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Global well-posedness for two dimensional semilinear wave equations

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Abstract. We consider the initial value problem (IVP) for certain semilinear wave equations in two dimensions. It is shown that global well-posedness holds in spaces of lower regularity than that suggested by the energy space $\dot{H}^1 \times L^2_x$. The technique to be used is adapted from a general scheme originally introduced by J. Bourgain to establish global well posedness of the cubic nonlinear Schrödinger equation.

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1. Introduction

We consider the initial value problem (IVP) for the semilinear wave equation

$$\begin{cases} \partial_t^2 u - \Delta_x u = -|u|^{k-1} u, & t \in \mathbb{R}\,, x \in \mathbb{R}^2, \\ u(0,x) = f(x), \\ u_t(0,x) = g(x), \end{cases} \tag{1.1}$$

where $k \geq 3$.

Our purpose in this paper is to establish the global well-posedness of this IVP in spaces of lower regularity than that dictated by the respective conservation law

$$\int_{\mathbb{R}^2} (u_t)^2 + (\nabla u)^2 + \frac{2}{k+1} \, |u|^{k+1}) = \text{constant} \tag{1.2}$$

and that, in addition with the existent local theory, guarantee global well-posedness of this IVP in the energy space.

The notion of local well-posedness for IVP (1.1) in a function space X includes local existence, uniqueness, continuous dependence upon the data, and persistence; i.e., that for $u_0 \in X$, the corresponding solution describes a continuous curve in X. The global well-posedness requires, in addition, that the notion above holds for every time interval. In this regard we have the following local well-posedness result.

Theorem 1.1. *Let $s \in (s(k), 1]$ with $s(k)$ given by*

$$s(k) = \begin{cases} \dfrac{3k-7}{4(k-1)}, & \text{if } 3 \le k \le 5, \\ \dfrac{k-3}{k-1}, & \text{if } k \ge 5. \end{cases} \tag{1.3}$$

Then IVP (1.1) is locally well-posed in $\dot{H}^s(\mathbb{R}^2) \times \dot{H}^{s-1}(\mathbb{R}^2)$. Furthermore, the time T of existence of the solution depends on the size of the initial data; i.e., $T = T(\|f\|_{\dot{H}^s(\mathbb{R}^2)}, \|g\|_{\dot{H}^{s-1}(\mathbb{R}^2)})$.

The proof of this theorem is carried out in [8], where three distinguished cases are considered: the conformal case $k = 5$ and the superconformal and subconformal cases $k > 5$ and $k < 5$, respectively. It is worth mentioning that this result is sharp in the sense that for initial data with lower regularity than that in (1.3), IVP (1.1) is ill posed (see [7], [8]). Notice that in the conformal and superconformal cases, the required regularity for the initial data to obtain local well posedness is the one suggested by the scaling argument: if $u(x, t)$ solves IVP (1.1) then $\lambda^{\frac{2}{k-1}} u(\lambda x, \lambda t)$ also solves the equation in (1.1), and there is invariance of the $\dot{H}^s \times \dot{H}^{s-1}$ norm of the initial data exactly for $s = \frac{k-3}{k-1}$.

It follows from this theorem and from the conservation law in (1.2) that IVP (1.1) is globally well-posed in $\dot{H}^1 \bigcap L_x^{k+1} \times L^2$.

Our main result can now be stated:

Theorem 1.2. *Let $s \in (s(k), 1]$ with $s(k)$ given by*

$$s(k) = \begin{cases} \dfrac{k-2}{k-1}, & \text{if } 3 \le k \le 5, \\ \dfrac{k-1}{k}, & \text{if } \ k > 5. \end{cases} \tag{1.4}$$

Then, for any $(u_0,u_1)\in \dot H^s(\mathbb{R}^2)\bigcap L^{k+1}_x\times \dot H^{s-1}(\mathbb{R}^2)$ *IVP (1.1) has a unique global solution* $u(t)$. *For a given time interval* $[0,T]$, u *can be expressed as*

$$u(t)=\partial_t W(t)u_0+W(t)u_1+I(t), \tag{1.5}$$

with I *more regular than the initial data, more precisely,* $I(t)\in \dot H^1(\mathbb{R}^2)$ *and satisfies the polynomial growth condition*

$$\sup_{t\in[0,T]}\|I(t)\|_{\dot H^1}\leq\ c\,T^{\frac{1-s}{(k-1)s-(k-2)}+}, \tag{1.6}$$

and $W(t)$ *is defined below.*

We restrict our attention to two dimensions, but the same problem can be studied in other dimensions as well. In fact, Kenig, Ponce and Vega [5] obtain similar results for the three dimensional case, but a much more restrictive range for the nonlinearity has to be imposed, namely, $2\leq k<5$. This is a required condition in the local theory for such IVP's in three dimensions.

