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depend continuously on its coefficients
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THE ROOTS OF A POLYNOMIAL DEPEND CONTINUOUSLY ON ITS COEFFICIENTS

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Abstract. An elementary proof is given of the continuous dependence of the roots of a polynomial on its coefficients.

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Definition 1. *We say that the complex number $u = \alpha + i\beta$ is lexicographically less than the complex number $v = \gamma + i\delta$ if $\alpha < \gamma$ or $\delta = \gamma$ and $\beta < \delta$. We denote this by writing $u \prec v$. The notation $u \preceq v$ means that $u \prec v$ or $u = v$.*

With $\mathcal{C}_n$ we denote the set of n–tuples of complex numbers lexicographically ordered from less to greater. Thus $(x_1, x_2, \ldots, x_n) \in \mathcal{C}_n$ iff $x_1 \preceq x_2 \preceq \ldots \preceq x_n$. With $\vec{0}$ we denote the n–tuple $(0, 0, \ldots, 0)$.

On the set $\mathcal{C}_n$ we define the metric

$$d(x,y) = \sqrt{\sum_{j=1}^{n} |x_j - y_j|^2}.$$

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The pair $(\mathcal{C}_n, d)$ is a complete metric space. In fact, $\mathcal{C}_n$ is a closed subset of the normed complex euclidean space, *i.e.*, of $(\mathbb{C}^n, \|\cdot\|)$, where the norm of $\vec{a} \in \mathbb{C}^n$ is

$$\|\vec{a}\| = \sqrt{\sum_{j=1}^{n} |a_j|^2}.$$

Now let

$$P(z) = z^n - a_1 z^{n-1} + \ldots + (-1)^n a_n$$

be a polynomial, and consider its coefficients as a vector in $\mathbb{C}^n$:

$$\vec{a} = (a_1, a_2, \ldots, a_n).$$

From the fundamental theorem af algebra [1] we know that $p(x)$ has n roots. We will denote these roots by $\lambda_i, i = 1, 2, \ldots, n$, and assume that $\lambda_i \preceq \lambda_{i+1}$, so that the vector

$$\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$$

is in $\mathcal{C}_n$. From the well known formulae of VIÈTE we have the identities

$$\begin{aligned} a_1 &= \lambda_1 + \lambda_2 + \cdots + \lambda_n \\ a_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n \\ a_n &= \lambda_1\lambda_2 \cdots \lambda_n, \end{aligned}$$

by means of which we can define in an obvious manner a continuous map

$$\mathcal{T} : \mathcal{C}_n \to \mathbb{C}^n;\ \vec{\lambda} \mapsto \mathcal{T}(\vec{\lambda}) = \vec{a}$$

From the fundamental theorem of algebra this function is one to one and onto, *i.e.*, $\mathcal{T}$ establishes a biyective correspondence between $\mathcal{C}_n$ and $\mathbb{C}^n$.

Let $\mathcal{S}$ denote the inverse mapping of $\mathcal{T}$:

$$\mathcal{S} := \mathcal{T}^{-1} : \mathbb{C}^n \to \mathcal{C}_n$$

Lemma 1.

$$d(\mathcal{S}(\vec{a}), \vec{0}) \leq 2n \max\{1, \|\vec{a}\|\}$$

Proof: Let $\mathcal{S}(\vec{a}) = (\lambda_1, \ldots, \lambda_n)$. Then

$$|\lambda_1|^n \leq \sum_{j=i}^{n} |a_j| \big(1 + |\lambda_1|^{n-j}\big). \tag{1}$$

Now, if $|\lambda_i| \leq 1$, then

$$|\lambda_i| \leq 2\sqrt{n} \max\{1, \|\vec{a}\|\}, \tag{2}$$

and if $|\lambda_i| \geq 1$, then, dividing (1) by $|\lambda_i|^{n-1}$, we obtain

$$\begin{aligned} |\lambda_i| &\leq \sum_{j=1}^{n} |a_j| \big(1 + |\lambda_i|^{1-j}\big) \\ &\leq 2 \sum_{j=1}^{n} |a_j| \leq 2\sqrt{n}\, \|\vec{a}\| \\ &\leq 2\sqrt{n} \max\{1, \|\vec{a}\|\}. \end{aligned} \tag{3}$$

From (2) and (3) the proof follows. □

Theorem 1. *The function* $\mathcal{S} : \mathbb{C}^n \to \mathcal{C}_n$ *is continuous.*

Proof: Assume that $\mathcal{S}$ is not continuous at a point $\vec{a}$. Then there exist $\delta > 0$ and a sequence $(\vec{a}_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \vec{a}_n = \vec{a}$ and

$$d\big(\mathcal{S}(\vec{a}_n), \mathcal{S}(\vec{a})\big) \geq \delta. \tag{4}$$

Because of Lemma 1, the sequence $\big(\mathcal{S}(\vec{a}_n)\big)_{n=1}^{\infty}$ is bounded, and therefore (passing to a subsequence, if necessary) we can assume that this sequence has a limit:

$$\lim_{n\to\infty} \mathcal{S}(\vec{a}_n) = \vec{\xi}.$$

But from the continuity of $\mathcal{T}$ we have

$$\mathcal{T}(\xi) = \lim_{n\to\infty} \mathcal{T}\big(\mathcal{S}(\vec{a}_n)\big) = \lim_{n\to\infty} \vec{a}_n = \vec{a},$$

and therefore $\vec{\xi} = \mathcal{S}(\vec{a})$. Hence, for n sufficiently large

$$d\big(\mathcal{S}(\vec{a}_n), \mathcal{S}(\vec{a})\big) < \delta,$$

which contradicts (4). □

References

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