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Categorical properties of iterated power

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Abstract. In [2], the class of the Lawson monads was introduced which contains sufficiently wide class of monads and have a functional representation. Unfortunately, the powers monads are not in this class. We introduce in this paper the iterated power monad and show that it is a Lawson monad.

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0. Introduction

The algebraic aspect of the theory of functors in categories of topological spaces and continuous maps was investigated rather recently. It is based, mainly, on the existence of monad (or triple) structure in the sense of S.Eilenberg and J.Moore [1].

In [2], the class of the Lawson monads was introduced which contains sufficiently wide class of monads. Lawson monads have a functional representation, i.e., their functorial part FX can be naturally imbedded in $\mathbb{R}^{CX}$. It was shown in [3] that the power monad is not a Lawson monad. In this paper we introduce the iterated power monad and show that it is a Lawson monad.

The paper is arranged in the following manner. In 1 we give some necessary definition. In 2 we introduce the iterated power monad in the category of $\mathcal{T}ych$ and in 3 we use some completion to obtain a monad in $\mathcal{C}omp$ and show that this monad is Lawson.

1. Preliminaries

By $\mathcal{C}omp$ ($\mathcal{T}ych$) we denote the category of compact Hausdorff spaces (Tychonov spaces) and continuous maps.

We need some definitions concerning monads and algebras. A *monad* $\mathbb{T} = (T, \eta, \mu)$ in a category $\mathcal{E}$ consists of an endofunctor $T : \mathcal{E} \to \mathcal{E}$ and natural transformations $\eta : \mathrm{Id}_{\mathcal{E}} \to T$ (unity), $\mu : T^2 \to T$ (multiplication) satisfying the relations $\mu \circ T\eta = \mu \circ \eta T = \mathbf{1}_T$ and $\mu \circ \mu T = \mu \circ T\mu$.

Let $\mathbb{T} = (T, \eta, \mu)$ be a monad in a category $\mathcal{E}$. The pair (X, ξ) is called a $\mathbb{T}$-*algebra* if $\xi \circ \eta X = id_X$ and $\xi \circ \mu X = \xi \circ T\xi$. Let (X, ξ), (Y, ξ') be two $\mathbb{T}$-algebras. A map $f : X \to Y$ is called a $\mathbb{T}$-algebras morphism if $\xi' \circ Tf = f \circ \xi$.

For any real $t \geq 0$,we denote by I_t the segment $[-t, t]$. If t_1, t_2 are real numbers with $0 \leq t_1 \leq t_2$, by $j_{t_1}^{t_2}$ we denote the natural embedding $j_{t_1}^{t_2} : I_{t_1} \to I_{t_2}$.

Let $\mathbb{T} = (T, \eta, \mu)$ be a monad in the category $\mathcal{C}omp$. A family of $\mathbb{T}$-algebras $\{\xi_t : TI_t \to I_t \mid t \geq 0\}$ is called *coherent* iff for each t_1, $t_2 \in \mathbb{R}$ with $0 \leq t_1 \leq t_2$ the embedding $j_{t_1}^{t_2}$ is an $\mathbb{T}$-algebras morphism. A monad $\mathbb{T} = (T, \eta, \mu)$ is called *Lawson* if there exists a coherent family of $\mathbb{T}$-algebras $\{\xi_t : TI_t \to I_t \mid t \geq 0\}$ such that for each $X \in \mathcal{C}omp$ there exists a point-separating family of $\mathbb{T}$-algebras morphisms $\{f_\alpha : (TX, \mu X) \to (I_{t(\alpha)}, \xi_{t(\alpha)}) \mid \alpha \in A\}$ (see [2]).

For $X \in \mathcal{C}omp$ and $n \in \mathbb{N}$ by $D_n X$ we denote the compactum X^n. For a map $f : X \to Y$ we define the map $D_n f : D_n X \to D_n Y$ by the rule $D_n f(x_1, \dots, x_n) = (f(x_1), \dots, f(x_n))$. One can check that D_n is a covariant functor on $\mathcal{C}omp$.

For $X \in \mathcal{C}omp$ define the maps $\gamma X : X \to D_n X$ and $\mu X : D_n^2 X \to D_n X$ by the formulas

$$\gamma X(x) = (x, \dots, x)$$

for $x \in X$ and

$$\mu X((x_1^1, \dots, x_n^1), \dots, (x_1^n, \dots, x_n^n)) = (x_1^1, \dots, x_n^n)$$

for $((x_1^1, \dots, x_n^1), \dots, (x_1^n, \dots, x_n^n)) \in D_n^2 X = D_n(D_n X)$.

