

A method of discretization for the Lagerström-Cole model equation

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ABSTRACT. A method of discretization for a non linear singularly perturbed boundary value problem is considered. It involves a certain number of steps, one of them including the application of Petrov-Galerkin finite element methods. The resulting scheme is called adjoint method scheme and is in some way related to Nijijima's scheme (cf. [11]). It is proved that this discretization provides existence and uniqueness of solution for a problem defined by the Lagerström-Cole model equation. Finally some numerical experiments compare the results obtained when the adjoint method scheme is used, as well as when Nijijima's scheme or a direct finite element discretization are applied.

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1. Introduction

In order to solve a boundary value problem defined by the Lagerström-Cole model equation several authors have proposed different types of discretizations based on both finite difference and finite element methods. Regarding the first type, worth mentioning are the scheme proposed by Abrahamsson and Osher in [1] and the one introduced by Nijijima in [11]. The scheme we shall propose below can be considered as a generalization of Nijijima's for a certain class of problems, even if the process to obtain it makes use of the finite element method after a previous manipulation of the original equation. This procedure

was already used applied to a finite difference discretization by Lisbona in [8]. Let us consider a boundary value problem of the type

$$\begin{cases} \mathcal{L}u = f & \text{on } \Omega, \\ g(u) = g_0 & \text{on } \Gamma = \partial\Omega, \end{cases} \quad (1.1)$$

where the operator \mathcal{L} is not necessarily linear. Once two Sobolev spaces H_1 and H_2 have been chosen, we apply to the previous problem a variational formulation, which yields the following variational problem:

$$u \in H_1 \text{ such that } a(u, v) = l(v) \quad \forall v \in H_{2,0}, \quad (1.2)$$

where $H_{2,0} = \{v \in H_2; v(0) = v(1) = 0\}$, l is a linear form such that $l(v) = \int_0^1 f v dx$ and $a(\cdot, \cdot)$ is an application defined from $H_1 \times H_2$ on \mathbb{R} and bilinear whenever \mathcal{L} is linear. Let us suppose as well that we have $l + 1$ different grids in Ω denoted as R_d for every $d \in \{0, 1, \dots, l\}$ and such that $R_d \subset R_{d+1}$, if $0 \leq d \leq l - 1$. Each grid is associated to a pair of subspaces \mathcal{H}_d and \mathcal{K}_d , the first one belonging to H_1 and the second one to $H_{2,0}$. It is widely known (see, for instance [4] or [10]) that every pair of subspaces and every grid give rise to a discretized problem whose formulation is

$$u_d \in \mathcal{H}_d \text{ such that } a(u_d, v) = l(v) \quad \forall v \in \mathcal{K}_d. \quad (1.3)$$

This kind of problem can be expressed in terms of discrete operators as follows:

$$u_d \in \mathcal{H}_d \text{ such that } \mathcal{L}_d u_d = f_d. \quad (1.4)$$

From now on we shall limit ourselves to study problems of the type (1.1) defined on the interval $[0, 1]$ by the operator of the Lagerström-Cole model equation \mathcal{L}_B , that is to say,

$$\mathcal{L}_B(u) = -\varepsilon \frac{d^2 u}{dx^2} + u \frac{du}{dx} + u, \quad (1.5)$$

and by the function $f = 0$, as well as by Dirichlet boundary conditions. We shall apply two different discretizations to it: the first one, a finite element discretization; the second one is the principal aim of this paper and shall be deduced in the following section. For example, let us suppose that the boundary conditions are $u(0) = 0$ and $u(1) = 1$. Then, the solution for this problem presents a boundary layer close to 1. When applying to this problem a variational formulation, application $a(\cdot, \cdot)$ from (1.2) turns out to be

$$a(u, v) = \varepsilon \int_0^1 u' v' dx + \int_0^1 u u' v dx + \int_0^1 u v dx,$$

and, if we apply the finite element method and search for a solution $\tilde{u} = \sum_{i=0}^{N+1} u_i \phi_i(x)$, where $\phi_i(x)$ are the classical Galerkin functions, we attain the