Our proof follows ideas recently introduced by Bourgain [1], where he obtained that the IVP associated with the two dimensional cubic Schrödinger is globally well-posed in Sobolev spaces of indexes in between those where the conservation laws hold. He also applied it for the PBVP for the Klein-Gordon equation (see [2]). This method has been extensively applied to many other equations as well(see [3], [4], [5]).

The sketch of the proof is the following: we split the initial data following the ideas in [1], that is, $u_0(x)=u_{0,1}+u_{0,2}$ and $u_1(x)=u_{1,1}+u_{1,2}$, where $(u_{0,1},u_{1,1})\in \dot H^1\bigcap L^{k+1}_x\times L^2_x$ and $(u_{0,2},u_{1,2})\in \dot H^s\bigcap L^{k+1}_x\times \dot H^{s-1}$. Then we consider IVP (1.1) with data $(u_{0,1},u_{1,1})$, for which the global theory allows to obtain a solution, $v(t)$, defined in an interval $[0,\Delta T]$ with ΔT an appropriate time. Next, we consider an IVP with variable coefficients for $z(t)=u(t)-v(t)$ whose solution, $z(t)$, is also defined in $[0,\Delta T]$ and can additionally be expressed, via Duhamel's principle, as $z(t)=y(t)+I(t)$, with $I(t)$ more regular than the initial data, more precisely, $(\nabla I,\partial_t I,I)\in L^2_x\times L^2_x\times L^{k+1}_x$. Then the argument is re-applied in the interval $[\Delta T,2\Delta T]$ having initial data $(v(\Delta T)+I(\Delta T),\partial_t v(\Delta T)+\partial_t v(\Delta T))$ and $(y(\Delta T),\partial_t y(\Delta T))$ for their respective IVP's. The process is continued until reaching any given time $T\gg 1$. Since at each step the Sobolev norms involved grow, we have to obtain some estimates to keep the process uniform.

Notations.

- D denotes the homogeneous derivative $\sqrt{-\Delta_x}$.
- $W(t)$ denotes the operator $W(t)g=\frac{\sin(Dt)}{D}g$, which solves the linear wave equation with initial data $(0,g)$.
- $\|\cdot\|$ denotes the L^2_x norm.

- The mixed space–time norm is denoted by

$$\|f\|_{L^q_T L^p_x} = \left(\int_0^T \left(\int_{-\infty}^{\infty} |f(x,t)|^p\,dx\right)^{q/p} dt\right)^{1/q}.$$

  In case that $T=\infty$, we will use $\|.\|_{L^q_t L^p_x}$.
- The letter c denotes a constant that may change from line to line.

2. Proof of Theorem 1.2

Before we proceed with the proof of our main result, we state some Strichartz estimates for solutions of the linear wave IVP in two dimensions given by

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0, & t\in\mathbb{R}\,, x\in\mathbb{R}^2, \\ u(0,x)=u_0(x), & u_t(0,x)=u_1(x). \end{cases} \tag{2.1}$$

Theorem 2.1. *Let $\theta\in[0,3/4), \gamma\in\mathbb{R}$. Then, for any pair of functions $(f,g)\in \dot{H}^\gamma(\mathbb{R}^2)\times\dot{H}^{\gamma-1}(\mathbb{R}^2)$, the solution of IVP (2.1) satisfies*

$$\|D^{\gamma-\theta}\, u\|_{L_t^{\frac{3}{\theta}} L_x^{\frac{6}{3-4\theta}}} \le c_\theta(\|D^\gamma u_0\| + \|D^{\gamma-1}u_1\|). \tag{2.2}$$

The proof of this estimate and of its versions in higher dimensions can be found in [9].