It is known that the triple $\mathbb{D}_n = (D_n, \gamma, \mu)$ is a monad in the category $\mathcal{C}omp$, where $\gamma = \{\gamma X\} : \mathrm{Id}_{\mathcal{C}omp} \to D_n$ and $\mu = \{\mu X\} : D_n^2 \to D_n$ are the natural transformations defined above [4]. This monad is called the power monad. It was shown in [3] that $\mathbb{D}_n$ is not Lawson.

2. The iterated power construction

Let us consider any $X \in \mathcal{T}ych$. For each $n \in \mathbb{N} \cup \{0\}$ we consider the product X^{2^n} as a subset of $X^{2^{n+1}}$ with the natural inclusion $X^{2^n} = \{(x, y) \in X^{2^{n+1}} \mid x = y\}$. Put $X_\infty = \bigcup_{i=0}^\infty X^{2^i}$.

Let us define a function $f:\mathbb{R}_\infty\to\mathbb{R}$ by the induction: put $f_0=\mathrm{id}_\mathbb{R}:\mathbb{R}\to\mathbb{R}$. Assume that we have defined $f_i:\mathbb{R}^{2^i}\to\mathbb{R}$ for each $i<n\geq 1$. Let us define $f_n:\mathbb{R}^{2^n}\to\mathbb{R}$ by the formula

$$\begin{aligned}f_n(a_1,a_2)=\frac{1}{3}\min\{1,|f_{n-1}(a_1)-f_{n-1}(a_2)|\}f_{n-1}(a_1)\\+(1-\frac{1}{3}\min\{1,|f_{n-1}(a_1)-f_{n-1}(a_2)|\})f_{n-1}(a_2)\end{aligned}$$

for $(a_1,a_2)\in\mathbb{R}^{2^n}$. We can define the function $f:\mathbb{R}_\infty\to\mathbb{R}$ by the condition $f|\mathbb{R}^{2^i}=f_i$ for each $i\in\mathbb{N}\cup\{0\}$.

Lemma 1. *For each $(a,b),(c,d)\in\mathbb{R}^2$ with $a\leq c$ and $b\leq d$ we have $f_2(a,b)\leq f_2(c,d)$.*

Proof. Firstly we will show that $f_2(a,b)\leq f_2(a,d)$ whenever $b\leq d$. We can assume that $a=0$. Consider the case $b\geq 0$. If $b\geq 1$, the inequality is obvious. If $b\leq 1\leq c$, we have $f_2(0,b)=(1-\frac{1}{3}b)b\leq\frac{2}{3}<1\leq f_2(0,c)$. In the case $b\leq c\leq 1$ we have

$$\begin{aligned}f_2(0,c)-f_2(0,b)&=(c-b)-\frac{1}{3}(c^2-b^2)\\&=(c-b)(1-\frac{1}{3}(c+b))\geq 0.\end{aligned}$$

If $b\leq 0\leq c$ the inequality is obvious. The proof of the case $c\leq 0$ is similar to the case $b\geq 0$.

Let us prove that $f_2(a,b)\leq f_2(c,b)$, whenever $a\leq c$. We can assume $b=0$. Then $f_2(a,0)=\frac{1}{3}|a|a\leq\frac{1}{3}|c|c=f_2(c,0)$.

Finally, if $a\leq c$ and $b\leq d$ we have $f_2(a,b)\leq f_2(a,d)\leq f_2(c,d)$. The lemma is proved. ☑

We consider in $\mathbb{R}_\infty$ the coordinate-wise partial order. Using the induction to Lemma 1 one can obtain the following lemma.

Lemma 2. *For each $a,b\in\mathbb{R}_\infty$ with $a\leq b$ we have $f(a)\leq f(b)$.*

Consider any $X\in\mathcal{T}ych$. By $BC(X)$ we denote the set of all continuous bounded functions from X to $\mathbb{R}$.

Let $g:X\to Y$ be any morphism in $\mathcal{T}ych$. Let us define a function $g_\infty:X_\infty\to Y_\infty$ as follows. If the function $g_\infty|X^{2^i}:X^{2^i}\to Y^{2^i}$ is defined, define $g_\infty|X^{2^{i+1}}:X^{2^{i+1}}\to Y^{2^{i+1}}$ by the formula $g_\infty|X^{2^{i+1}}(x,y)=(g_\infty|X^{2^i}(x),f_\infty|X^{2^i}(y))$ for each $(x,y)\in X^{2^{i+1}}$.

