum_{i=1}^{3} (A_i\alpha_i + 1) \right)^{-4} dA \wedge \cdots \wedge dP \wedge dA_1 \wedge \cdots \wedge d\alpha_3.$$

Remark 2.5. If H_r is not transitive, the family $\mathcal{F}_q$ is not measurable, even though $r \geq q$ and the Deltheil system associated to H_r have a solution, as this solution is not unique. Here we have an unusual kind of non-measurability, different from that pointed out by the Stoka's second condition [8], since in this case the family $\mathcal{F}_q$ has different measures with respect to the same group, namely, the maximal group of invariance.

3. Iterated unions on a family of varieties

A special kind of independent union of families of varieties can be obtained by replacing the parameters $A_1, A_2, \dots, A_t$ of a family $\mathcal{F}_t$ by new parameters $B_1, B_2, \dots, B_t$ having no relation with the former, and by taking $\mathcal{F}_q(\underline{A}, \underline{B}) = \mathcal{F}_t(\underline{A}) + \mathcal{F}_t(\underline{B})$. In this case we write $\mathcal{F}_q = 2\mathcal{F}_t$. We can easily see that the groups of invariance of $\mathcal{F}_q$ and $\mathcal{F}_t$ are the same. Hence $\mathcal{F}_q$ actually is an independent union. By iterating this construction we can build up the family $\mathcal{F}_q(\underline{A}^1, \underline{A}^2, \dots, \underline{A}^m) = m\mathcal{F}_t$ that is the independent union of m copies of the family $\mathcal{F}_t$.

Let $\mathcal{F}_q = 2\mathcal{F}_t$ and assume $\mathcal{F}_t$ to be measurable with respect to the maximal group of invariance G_r, $r \geq 2t$ (so that $\mathcal{F}_t$ is measurable), with density

$$\psi_t = \phi_t(A_1, A_2, \dots, A_t) dA_1 \wedge dA_2 \wedge \dots \wedge dA_t.$$

Further assume that the coefficients of the infinitesimal transformations ξ_h^k, $k = 1, 2, \dots, t$; $h = 1, 2, \dots, r$, are polynomials of degree $d \leq 1$, that is

$$\xi_h^k = f_h^k(A_1, A_2, \dots, A_t) = \sum_{j=1}^{t} \lambda_{hj}^k A_j + \mu_h^k, \quad \lambda_{hj}^k,\ \mu_h^k \in \mathbb{R}. \tag{8}$$

The Deltheil system associated to $\mathcal{F}_q$ is

$$\sum_{i=1}^{q} \frac{\partial(\xi_h^i \Phi)}{\partial C_i} = \sum_{k=1}^{t} \frac{\partial(\xi_h^k \Phi)}{\partial A_k} + \sum_{k=1}^{t} \frac{\partial(\eta_h^k \Phi)}{\partial B_k}, \quad h = 1, 2, \dots, r, \tag{9}$$

where $\eta_h^k = f_h^k(B_1, B_2, \dots, B_t)$.

Now let $\Phi = [\Phi_t(A_1, A_2, \dots, A_t)]^\alpha [\Phi_t(B_1, B_2, \dots, B_t)]^{2-\alpha}$. Replacing in (9) we get

$$\sum_{k=1}^{t} \frac{\partial \left[\xi_h^k \Phi_t(\underline{A})^\alpha \Phi_t(\underline{B})^{2-\alpha}\right]}{\partial A_k} + \sum_{k=1}^{t} \frac{\partial \left[\eta_h^k \Phi_t(\underline{A})^\alpha \Phi_t(\underline{B})^{2-\alpha}\right]}{\partial B_k} =$$

$$
\begin{aligned}
&= \sum_{k=1}^{t} \left[\frac{\partial \xi_h^k}{\partial A_k} \Phi_t(\underline{A})^\alpha \Phi_t(\underline{B})^{2-\alpha} + \xi_h^k \alpha \Phi_t(\underline{A})^{\alpha-1} \frac{\partial \Phi_t(\underline{A})}{\partial A_k} \Phi_t(\underline{B})^{2-\alpha} + \right. \\
&\qquad \left. + \frac{\partial \eta_h^k}{\partial B_k} \Phi_t(\underline{A})^\alpha \Phi_t(\underline{B})^{2-\alpha} + \eta_h^k (2-\alpha) \Phi_t(\underline{A})^\alpha \Phi_t(\underline{B})^{1-\alpha} \frac{\partial \Phi_t(\underline{B})}{\partial B_k} \right] \\
&= \Phi_t(\underline{A})^\alpha \Phi_t(\underline{B})^{2-\alpha} \sum_{k=1}^{t} \left[\frac{\partial f_h^k(\underline{A})}{\partial A_k} + \alpha f_h^k(\underline{A}) \Phi_t(\underline{A})^{-1} \frac{\partial \Phi_t(\underline{A})}{\partial A_k} + \right. \\
&\qquad \left. + \frac{\partial f_h^k(\underline{B})}{\partial B_k} + (2-\alpha) f_h^k(\underline{B}) \Phi_t(\underline{B})^{-1} \frac{\partial \Phi_t(\underline{B})}{\partial B_k} \right].
\end{aligned}
\tag{10}
$$