Let us start with the proof of our Theorem 1.2. We split the initial data $(u_0,u_1)\in \dot{H}^s\bigcap L_x^{k+1}\times\dot{H}^{s-1}$ following the scheme in [1], that is,

$$u_0(x) = (\chi_{\{|\xi|<N\}}\,\hat{f})^\vee(x) + (\chi_{\{|\xi|\ge N\}}\,\hat{u}_0)^\vee(x) = u_{0,1}(x)+u_{0,2}(x) \tag{2.3}$$

and

$$u_1(x) = (\chi_{\{|\xi|<N\}}\,\hat{f})^\vee(x) + (\chi_{\{|\xi|\ge N\}}\,\hat{u}_0)^\vee(x) = u_{1,1}(x)+u_{1,2}(x), \tag{2.4}$$

where N is a large number to be chosen later.

From this splitting it follows that

$$\|D^\rho u_{0,1}\|, \|D^{\rho-1}u_{1,1}\| \le cN^{\rho-s} \qquad \text{for} \quad \rho\ge s, \tag{2.5}$$

and

$$\|D^\rho u_{0,2}\|, \|D^{\rho-1}u_{1,2}\| \le cN^{\rho-s} \qquad \text{for} \quad 0\le\rho\le s. \tag{2.6}$$

As we already pointed out, for initial data $u_{0,1}, u_{1,1}$, IVP (1.1) has a unique global solution in $\dot{H}^1 \bigcap L_x^{k+1} \times L^2$. Moreover, this solution is expressed via Duhamel's principle in the integral form

$$v(t) = \partial_t W(t) u_{0,1} + W(t) u_{1,1} - \int_0^t W(t-t')(|v|^{k-1}v)(t')\,dt', \tag{2.7}$$

and it satisfies the conservation law (1.2), which in terms of the size of the initial data yields the estimate

$$\|v_t(t)\| + \|\nabla v(t)\| + \|v(t)\|_{L_x^{k+1}}^{\frac{k+1}{2}} \sim N^{1-s}. \tag{2.8}$$

Next, we consider the IVP with variable coefficients given by

$$\begin{cases} \partial_t^2 z - \Delta_x z = -|z+v|^{k-1}(z+v) + |v|^{k-1}v, & t \in \mathbb{R}\,, x \in \mathbb{R}^2, \\ z(0,x) = u_{0,2}(x) \in \dot{H}^s, \\ z_t(0,x) = u_{1,2}(x) \in \dot{H}^{s-1}, \end{cases} \tag{2.9}$$

with the initial data satisfying

$$\|D^\rho u_{0,2}\|, \|D^{\rho-1} u_{1,2}\| \le c N^{\rho - s} \qquad \text{for} \quad 0 \le \rho \le s,$$

and its integral version

$$z(t) = \partial_t W(t) u_{0,1} + W(t) u_{1,1} - \int_0^t W(t-t') F(t')\,dt', \tag{2.10}$$

where

$$F = -|z+v|^{k-1}(z+v) + |v|^{k-1}v. \tag{2.11}$$

We will obtain the following estimates for the size of the solution of IVP (2.9) in terms of the size of the initial data. The local well posedness of IVP (2.9) easily follows from these estimates and the contraction principle applied to the integral equation (2.10). For the sake of simplicity, the details are omitted.

First, we consider the conformal and subconformal cases $3 \le k \le 5$. Let us define

$$\|z\|_\gamma = \|D^\gamma z\|_{L^\infty_{\Delta T} L^2_x} + \|z\|_{L^{3/\gamma}_{\Delta T} L_x^{6/(3-4\gamma)}}, \tag{2.12}$$

with $\frac{3k-6}{4k-3} \le \gamma \le s$.

From Strichartz estimate (2.2) and (2.5) we obtain

$$\begin{aligned} \|z\|_{L^{3/\gamma}_{\Delta T} L_x^{6/(3-4\gamma)}} &\le c(\|D^\gamma u_{0,2}\| + \|D^{\gamma-1} u_{1,2}\|) + c\int_0^{\Delta T} \|D^{\gamma-1}(F(t'))\|\,dt' \\ &\le cN^{\gamma - s} + c\int_0^t \|D^{\gamma-1} F(t')\|\,dt' \\ &\le cN^{\gamma-s} + c\|F\|_{L^1_{\Delta T} L_x^{2/(2-\gamma)}}. \end{aligned} \tag{2.13}$$

In order to estimate the last term in (2.13), we observe that from (2.11) it is enough to estimate the terms $F_1 = |z|^k$ and $F_k = |z||v|^{k-1}$.