Lemma 3. *For each* $x = (x_1, \dots, x_{2^i}), y = (y_1, \dots, y_{2^i}) \in X^{2^i} \subset X_\infty$ *such that* $x_{2^i} \neq y_{2^i}$ *there exists a function* $\varphi \in BC(X)$ *such that* $f \circ \varphi_\infty(x) \neq f \circ \varphi_\infty(y)$.

Proof. Choose $a \in (0, 1]$ such that $(1 - \frac{a}{3})^i > \frac{1}{2}$. Consider a function $\varphi : X \to [0, a]$ such that $\varphi(x_{2^i}) = 0$ and $\varphi(y_{2^i}) = a$. Then we have

$$\begin{aligned} f \circ \varphi_\infty(x) &= f(\varphi(x_1), \dots, \varphi(x_{2^i-1}), 0) \\ &\leq f(a, \dots, a, 0) \leq (1 - (1 - \frac{a}{3})^i)a \\ &< \frac{a}{2} < (1 - \frac{a}{3})^i a \leq f(0, \dots, 0, a) \\ &\leq f(\varphi(y_1), \dots, \varphi(y_{2^i})) = f \circ \varphi_\infty(y). \end{aligned}$$

The lemma is proved. ☑

Let $X \in \mathcal{T}ych$. Consider an equivalence relation ρ in the set X_∞ defined by $x\rho y$ iff $f \circ \varphi_\infty(x) = f \circ \varphi_\infty(y)$ for each $\varphi \in BC(X)$. Denote by $S_\infty X$ the identification set X_∞/ρ. The class of equivalence with a representative $x \in X_\infty$ will be denoted by $[x]$.

We are going to define a topology of $S_\infty X$. For each $\varphi \in BC(X)$ define a function $\varphi_f : S_\infty X \to \mathbb{R}$ by the formula $\varphi_f([x]) = f \circ \varphi_\infty(x)$. Define the map $p_i : X^{2^i} \to X$ by the formula $p_i(x_1, \dots, x_{2^i}) = x_{2^i}$. The condition $p|X^{2^i} = p_i$ defines the map $p : X_\infty \to X$. Let us define for each $\varphi \in BC(X)$ the function $\varphi_p : S_\infty X \to \mathbb{R}$ by the formula $\varphi_p([x]) = \varphi(p(x))$. It follows from Lemma 3 that the map φ_p is well-defined. We will consider $S_\infty X = \bigcup_{n=0}^\infty S_n X$ where $S_n X = \{[x] \in S_\infty X \mid \text{there exists } y \in X^{2^i} \text{ such that } y \in [x]\}$.

Consider a family $\mathcal{F}(X) \subset BC(S_\infty X)$ defined as

$$\mathcal{F}(X) = \{\varphi_f \mid \varphi \in BC(X)\} \cup \{\varphi_p \mid \varphi \in BC(X)\}.$$

For $\varphi_1, \dots, \varphi_n \in \mathcal{F}(X)$ define the function $d_{\varphi_1,\dots,\varphi_n} : S_\infty \times S_\infty \to \mathbb{R}$ by the formula

$$d_{\varphi_1,\dots,\varphi_n}(x, y) = \max\{|\varphi_i(x) - \varphi_i(y)| : i \in \{1, \dots, n\}\}.$$

It is easy to check that $d_{\varphi_1,\dots,\varphi_n}$ is a pseudometric of $S_\infty X$. The family of pseudometrics $\{d_{\varphi_1,\dots,\varphi_n} \mid n \in \mathbb{N} \text{ and } \varphi_1, \dots, \varphi_n \in \mathcal{F}(X)\}$ defines a uniformity $\mathcal{U}_{\mathcal{F}(X)}$ of $S_\infty X$. We consider $S_\infty X$ with the topology generated by this uniformity.

By $\eta X : X \to S_\infty X$ we denote the continuous map defined by the formula $\eta X(x) = [(x)]$ for $x \in X$. Let us remark that $\eta X(X) = S_0 X$.

For each continuous map $g : X \to Y$ define a function $S_\infty g : S_\infty X \to S_\infty Y$ by the formula $S_\infty g([x]) = [g_\infty(x)]$ for $x \in X_\infty$. It is easy to see that the function $S_\infty g$ is well defined.

Lemma 4. *For each* $\varphi \in BC(X)$ *we have* $\varphi_f \circ S_\infty g = (\varphi \circ g)_f$ *and* $\varphi_p \circ S_\infty g = (\varphi \circ g)_p$.