Since $\mathcal{F}_t$ is measurable, we have

$$
\sum_{k=1}^{t} \left[f_h^k(\underline{A}) \frac{\partial \Phi_t(\underline{A})}{\partial A_k} + \frac{\partial f_h^k(\underline{A})}{\partial A_k} \Phi_t(\underline{A}) \right] = 0, \tag{11}
$$

and

$$
\sum_{k=1}^{t} \left[f_h^k(\underline{B}) \frac{\partial \Phi_t(\underline{B})}{\partial B_k} + \frac{\partial f_h^k(\underline{B})}{\partial B_k} \Phi_t(\underline{B}) \right] = 0. \tag{12}
$$

By (11) and (12) we have

$$
\sum_{k=1}^{t} \frac{\partial f_h^k(\underline{A})}{\partial A_k} = - \sum_{k=1}^{t} f_h^k(\underline{A}) \left[\Phi_t(\underline{A})\right]^{-1} \frac{\partial \Phi_t(\underline{A})}{\partial A_k}, \tag{13}
$$

and

$$
\sum_{k=1}^{t} \frac{\partial f_h^k(\underline{B})}{\partial B_k} = - \sum_{k=1}^{t} f_h^k(\underline{B}) \left[\Phi_t(\underline{B})\right]^{-1} \frac{\partial \Phi_t(\underline{B})}{\partial B_k}. \tag{14}
$$

Moreover, from (8) we obtain

$$
\frac{\partial f_h^k(\underline{A})}{\partial A_k} = \frac{\partial f_h^k(\underline{B})}{\partial B_k} = \lambda_{hk}^k. \tag{15}
$$

Therefore, by replacing (13), (14) and (15) in (10), we get

$$
\begin{aligned}
&\Phi_t(\underline{A})^\alpha \Phi_t(\underline{B})^{2-\alpha} \sum_{k=1}^{t} \left[\frac{\partial f_h^k(\underline{A})}{\partial A_k} - \alpha \frac{\partial f_h^k(\underline{A})}{\partial A_k} + \frac{\partial f_h^k(\underline{B})}{\partial B_k} - (2-\alpha) \frac{\partial f_h^k(\underline{B})}{\partial B_k} \right] \\
&\qquad = (\alpha - 1) \Phi_t(\underline{A})^\alpha \Phi_t(\underline{B})^{2-\alpha} \sum_{k=1}^{t} \left[\frac{\partial f_h^k(\underline{B})}{\partial B_k} - \frac{\partial f_h^k(\underline{A})}{\partial A_k} \right] = 0.
\end{aligned}
$$

Consequently, the following result holds.

Theorem 3.1. *Let $\mathcal{F}_t = \mathcal{F}_t(A_1, A_2, \dots, A_t)$ be a family of varieties which is measurable with respect to its maximal group of invariance G_r, with density*

$$\Phi_t(A_1, A_2, \dots, A_t) dA_1 \wedge dA_2 \wedge \cdots \wedge dA_t.$$

If the coefficients of the infinitesimal transformations are polynomials of degree $d \leq 1$, then the independent union $\mathcal{F}_q = 2\mathcal{F}_t$, $r \geq q = 2t$, is a non-trivially non-measurable family, and it assumes at least the densities

$$[\Phi_t(A_1, A_2, \dots, A_t)]^{\alpha} [\Phi_t(B_1, B_2, \dots, B_t)]^{2-\alpha} dA_1 \wedge \cdots \wedge dA_t \wedge dB_1 \wedge \cdots \wedge dB_t,$$

for every $\alpha \in \mathbb{R}$.

Remark 3.2. A family $\mathcal{F}_q$ is said to be a trivially non-measurable family of varieties, if its maximal group of invariance G_r depends on $r < q$ parameters. In this case the group H_r isomorphic to G_r is not transitive ([1], [2]).

By an analogous argument we can also state the following theorem.