For F_1 we have

$$\begin{aligned}\|F_1\|_{L^1_{\Delta T}L^{2/(2-\gamma)}_x} &\leq \|z\|^k_{L^k_{\Delta T}L^{2k/(2-\gamma)}_x}\\ &\leq c\|D^\gamma z\|^{\frac{6-3k-\gamma(3-4k)}{\gamma}}_{L^k_{\Delta T}L^2_x}\|z\|^{\frac{3k-6+\gamma(3-3k)}{\gamma}}_{L^k_{\Delta T}L^{6/(3-4\gamma)}_x}\\ &\leq c\Delta T^{3-k+\gamma(k-1)}\|D^\gamma z\|^{\frac{6-3k-\gamma(3-4k)}{\gamma}}_{L^\infty_{\Delta T}L^2_x}\|z\|^{\frac{3k-6+\gamma(3-3k)}{\gamma}}_{L^{3/\gamma}_{\Delta T}L^{6/(3-4\gamma)}_x}\\ &\leq c\Delta T^{3-k+\gamma(k-1)}|||z|||^k_\gamma,\end{aligned}\tag{2.14}$$

and similarly for F_k,

$$\begin{aligned}\|F_k\|_{L^1_{\Delta T}L^{2/(2-\gamma)}_x} &\leq \|zv^{k-1}\|_{L^1_{\Delta T}L^{2/(2-\gamma)}_x}\\ &\leq c\Delta T\|z\|_{L^\infty_{\Delta T}L^{2/(1-\gamma)}_x}\|v\|^{k-1}_{L^\infty_{\Delta T}L^{2(k-1)}_x}\\ &\leq c\Delta T\|D^\gamma z\|_{L^\infty_{\Delta T}L^2_x}\|Dv\|^{\frac{k-3}{2}}_{L^\infty_{\Delta T}L^2_x}\|v\|^{\frac{k+1}{2}}_{L^\infty_{\Delta T}L^{(k+1)}_x}\\ &\leq c\Delta T N^{\frac{(k-3)(1-s)}{2}}N^{1-s}|||z|||_\gamma \leq c\Delta T N^{\frac{(k-1)(1-s)}{2}}|||z|||_\gamma.\end{aligned}\tag{2.15}$$

We remark that in order to incorporate the appropriate time space in (2.14), the above given restriction on k is needed.

The estimate for $\|D^\gamma z\|_{L^\infty_{\Delta T}L^2_x}$ is exactly the same as in (2.13). Hence, it follows from (2.13), (2.14) and (2.15) and from the choice of ΔT that for any $\gamma \in (\frac{3k-6}{4k-3}, s)$,

$$|||z|||_\gamma \leq cN^{\gamma-s}.\tag{2.16}$$

It is also important to estimate the L^2_x norm of the solution of IVP (2.9). Thus, from (2.6) and Sovolev's inequality,

$$\|z\|_{L^\infty_{\Delta T}L^2_x} \leq \|u_{0,2}\| + \|u_{1,2}\| + \|D^{-1}F\|_{L^1_{\Delta T}L^2_x} \leq cN^{-s} + c\|F\|_{L^1_{\Delta T}L^1_x}.\tag{2.17}$$

To estimate F_1 we apply (2.16) and a Gagliardo-Nirenberg type inequality, to obtain that

$$\begin{aligned}\|F_1\|_{L^1_{\Delta T}L^1_x} &\leq \Delta T\|z^k\|_{L^\infty_{\Delta T}L^1_x} \leq c\Delta T\|z\|^k_{L^\infty_{\Delta T}L^k_x}\\ &\leq c\Delta T(\|z\|^{\frac{1}{k}}_{L^\infty_{\Delta T}L^2_x}\|D^{\frac{k-2}{k-1}}z\|^{\frac{k-1}{k}}_{L^\infty_{\Delta T}L^2_x})^k\\ &\leq c\Delta T\|z\|_{L^\infty_{\Delta T}L^2_x}\|D^\gamma z\|^{k-1}_{L^\infty_{\Delta T}L^2_x}\\ &\leq c\Delta T\|z\|_{L^\infty_{\Delta T}L^2_x}|||z|||^{k-1}_\gamma\\ &\leq cN^{-\frac{(k-1)(1-s)}{2}}N^{(k-1)(\gamma-s)}\|z\|_{L^\infty_{\Delta T}L^2_x}\\ &\leq cN^{\frac{(k-3)}{2}-\frac{(k-1)}{2}s}\|z\|_{L^\infty_{\Delta T}L^2_x},\end{aligned}\tag{2.18}$$

where $\gamma = \frac{k-2}{k-1}$. Since N is large and the exponent is negative, this part can be absorbed into the left hand side of (2.17).