Proof. Let $[x] \in S_\infty X$. We have $\varphi_f \circ S_\infty g([x]) = \varphi_f[g_\infty(x)] = f(\varphi_\infty(g_\infty(x))) = f((\varphi \circ g)_\infty(x)) = (\varphi \circ g)_f([x])$ and $\varphi_p \circ S_\infty g([x]) = \varphi_p[g_\infty(x)] = f(p(g_\infty(x))) = f \circ g(p(x)) = (\varphi \circ g)_p([x])$. The lemma is proved. ☑

Corollary 1. *The map* $S_\infty g : (S_\infty X, \mathcal{U}_{\mathcal{F}(X)}) \to (S_\infty Y, \mathcal{U}_{\mathcal{F}(Y)})$ *is uniformly continuous.*

In particular, we have that the map $S_\infty g$ is continuous and is easy to check that S_∞ is a covariant functor in the category $\mathcal{T}ych$.

For each $a, b \in X^{2^n}$ we can consider the element $(a, b) \in X^{2^{n+1}}$. Define the map $s : S_\infty X \times S_\infty X \to S_\infty X$ by the formula $s([a], [b]) = [(a, b)]$ for $[a], [b] \in S_\infty X$. It is easy to check that the map s is well-defined.

By S_∞^2 we denote the iteration $S_\infty \circ S_\infty$ of the functor S_∞. Now we are going to define a map $\mu x : S_\infty^2 X \to S_\infty X$ for each $X \in \mathcal{T}ych$. Put $\mu X | S_0(S_\infty X) = (\eta S_\infty X)^{-1}$. Assume that we have defined $\mu X | S_i(S_\infty X)$ for each $i < n \geq 1$. Consider any $[x] \in S_n(S_\infty X)$. Then $[x] = [(x_1, x_2)]$ where $x_1, x_2 \in (S_\infty X)^{2^{n-1}}$. Put $\mu X([x]) = s(\mu X([x_1]), \mu X([x_2]))$.

Lemma 5. *For each* $\varphi \in BC(X)$ *we have* $\varphi_f \circ \mu X = (\varphi_f)_f$ *and* $\varphi_p \circ \mu X = (\varphi_p)_p$.

Proof. Consider any $[x] \in S_\infty^2 X$. For $[x] \in S_0(S_\infty X)$ we have $(\varphi_f)_f([x]) = \varphi_f(\eta X^{-1}([x])) = \varphi_f(\mu X([x])$.

Assume that we have proved the equality for $[x] \in S_i(S_\infty X)$ for each $i < n \geq 1$. Consider any $[x] \in S_n(S_\infty X)$. Then $[x] = [(x_1, x_2)]$ where $x_1, x_2 \in (S_\infty X)^{2^{n-1}}$. Then we have

$$\begin{aligned}
\varphi_f \circ \mu X([x]) &= \varphi_f(s(\mu X([x_1]), \mu X([x_2]))) \\
&= f(\varphi_f \circ \mu_X([x_1]), \varphi_f \circ \mu_X([x_2])) \\
&= f((\varphi_f)_f([x_1]), (\varphi_f)_f([x_2])) = (\varphi_f)_f([x]).
\end{aligned}$$

The equality $\varphi_p \circ \mu X = (\varphi_p)_p$ can be proved analogously. ☑

Corollary 2. *The map* $\mu X : (S_\infty^2 X, \mathcal{U}_{\mathcal{F}(S_\infty X)}) \to (S_\infty X, \mathcal{U}_{\mathcal{F}(X)})$ *is uniformly continuous.*

It is easy to check that the maps $\eta X : X \to S_\infty X$ and $\mu X : S_\infty^2 X \to S_\infty X$ are the components of the natural transformations $\eta : Id_{\mathcal{T}ych} \to S_\infty$ and $\mu : S_\infty^2 \to S_\infty$.

Theorem 1. *The triple* (S_∞, η, μ) *forms a monad in the category* $\mathcal{T}ych$.

Proof. Let $X \in \mathcal{T}ych$. The equality $\mu X \circ \eta S_\infty X = \mathrm{id}_{S_\infty X}$ follows from the definition. Let us prove the equality $\mu X \circ \eta S_\infty X = \mathrm{id}_{S_\infty X}$. For $[x] \in S_0 X$ we have $S_\infty \eta X = \eta S_\infty X([x])$, so $\mu X \circ S_\infty \eta X([x]) = [x]$.