Theorem 3.3. *Let $\mathcal{F}_t = \mathcal{F}_t(A_1, A_2, \dots, A_t)$ be a family of varieties which is measurable with respect to its maximal group of invariance G_r, with density*

$$\Phi_t(A_1, A_2, \dots, A_t) dA_1 \wedge dA_2 \wedge \cdots \wedge dA_t.$$

If the coefficients of the infinitesimal transformations are polynomials of degree $d \leq 1$, then the independent union $\mathcal{F}_q = m\mathcal{F}_t$, $r \geq q = mt$, is, for every integer $m \neq 1$, a non-trivially non-measurable family, and it assumes at least the densities

$$\left[\Phi_t(\underline{A}^1)\right]^{\alpha_1} \left[\Phi_t(\underline{A}^2)\right]^{\alpha_2} \cdots [\Phi_t(\underline{A}^m)]^{\alpha_m} \wedge^t d\underline{A}^1 \wedge^t d\underline{A}^2 \wedge^t \cdots \wedge^t d\underline{A}^m,$$

where $\underline{A}^i = (A_1^i, A_2^i, \dots, A_t^i)$, $i = 1, 2, \dots, m$, is the set of parameters of the i-th copy of $\mathcal{F}_t$, $\wedge^t d\underline{A}^i = dA_1^i \wedge dA_2^i \wedge \cdots \wedge dA_t^i$, and $\alpha_1, \alpha_2, \dots, \alpha_m$ are real numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_m = m$.

Example 3.4. From [11] we know that the family $\mathcal{F}_{n+1}$ of hyperspheres of the n-dimensional projective space $\mathbb{P}_n$

$$x_1^2 + x_2^2 + \cdots + x_n^2 - 2u_1x_1 - 2u_2x_2 - \cdots - 2u_nx_n + v = 0,$$

is measurable with respect to its maximal group of invariance, namely the group of similarities $G_{\frac{n(n+1)}{2}+1}(\rho, \underline{\alpha}, \underline{\theta})$.

Here $\rho, \underline{\alpha} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and

$$\underline{\theta} = \{\theta_{12}, \dots, \theta_{1n}, \theta_{23}, \dots, \theta_{2n}, \dots, \theta_{(n-1)n}\}$$

are, respectively, the homothety, translation and rotation parameters. The density is

$$d\psi_{n+1} = R^{-(n+1)} du_1 \wedge du_2 \wedge \cdots \wedge du_n \wedge dR,$$

where

$$R = \left[\left(\sum_{j=1}^{n} u_j^2\right) - v\right]^{-\frac{1}{2}}$$

is the radius of the hyperspheres. The coefficients of the infinitesimal transformations are ([11])

$$\left(\frac{\partial u_j'}{\partial \rho}\right)\Big|_0 = -u_j, \quad j = 1, 2, \ldots, n,$$

$$\left(\frac{\partial u_j'}{\partial \alpha_i}\right)\Big|_0 = \begin{cases} 0, & \text{if } \ j \neq i, \\ -1, & \text{if } \ j = i, \quad j, i = 1, 2, \ldots, n, \end{cases}$$

$$\left(\frac{\partial u_j'}{\partial \theta_{hk}}\right)\Big|_0 = \begin{cases} 0, & \text{if } \ j \neq h, k, \\ u_k, & \text{if } \ j = k, \\ -u_h, & \text{if } \ j = h, \quad j = 1, 2, \ldots, n, \quad h < k = 1, 2, \ldots, n, \end{cases}$$

$$\left(\frac{\partial v'}{\partial \rho}\right)\Big|_0 = -2v,$$

$$\left(\frac{\partial v'}{\partial \alpha_i}\right)\Big|_0 = -2u_i, \quad i = 1, 2, \ldots, n,$$

$$\left(\frac{\partial v'}{\partial \theta_{hk}}\right)\Big|_0 = 0, \quad h < k = 1, 2, \ldots, n.$$

Note that all of them are polynomials of degree $d \leq 1$.

Let m, n be two integers such that

$$m(n+1) \leq \frac{n(n+1)}{2} + 1.$$

For fixed m this holds for any $n \geq n_0$, where n_0 is the least integer greater than

$$\frac{2m + 1 + \sqrt{4m^2 + 4m - 7}}{2},$$

so that $n \geq 2m$. By Theorem 3.3, the independent union $\mathcal{F}_{m(n+1)} = m\mathcal{F}_{n+1}$,

$$(x_1 - u_{11})^2 + (x_2 - u_{12})^2 + \cdots + (x_n - u_{1n}^2) = R_1^2,$$
$$(x_1 - u_{21})^2 + (x_2 - u_{22})^2 + \cdots + (x_n - u_{2n}^2) = R_2^2,$$
$$\vdots$$
$$(x_1 - u_{m1})^2 + (x_2 - u_{m2})^2 + \cdots + (x_n - u_{mn}^2) = R_m^2,$$

is, for every integer $m \geq 2$, a non-trivially non-measurable family of varieties of the projective space $\mathbb{P}_{2m}$. This family assumes, at least, the densities

$$[R_1^{a_1} R_2^{a_2} \cdots R_m^{a_m}]^{-(n+1)}\, du_{11} \wedge \cdots \wedge du_{mn} \wedge dR_1 \wedge \cdots \wedge dR_m, \quad \sum_{s=1}^{m} a_s = m.$$

If $a_1 = a_2 = \cdots = a_m = 1$, this yields the product measure.

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