following non-linear system of unknowns u_1, \dots, u_N :

$$\begin{aligned} \varepsilon \int_0^1 \left(\sum_{i=0}^{N+1} u_i \phi_i'(x) \right) \psi_j'(x) dx + \int_0^1 \left(\sum_{i=0}^{N+1} u_i \phi_i(x) \right) \left(\sum_{i=0}^{N+1} u_i \phi_i'(x) \right) \psi_j(x) dx \\ + \int_0^1 \left(\sum_{i=0}^{N+1} u_i \phi_i(x) \right) \psi_j(x) dx = 0. \end{aligned} \quad (1.6)$$

The choice for the test functions $\psi_j(x)$ depends not only on the operator \mathcal{L}_B but also on the problem and the predicted solution. As for the problem considered above, adequate functions seem to be

$$\psi_j(x) = \begin{cases} \frac{1 - e^{-b_0^j(x)}}{1 - e^{-\beta_0^j}}, & \text{if } x \in I_{j-1}, \\ \frac{e^{-b_1^j(x)} - e^{-\beta_1^j}}{1 - e^{-\beta_1^j}}, & \text{if } x \in I_j, \\ 0, & \text{if } x \in I - (I_{j-1} \cup I_j), \end{cases} \quad \forall j \in \{1, \dots, N\}, \quad (1.7)$$

where

$$\begin{aligned} \beta_k^j = u_j \frac{h_{j+k-1}}{2\varepsilon}, \quad b_k^j(x) = \frac{\beta_k^j(x - x_{j+k-1})}{h_{j+k-1}}, \quad \forall j \in \{1, \dots, N\}, \quad \forall k \in \{0, 1\}, \\ I_j = [x_j, x_{j+1}], \quad h_j = x_{j+1} - x_j, \quad \forall j \in \{0, \dots, N\}. \end{aligned}$$

The choice for the parameters has been inspired following the recommendations given by Hemker in [5] for equations of the type $-\varepsilon u'' + (a(x)u)' + b(x)u = 0$, for which a good value of β is pointed out to be

$$h \left(\frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)} \right).$$

In our case $b(x) = 1$, while $a(x)$ should be substituted by $a(x, u) = u/2$. If we truncate the expression of β by eliminating the second term, we can assure the continuity of β in the case when u vanishes. On the other hand, we would also like to note that if u_j cancels, then the corresponding function $\psi_j(x)$ is a classical Galerkin one, while, if u_j is negative instead of positive, the orientation of the functions changes. Nevertheless, the expression for functions (1.7) can be simplified by replacing u_j in β_k^j just by 1 or -1 , depending on whether u_j has positive or negative sign. In the next section we show how to develop our method of discretization when a boundary value problem (1.1) with operator \mathcal{L}_B is considered. To abbreviate we shall call the resulting scheme the adjoint method scheme. In Section 3 sufficient conditions are given for the existence and uniqueness of solution of the adjoint method scheme, while some numerical experiments which illustrate the theoretical results as far obtained are carried out in Section 5. As for Section 4, it is a preliminary to the last part of this paper, where the adaption method for the grid used in the numerical experiments is explained.

2. The adjoint method discretization

As has been pointed out in the previous section, there exist various possibilities of discretizing a problem of the type (1.1). Among them we outline Nijjima's scheme, as introduced in [11]. This scheme is specially fitted to solve problems given by equations of the kind $-\varepsilon u'' + (g(u))' + b(x, u) = 0$, where both g and b are taken to be continuous functions. If we consider a problem of the type 1.1 given by operator \mathcal{L}_B , we have that $g(u) = u^2/2$ and $b(x, u) = u$. Then, the expression for Nijjima scheme given a uniform grid is:

$$\frac{-\varepsilon}{h^2} \left(\int_{u_j}^{u_{j+1}} \xi\left(\frac{sh}{2\varepsilon}\right) ds - \int_{u_{j-1}}^{u_j} \xi\left(\frac{sh}{2\varepsilon}\right) ds \right) + \frac{1}{4h} (u_{j+1}^2 - u_{j-1}^2) + u_j = f_j, \quad (2.1)$$