In a similar manner, we have for F_k

$$\begin{aligned}\|F_k\|_{L^1_{\Delta T}L^1_x} &\le \|zv^{k-1}\|_{L^1_{\Delta T}L^1_x} \le \Delta T\ \|z\|_{L^\infty_{\Delta T}L^2_x}\|v\|^{k-1}_{L^\infty_{\Delta T}L^{2(k-1)}_x}\\ &\le c\Delta T\ \|z\|_{L^\infty_{\Delta T}L^2_x}\|Dv\|^{\frac{k-3}{2}}_{L^\infty_{\Delta T}L^2_x}\|v\|^{\frac{k+1}{2}}_{L^\infty_{\Delta T}L^{k+1}_x}\\ &\le c\Delta T\ N^{\frac{(k-3)(1-s)}{2}}N^{1-s}\|z\|_{L^\infty_{\Delta T}L^2_x}\\ &\le c\Delta T N^{-\frac{(k-1)(1-s)}{2}}\|z\|_{L^\infty_{\Delta T}L^2_x},\end{aligned} \tag{2.19}$$

and therefore, from the choice of ΔT, this term can also be absorbed into the left hand side of (2.17). Summing up, we have the estimate

$$\|z\|_{L^\infty_{\Delta T}L^2_x} \le cN^{-s}. \tag{2.20}$$

Now it comes the nontrivial part of showing that the integral part $I(t) = \int_0^t W(t-t')F(t')\,dt'$ in (2.10) is more regular than the initial data. More precisely, $\partial_t I(t)$ and $\nabla I(t)$ live in L^2_x, whereas the initial data are merely in $\dot H^s \times \dot H^{s-1}$ with $s<1$. Furthermore, the estimates in (2.16) and (2.20) will allow to measure the size of this piece of the solution of the IVP (2.9). In fact, Minkowski's inequality yields

$$\|(\partial_t I, \nabla I)(t)\|_{L^2_x} \le\ \|F\|_{L^1_{\Delta T}L^2_x}. \tag{2.21}$$

For F_1 we have

$$\begin{aligned}\|F_1\|_{L^1_{\Delta T}L^2_x} &\le \|z^k\|_{L^1_{\Delta T}L^2_x} \le \|z\|^k_{L^k_{\Delta T}L^{2k}_x} \le \Delta T^{\frac{5-k}{4}}\|z\|^k_{L^{4k/(k-1)}_{\Delta T}L^{2k}_x}\\ &\le \Delta T^{\frac{5-k}{4}}\|z\|^k_{L^{3/\gamma}_{\Delta T}L^{6/(3-4\gamma)}_x} \le \Delta T^{\frac{5-3k}{4}}\|z\|^k_{\gamma},\end{aligned} \tag{2.22}$$

where we have chosen $\gamma = \frac{3k-3}{4k}$.

Then, the choice of ΔT and (2.16) give

$$\begin{aligned}\|F_1\|_{L^1_{\Delta T}L^2_x} &\le cN^{-\frac{(5-k)(k-1)(1-s)}{8}}N^{k(\frac{3k-3}{4k}-s)}\\ &\le\ cN^{\frac{(k-1)(k+1)}{8}-\frac{s}{8}(k^2+2k+5)} \le\ cN^{\frac{k-1}{2}-\frac{(k+1)}{2}s}.\end{aligned} \tag{2.23}$$

The last inequality holds since $s > \frac{k-3}{k-1}$.

For F_k we use the Sobolev and a Gagliardo-Nirenberg type inequalities, and the estimates in (2.8), (2.16) and (2.20).

$$\begin{aligned}\|F_k\|_{L^1_{\Delta T}L^2_x} &\le \|zv^{k-1}\|_{L^1_{\Delta T}L^2_x} \le \Delta T\|z\|_{L^\infty_{\Delta T}L^{2/(1-2\varepsilon)}_x}\|v\|^{k-1}_{L^\infty_{\Delta T}L^{(k-1)/\varepsilon}_x}\\ &\le\ c\Delta T\|D^{2\varepsilon}z\|_{L^\infty_{\Delta T}L^2_x}\|Dv\|^{k-1-(k+1)\varepsilon}_{L^\infty_{\Delta T}L^2_x}\|v\|^{(k+1)\varepsilon}_{L^\infty_{\Delta T}L^{k+1}_x}\\ &\le\ cN^{-\frac{(k-1)(1-s)}{2}}N^{2\varepsilon-s}N^{(k-1-(k+1)\varepsilon)(1-s)}N^{2(1-s)\varepsilon}\\ &\le\ cN^{\frac{(k-1)}{2}-\frac{(k+1)}{2}s+\varepsilon(2-(k-1)(1-s))},\end{aligned} \tag{2.24}$$

where ε is a small positive number to be chosen.