Let us assume that we have proved the equality for $[x] \in S_iX$ for $i < n \geq 1$. Consider any $[x] \in S_nX$. Then $[x] = [(x_1, x_2)]$ where $x_1, x_2 \in X^{2^{n-1}}$. We have

$$\begin{aligned}\mu X \circ S_\infty \eta([x]) &= \mu X[((\eta X)_\infty([x_1]), \circ(\eta X)\infty([x_2]))] \\ &= s(\mu X \circ S_\infty \eta X([x_1]), \mu X \circ S_\infty \eta X([x_2]) \\ &= s([x_1], [x_2]) = [x].\end{aligned}$$

Now, let us prove the equality $\mu x \circ S_\infty \mu X = \mu X \circ \mu S_\infty X$. For $\alpha \in S_0(S^2_\infty X)$ we have $\mu X \circ S_\infty \mu X(\alpha) = \mu_X((\eta S^2_\infty X)^{-1}(\alpha)) = \mu X \circ \mu S_\infty X(\alpha)$.

Let us assume that we have proved the equality for $\alpha \in S_i(S^2_\infty X)$ for $i < n \geq 1$. Consider any $\alpha \in S_n(S^2_\infty X)$. Then $\alpha = [(\alpha_1, \alpha_2)]$ where $\alpha_1, \alpha_2 \in (S^2_\infty X)^{2^{n-1}}$. Then we have

$$\begin{aligned}\mu X \circ S_\infty \mu X([(\alpha_1, \alpha_2)]) &= s(\mu x \circ \mu S_\infty X[\alpha_1], \mu x \circ \mu S_\infty X[\alpha_2]) \\ &= \mu X \circ \mu S_\infty X([(\alpha_1, \alpha_2)]).\end{aligned}$$

The theorem is proved. ☑

3. The iterated power monad

In this section we will define a monad in the category $\mathcal{C}omp$ using the construction of the iterated product and we are going to prove that this monad is a Lawson monad.

It is easy to check that the uniformity $\mathcal{U}_{\mathcal{F}(X)}$ defined in $S_\infty X$ is totally bounded. For each $X \in \mathcal{C}omp$ define a uniform space $(SX, \mathcal{V}_{\mathcal{F}(X)})$ being a completion of the uniform space $(S_\infty X, \mathcal{U}_{\mathcal{F}(X)})$. Then we have $SX \in \mathcal{C}omp$. We consider $S_\infty X$ as a subset of SX which is certainly dense.

Consider any morphism $f : X \to Y$. We have by Corollary 1 that the map $S_\infty f : S_\infty X \to S_\infty Y$ is uniformly continuous. Thus, there exists a unique extension $Sf : SX \to SY$. We have defined the functor $S : \mathcal{C}omp \to \mathcal{C}omp$.

For each $X \in \mathcal{C}omp$ define the map $hX : X \to SX$ by

$$hX(x) = \eta X(x) \in S_\infty X \subset SX.$$

Now, the set $S^2_\infty X$ is dense in S^2X. It follows from Corollary 2 that the map $\mu X : S^2_\infty X \to S_\infty X$ is uniformly continuous. Hence there exists a unique extension $mX : S^2X \to SX$. We have defined the natural transformations $h : \mathrm{Id}_{\mathcal{C}omp} \to S$ and $m : S^2 \to S$. Since the map mX is the extension of μX, the triple $\mathbb{S} = (S, h, \mu)$ is a monad in the category $\mathcal{C}omp$.

Theorem 2. *The monad $\mathbb{S}$ is a Lawson monad.*

Proof. For $t \geq 0$ define the map $f_t : S_\infty I_t \to I_t$ by the formula $f_t([x]) = (j_t)_f$ where $j_t : I_t \to \mathbb{R}$ is the natural embedding. It follows from the definition of the uniformity $\mathcal{U}_{\mathcal{F}(X)}$ that the map f_t is uniformly continuous (we consider I_t with the uniformity generated by the natural metric). Hence there exists a unique extension $\xi_t : SI_t \to I_t$.

Let us show that the pair (I_t, ξ_t) is an $\mathbb{S}$-algebra. Since $S_\infty X$ and $S^2_\infty X$ are dense subsets of SX and S^2X, it is enough to prove the equalities $f_t \circ \eta I_t = \mathrm{id}_{I_t}$ and $f_t \circ \mu X = f_t \circ S_\infty f_t$. The first equality is obvious. Let us remark that $f_t \circ S_\infty f_t = ((j_t)_f)_f$. Now, the second equality follows from Lemma 5.

It is easy to see that the family $\{(I_t, \xi_t) | t \geq 0\}$ is coherent.

Finally, for each $X \in \mathcal{C}omp$ the family $\{f_{\|\varphi\|} \circ S_\infty \varphi \mid \varphi \in CX\}$ separates points of $S_\infty X$ by the definition of $S_\infty X$. Hence, the family $\{\xi_{\|\varphi\|} \circ S_\infty \varphi \mid \varphi \in CX\}$ separates points of SX. The theorem is proved. ☑

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