where $\xi(x) = x \coth x$ and h is the distance between two consecutive grid points. It was proved in [8] that the Nijjima scheme 2.1 belongs to a more general kind of schemes. These are deduced following a process of discretization which makes use of finite differences. This process coincides with the one described below, except for the novelty that Petrov-Galerkin finite element methods are used instead of finite differences. The resulting process of discretization yields, as we shall call it from now on, the adjoint method scheme and consists of five steps. We illustrate the application of these steps by studying the particular case of a problem defined by the operator \mathcal{L}_B from (1.5).

Step 1. Calculus of the Fréchet derivative for the original differential operator, $D\mathcal{L}(u)$. For \mathcal{L}_B it yields:

$$(D\mathcal{L}_B u)(v) = -\varepsilon v'' + uv' + vu' + v.$$

Step 2. Calculus of the adjoint operator to the previous one, that is to say, $(D\mathcal{L}u)^*$:

$$(D\mathcal{L}_B u)^*(w) = -\varepsilon w'' - uw' + w.$$

Step 3. Discretization of the equation obtained in the immediately previous step. We first apply the variational formulation and then make use of Petrov-Galerkin methods. In the particular case of \mathcal{L}_B the variational formulation yields:

$$w \in H_1 \text{ such that } \varepsilon \int_0^1 w' \psi' - \int_0^1 uw' \psi + \int_0^1 w \psi = 0, \quad \forall \psi \in H_{2,0}. \quad (2.2)$$

Next, two finite-dimensional subspaces are chosen in H_1 and $H_{2,0}$, the trial and the test space. The first one is generated by the classical linear Galerkin functions, while the second one is generated by the following exponential functions:

$$\psi_j(x) = \begin{cases} \frac{e^{-b_0^j(x)} - e^{-\beta_0^j}}{1 - e^{-\beta_0^j}}, & \text{if } x \in I_{j-1}, \\ \frac{1 - e^{-b_1^j(x)}}{1 - e^{-\beta_1^j}}, & \text{if } x \in I_j, \\ 0, & \text{if } x \in I - (I_{j-1} \cup I_j) \end{cases}, \quad \forall j \in \{1, \dots, N\}, \quad (2.3)$$

where, making use of the studies developed in [5], for every $j \in \{1, \dots, N\}$ and $k \in \{0, 1\}$, we select $\beta_k^j = u_j \frac{h_{j+k-1}}{\varepsilon}$ and $b_k^j(x) = \beta_k^j (1 - \frac{x - x_{j+k-1}}{h_{j+k-1}})$.

We remark that the orientation of the functions $\psi_j(x)$ given by (2.3) is the opposite to that of the functions considered in (1.7). This is so because the coefficient multiplying w' in $(D\mathcal{L}_B u)^*(w)$ presents the opposite sign to the one which multiplies u' in the expression for $\mathcal{L}_B(u)$. Once the Petrov-Galerkin discretization has been carried out, the unknown u is substituted by u_j –that means, its approximate value on the grid point x_j – at each equation originated by ψ_j . As a result, the following difference scheme is obtained:

$$\begin{aligned} & [(\varphi_1 + b_1 - a_1)(u_j, h_j)] w_{j-1} + \\ & [(a_1 - \varphi_1)(u_j, h_j) + (a_2 - \varphi_2)(u_j, h_{j+1})] w_j + \\ & [(\varphi_2 + b_2 - a_2)(u_j, h_{j+1})] w_{j+1} = 0, \end{aligned}$$

where, taking into account that β stands for sh/ε , functions $\varphi_1, \varphi_2, a_1, a_2, b_1$ and b_2 are defined by the following formulas –we also quote their limits when s tends to 0–:

$$\begin{aligned} \varphi_1(s, h) &= -\frac{se^{-\beta}}{1 - e^{-\beta}} \longrightarrow -\frac{\varepsilon}{h}, \\ \varphi_2(s, h) &= -\frac{s}{1 - e^{-\beta}} \longrightarrow -\frac{\varepsilon}{h}, \\ b_1(s, h) &= h\left(\frac{1}{\beta} - \frac{e^{-\beta}}{1 - e^{-\beta}}\right) \longrightarrow \frac{h}{2}, \\ a_1(s, h) &= h\left(-\frac{1}{\beta^2} - \frac{e^{-\beta}}{2(1 - e^{-\beta})} + \frac{1}{\beta(1 - e^{-\beta})}\right) \longrightarrow \frac{h}{3}, \\ b_2(s, h) &= h - b_1(s, h) \longrightarrow \frac{h}{2}, \\ a_2(s, h) &= \frac{h}{2} - b_1(s, h) + a_1(s, h) \longrightarrow \frac{h}{3}. \end{aligned} \quad (2.4)$$

Step 4. Calculus of the scheme adjoint to the latter one. For the case of \mathcal{L}_B this yields:

$$\begin{aligned} & [(\varphi_2 + b_2 - a_2)(u_{j-1}, h_j)] v_{j-1} + \\ & [(-\varphi_1 + a_1)(u_j, h_j) + (-\varphi_2 + a_2)(u_j, h_{j+1})] v_j + \\ & [(\varphi_1 + b_1 - a_1)(u_{j+1}, h_{j+1})] v_{j+1} = 0. \end{aligned}$$

Step 5. Construction of a finite difference operator whose Fréchet derivative is the scheme in Step 4 and such that it is consistent with the original problem defined when using operator \mathcal{L} . In particular, the scheme obtained when we consider problem 1.1 with operator \mathcal{L}_B is:

$$\begin{aligned}
& \int_0^{u_{j+1}} \varphi_1(s, h_{j+1}) ds - \int_0^{u_j} \varphi_2(s, h_{j+1}) ds - \\
& \left(\int_0^{u_j} \varphi_1(s, h_j) ds - \int_0^{u_{j-1}} \varphi_2(s, h_j) ds \right) + \\
& \int_0^{u_{j+1}} (b_1 - a_1)(s, h_{j+1}) ds + \int_0^{u_j} a_2(s, h_{j+1}) ds + \\
& \int_0^{u_j} a_1(s, h_j) ds + \int_0^{u_{j-1}} (b_2 - a_2)(s, h_j) ds = h_j f_j, \quad \forall j \in \{1, \dots, N\},
\end{aligned} \tag{2.5}$$

where $f_j = f(x_j)$. We denote scheme (2.5) more briefly as

$$\overline{\mathcal{L}}_B u = f. \tag{2.6}$$

Both u and f denote no longer functions as in (1.1), but vectors belonging to \mathbb{R}^N . In the particular case of uniform grids, scheme (2.5) abridges to the following:

$$\begin{aligned}
& \frac{1}{h} \left(\int_{u_j}^{u_{j+1}} \varphi_1(s, h) ds - \int_{u_{j-1}}^{u_j} \varphi_2(s, h) ds \right) + \\
& \frac{1}{h} \left(\int_{u_j}^{u_{j+1}} (b_1 - a_1)(s, h) ds - \int_{u_{j-1}}^{u_j} (b_2 - a_2)(s, h) ds \right) + u_j = f_j.
\end{aligned}$$

Precisely, the first part of this scheme —that defined by functions φ_1 and φ_2 — coincides with Nijijima's discretization (2.1) for the terms $-\varepsilon u'' + uu'$ in operator \mathcal{L}_B , so that it can be assured that the adjoint method scheme preserves the exponential character of the problem. As for term u , a more complex expression has been obtained than that which appears in Nijijima's scheme, having been added to the term u_j the contributions by the functions a_1 , a_2 , b_1 and b_2 . Analogously, when we consider the same operator \mathcal{L}_B and we suppose that f equals 0 and that the boundary conditions are such that the boundary layer is located on the left of the interval, the functions $\psi_j(x)$ in Step 3 are taken an orientation opposite to that in (2.3). Also they have a similar expression to the functions from (1.7), except for the value of β_k^j , which now is replaced by $u_j h_{j+k-1}/\varepsilon$. So, when applying the adjoint method, we obtain the same scheme