In order to apply the scheme roughly described in the introduction, we shall also need an estimate on the growth of the L_x^{k+1} norm of the integral part $I(t)$ at time ΔT. We apply Sovolev's inequality to obtain

$$\begin{aligned}\|I(\Delta T)\|_{L_x^{k+1}} \leq& \int_0^{\Delta T} \|\frac{\sin\ D(\Delta T - t')}{D} F\|_{L_x^{k+1}}\ dt' \\ \leq&\ c\int_0^{\Delta T} \|D^{\frac{k-1}{k+1}} \frac{\sin\ D(\Delta T - t')}{D} F\|_{L_x^2} dt' \\ \leq&\ c\int_0^{\Delta T} \|D^{-\frac{2}{k+1}}\ F\|_{L_x^2} dt' \leq\ c\|D^{-\frac{2}{k+1}}\ F\|_{L^1_{\Delta T}L_x^2},\end{aligned} \tag{2.25}$$

and for F_1, Sovolev's inequality and the estimates in (2.16) with $\gamma = \frac{(k^2-3)}{k(k+1)}$ yield

$$\begin{aligned}\|D^{-\frac{2}{k+1}}\ F_1\|_{L^1_{\Delta T}L_x^2} \leq& \|D^{-\frac{2}{k+1}}\ |z|^k\|_{L^1_{\Delta T}L_x^2} \leq \Delta T\|z\|^k_{L^\infty_{\Delta T}L_x^{2k(k+1)/(k+3)}} \\ \leq&\ c\Delta T\|D^\gamma z\|^k_{L^\infty_{\Delta T}L_x^2} \leq\ c\Delta T|\!|\!|z|\!|\!|^k_\gamma \\ \leq&\ cN^{-\frac{k-1}{2}(1-s)}N^{(\gamma-s)k} \\ \leq&\ cN^{-\frac{k-1}{2}(1-s)}N^{(\frac{k^2-3}{k(k+1)}-s)k} \\ \leq&\ cN^{\frac{k^2-5}{2(k+1)}-\frac{k+1}{2}s} \leq\ cN^{\frac{k-1}{2}-\frac{(k+1)}{2}s}.\end{aligned} \tag{2.26}$$

The estimates in (2.8) and (2.16) yield for F_k that

$$\begin{aligned}\|D^{-\frac{2}{k+1}}\ F_k\|_{L^1_{\Delta T}L_x^2} \leq& \|D^{-\frac{2}{k+1}}\ (|z||v|^{k-1})\|_{L^1_{\Delta T}L_x^2} \\ \leq&\ c\|zv^{k-1}\|_{L^1_{\Delta T}L_x^{2(k+1)/(k+3)}} \\ \leq&\ c\Delta T\|z\|_{L^\infty_{\Delta T}L_x^2}\|v^{k-1}\|_{L^\infty_{\Delta T}L_x^{k+1}} \\ \leq&\ c\Delta T\|z\|_{L^\infty_{\Delta T}L_x^2}\|v\|^{k-1}_{L^\infty_{\Delta T}L_x^{k^2-1}} \\ \leq&\ c\Delta T\|z\|_{L^\infty_{\Delta T}L_x^2}\|Dv\|^{k-2}_{L^\infty_{\Delta T}L_x^2}\|v\|_{L^\infty_{\Delta T}L_x^{k+1}} \\ \leq&\ cN^{-\frac{k-1}{2}(1-s)}N^{-s}N^{(1-s)(k-2)}N^{\frac{2}{k+1}(1-s)} \\ \leq&\ cN^{\frac{k^2-2k+1}{2(k+1)}-\frac{k^2+3}{2(k+1)}s} \leq\ cN^{\frac{k-1}{2}-\frac{(k+1)}{2}s}.\end{aligned} \tag{2.27}$$