than (2.5), except for functions $\varphi_1, \varphi_2, a_1, a_2, b_1$ and b_2 , whose expressions are now:

$$\begin{aligned}
 \varphi_1(s, h) &= -\frac{s}{1 - e^{-\beta}} \longrightarrow -\frac{\varepsilon}{h}, \\
 \varphi_2(s, h) &= -\frac{se^{-\beta}}{1 - e^{-\beta}} \longrightarrow -\frac{\varepsilon}{h}, \\
 b_1(s, h) &= h\left(-\frac{1}{\beta} + \frac{1}{1 - e^{-\beta}}\right) \longrightarrow \frac{h}{2}, \\
 a_1(s, h) &= h\left(-\frac{1}{\beta^2} + \frac{1}{2(1 - e^{-\beta})} + \frac{e^{-\beta}}{\beta(1 - e^{-\beta})}\right) \longrightarrow \frac{h}{3}, \\
 b_2(s, h) &= h - b_1(s, h), \\
 a_2(s, h) &= \frac{h}{2} - b_1(s, h) + a_1(s, h).
 \end{aligned} \tag{2.7}$$

These functions decentralize the scheme in the opposite sense than the functions defined by (2.4). We also remark that for negative values of s the functions (2.7) coincide with those from (2.4) for positive values of the same parameter.

3. Existence and uniqueness of solution for the adjoint method scheme

A proof for the existence and uniqueness of solution for the original continuous problem (1.1) associated to operator (1.5) can be found in [8], pp. 9-14. We shall now concentrate on the obtention for sufficient conditions in order to assure that a scheme of the type (2.5) has a unique solution. With this purpose we rewrite system (2.5) –analogously, (2.6)–, whose unknowns are $(u_1, \dots, u_N) \in \mathbb{R}^N$, as

$$\begin{aligned}
 -\varepsilon \sum_{l=-1}^1 \alpha_{j,l} u_{j+l} + G(u_{j-1}, u_j, u_{j+1}, h_j, h_{j+1}) \\
 + B(u_{j-1}, u_j, u_{j+1}, h_j, h_{j+1}) = f_j, \forall j \in \{1, \dots, N\},
 \end{aligned} \tag{3.1}$$

where the functions and coefficients here introduced are defined as follows:

$$\alpha_{j,-1} = \frac{1}{h_j}, \quad \alpha_{j,1} = \frac{1}{h_{j+1}}, \quad \alpha_{j,0} = -\alpha_{j,-1} - \alpha_{j,1}, \quad \forall j \in \{1, \dots, N\},$$

$$\begin{aligned}
 G(y_{-1}, y_0, y_1, h_0, h_1) &= \int_0^{y_1} \varphi_1(s, h_1) ds - \int_0^{y_0} \varphi_2(s, h_1) ds - \\
 &\quad \left(\int_0^{y_0} \varphi_1(s, h_0) ds - \int_0^{y_{-1}} \varphi_2(s, h_0) ds \right),
 \end{aligned}$$

$$\begin{aligned}
B(y_{-1}, y_0, y_1, h_0, h_1) &= \varepsilon \sum_{l=-1}^1 (\alpha_{j,l} y_l) + \int_0^{y_1} (b_1 - a_1)(s, h_1) ds + \int_0^{y_0} a_2(s, h_1) ds \\
&\quad + \int_0^{y_0} a_1(s, h_0) ds + \int_0^{y_{-1}} (b_2 - a_2)(s, h_0) ds.
\end{aligned}$$

Taking into account this reformulation of scheme (2.6), we now set to study some properties satisfied by it. To begin with, we enunciate the next result.