Estimates (2.21)-(2.27) certainly prove our claim that the integral part in (2.10) is more regular than the initial data, and furthermore, that in the interval $[0, \Delta T]$ it grows like

$$\|(\partial_t I, \nabla I)(\Delta T)\| + \|I(\Delta T)\|^{\frac{k+1}{2}}_{L_x^{k+1}} \leq cN^{\frac{(k-1)}{2}-\frac{(k+1)}{2}s+\varepsilon(2-(k-1)(1-s))}. \tag{2.28}$$

For higher powers of k we slightly modify the $\|\!|\cdot|\!\|$ norm in (2.12) and consider

$$|\!|\!| z |\!|\!|_\gamma = \|D^\gamma z\|_{L^\infty_{\Delta T} L^2_x} + \|D^{\gamma-\theta} z\|_{L^{3/\theta}_{\Delta T} L^{6/(3-4\theta)}_x}, \tag{2.29}$$

where $\gamma > \frac{k-3}{k-1}$ and $\theta = 3 - 3\gamma + \frac{3\gamma-6}{k}$.

Once again, Strichartz's estimate yields

$$\begin{aligned} \|D^{\gamma-\theta} z\|_{L^{3/\gamma}_{\Delta T} L^{6/(3-4\gamma)}_x} &\le c(\|D^\gamma u_{0,2}\| + \|D^{\gamma-1} u_{1,2}\|) + c\int_0^{\Delta T} \|D^{\gamma-1}(F(t'))\|\, dt' \\ &\le\ cN^{\gamma-s} + c\|F\|_{L^1_{\Delta T} L^{2/(2-\gamma)}_x}. \end{aligned} \tag{2.30}$$

We proceed as in (2.13)-(2.15) to obtain that

$$\begin{aligned} \|F_1\|_{L^1_{\Delta T} L^{2/(2-\gamma)}_x} &\le \|z^k\|_{L^1_{\Delta T} L^{2/(2-\gamma)}_x} \le \|z\|^k_{L^k_{\Delta T} L^{2k/(2-\gamma)}_x} \\ &\le\ c\|D^{\gamma-\theta} z\|^k_{L^k_{\Delta T} L^{6/(3-4\theta)}_x} \\ &\le\ c\Delta T^{\frac{3-k\theta}{3}} \|D^{\gamma-\theta} z\|^k_{L^{3/\theta}_{\Delta T} L^{6/(3-4\theta)}_x} \le\ c\Delta T^{\frac{3-k\theta}{3}} |\!|\!| z |\!|\!|^k_\gamma \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} \|F_k\|_{L^1_{\Delta T} L^{2/(2-\gamma)}_x} &\le \|z v^{k-1}\|_{L^1_{\Delta T} L^{2/(2-\gamma)}_x} \\ &\le\ c\Delta T \|z\|_{L^\infty_{\Delta T} L^{2/(1-\gamma)}_x} \|v\|_{L^\infty_{\Delta T} L^{2(k-1)}_x} \\ &\le\ c\Delta T \|D^\gamma z\|_{L^\infty_{\Delta T} L^2_x} \|Dv\|^{\frac{k-3}{2}}_{L^\infty_{\Delta T} L^2_x} \|v\|^{\frac{k+1}{2}}_{L^\infty_{\Delta T} L^{(k+1)}_x} \\ &\le\ c\Delta T N^{-\frac{(k-1)(1-s)}{2}} |\!|\!| z |\!|\!|_\gamma. \end{aligned} \tag{2.32}$$

From (2.30)-(2.32), we get

$$\|D^{\gamma-\theta} z\|_{L^{3/\gamma}_{\Delta T} L^{6/(3-4\gamma)}_x} \le cN^{\gamma-s} + c\Delta T^{\frac{3-k\theta}{3}} |\!|\!| z |\!|\!|^k_\gamma + c\Delta T N^{-\frac{(k-1)(1-s)}{2}} |\!|\!| z |\!|\!|_\gamma, \tag{2.33}$$

and therefore, from the choice of ΔT it follows that

$$|\!|\!| z |\!|\!|_\gamma \le cN^{\gamma-s}, \tag{2.34}$$

for $\gamma \in (\frac{k-3}{k-1}, s)$.

The estimates for the $L^\infty_{\Delta T} L^2_x$ norms found in (2.17)-(2.20) for the subconformal and conformal cases are the same for the superconformal case, and therefore

$$\|z\|_{L^\infty_{\Delta T} L^2_x} \le c\, N^{-s}. \tag{2.35}$$

The same also holds in the analysis of the smoothing property of the integral part $I(t)$ of the solution of the IVP (2.9), for all the estimates except those in (2.22) and (2.23), that require low values of k. For values of $k > 5$, we can give the following estimate, that demands additional regularity of the initial data in the statement of Theorem 1.2 in the superconformal case:

$$\begin{aligned} \|F_1\|_{L^1_{\Delta T} L^2_x} &\le \|z^k\|_{L^1_{\Delta T} L^2_x} \le \Delta T \|z\|^k_{L^\infty_{\Delta T} L^{2k}_x} \\ &\le c\Delta T \|D^{\frac{k-1}{k}} z\|^k_{L^\infty_{\Delta T} L^2_x} \le c\Delta T \|\!|z|\!\|^k_\gamma \\ &\le cN^{-\frac{k-1}{2}(1-s)} N^{(\frac{k-1}{k}-s)k} \le \ cN^{\frac{k-1}{2}-\frac{(k+1)}{2}s}, \end{aligned} \tag{2.36}$$

where $s > \gamma = \frac{k-1}{k}$.

With these estimates at hand, we obtain the same growth condition as in (2.28):

$$\|(\partial_t I, \nabla I)(\Delta T)\| + \|I(\Delta T)\|^{\frac{k+1}{2}}_{L^{k+1}_x} \le cN^{\frac{(k-1)}{2}-\frac{(k+1)}{2}s+\varepsilon(2-(k-1)(1-s))}. \tag{2.37}$$

The closing argument that allows to yield the solution in the interval $[0, T]$ for any $T \gg 1$, is based in an iterative process along with the growth condition (2.37).

In the interval $[0, \Delta T]$, we have that the solution of IVP (1.1) is given by

$$\begin{aligned} u(t) &= v(t) + z(t) \\ &= v(t) + I(t) + \partial_t W(t) u_{0,2} + W(t) u_{1,2}, \end{aligned} \tag{2.38}$$

where $I(t) = -\int_0^t W(t-t')(F)(t')dt'$, with F as in (2.11).

Next, we solve the IVP (1.1) in the interval $[\Delta T, 2\Delta T]$ with initial data $(v(\Delta T) + I(\Delta T),\ \partial_t v(\Delta T) + \partial_t I(\Delta T)) \in \dot{H}^1 \bigcap L^{k+1}_x \times L^2_x$, which makes sense because the smoothing effect that takes place in the integral part $I(t)$. With this solution at hand, we consider IVP (2.9) with initial data

$$(\partial_t W(\Delta T) u_{0,2} + W(\Delta T) u_{1,2}, -\Delta W(\Delta T) u_{0,2} + \partial_t W(\Delta T) u_{1,2}). \tag{2.39}$$

This iterative procedure is continued until any given time $T \gg 1$ is reached. In order to accomplish this goal we have to be sure that the process can be carried out uniformly, that is , that the growth condition in (2.8) should hold at all the steps up to reaching time T. It is here where estimate (2.37) turns out to be essential.

To reach time T, the number of iterations is

$$\frac{T}{\Delta T}. \tag{2.40}$$

For the problem with initial data in the energy space there is some growth of the initial data in the $\|(\partial_t\cdot,\nabla\cdot)\|+\|\cdot\|^{\frac{k+1}{2}}_{L_x^{k+1}}$-norm during each iteration, coming from the integral part $I(t)$ and measured in (2.28). Since we want to keep (2.9) uniform, the following estimate should hold:

$$\frac{T}{\Delta T}N^{\frac{(k-1)}{2}-\frac{(k+1)}{2}s+\varepsilon(2-(k-1)(1-s))}\leq cN^{1-s}; \tag{2.41}$$

and plugging into (2.39) the size of ΔT, we have to check that

$$TN^{\frac{k-1}{2}(1-s)}N^{\frac{(k-1)}{2}-\frac{(k+1)}{2}s+\varepsilon(2-(k-1)(1-s))}\leq cN^{1-s}. \tag{2.42}$$

Inequality (2.42) holds provided $s>\frac{k-2}{k-1}$, ε is sufficiently small and $N=N(T)$ is sufficiently large. We remark however, that (2.36) imposes an additional restriction on s for the superconformal case, namely, $s>\frac{k-1}{k}$. This completes the proof of Theorem 1.2. ☑

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