Lemma 3.1. *For every vector $u = (u_1, \dots, u_j, \dots, u_N) \in \mathbb{R}^N$ and for every h_j and h_{j+1} which satisfy relationships*

$$h_j^2 h_{j+1} - 2\varepsilon(h_j + h_{j+1}) > 0, \quad h_j h_{j+1}^2 - 2\varepsilon(h_j + h_{j+1}) > 0, \quad \forall j \in \{1, \dots, N\},$$

scheme (2.6) satisfies the following five properties:

- (i) $\alpha_{j,l} \geq 0$, for $l = -1, 1$,
- (ii) $\sum_{l=-1}^1 \alpha_{j,l} = 0$,
- (iii) $\frac{\partial G}{\partial y_l}(u_{j-1}, u_j, u_{j+1}, h_j, h_{j+1}) \leq 0$, for $l = -1, 1$,
- (iv) $G(y, y, y, h, h) = 0$,
- (v) $\frac{\partial B}{\partial y_l}(u_{j-1}, u_j, u_{j+1}, h_j, h_{j+1}) \geq \mu > 0$, for $l = -1, 0, 1$.

Proof. Properties (i), (ii) and (iv) are straightforward deduced. As for Property (iii), it suffices to take into account that both φ_1 and φ_2 are negative functions and that

$$\begin{aligned}
\frac{\partial G}{\partial y_{-1}}(u_{j-1}, u_j, u_{j+1}, h_j, h_{j+1}) &= \varphi_1(u_{j+1}, h_{j+1}), \\
\frac{\partial G}{\partial y_1}(u_{j-1}, u_j, u_{j+1}, h_j, h_{j+1}) &= \varphi_2(u_{j-1}, h_j).
\end{aligned}$$

Finally we prove the slightly more complex Property (v). It can be checked that

$$\frac{\partial B}{\partial y_{-1}}(u_{j-1}, u_j, u_{j+1}, h_j, h_{j+1}) = (b_2 - a_2)(u_{j-1}, h_j) + \frac{\varepsilon}{h_j} \geq \frac{\varepsilon}{h_j} > 0,$$

and also that

$$\frac{\partial B}{\partial y_1}(u_{j-1}, u_j, u_{j+1}, h_j, h_{j+1}) = (b_1 - a_1)(u_{j+1}, h_{j+1}) + \frac{\varepsilon}{h_{j+1}} > \frac{\varepsilon}{h_{j+1}} > 0,$$

while the other partial derivative of B takes the value

$$\frac{\partial B}{\partial y_0}(u_{j-1}, u_j, u_{j+1}, h_j, h_{j+1}) = a_1(u_j, h_j) + a_2(u_j, h_{j+1}) - \frac{\varepsilon}{h_j} - \frac{\varepsilon}{h_{j+1}}.$$

It can be checked that sum $a_1 + a_2$ attains its minimal value either when u_j tends to $+\infty$ (whose limit is then $h_{j+1}/2$) or else when u_j tends to $-\infty$ (whose limit is $h_j/2$). As a result, if

$$\min \left\{ \frac{h_{j+1}}{2} - \frac{\varepsilon}{h_j} - \frac{\varepsilon}{h_{j+1}}, \frac{h_j}{2} - \frac{\varepsilon}{h_j} - \frac{\varepsilon}{h_{j+1}} \right\} > 0, \quad (3.2)$$

then the partial derivative $\partial B/\partial y_0$ is also positive. As a matter of fact, condition (3.2) is satisfied whenever the assumptions of Lemma 3.1 concerning h_j hold for every j . Furthermore, if we adjust a little these assumptions it can be ensured that the number from (3.2) is lowerly bounded by a sufficiently small positive number μ_0 . Then, it is enough to take

$$\mu = \min \left\{ \frac{\varepsilon}{h_1}, \dots, \frac{\varepsilon}{h_{N+1}}, \mu_0 \right\}$$

for Property (v) to hold. □

Result 3.2. *When the same operator \mathcal{L}_B defined by (1.5) is discretized by a direct finite element method by using the functions introduced in (1.7), it arises a scheme of the type (1.6). Then, simultaneous verification of all the Properties (i)-(v) stated in Lemma 3.1 turns out to be impossible.*

Proof. If scheme (1.6) is rewritten in the form (3.1), the part of it derived from the term $-\varepsilon u''$ presents the expected form $-\varepsilon \sum_{l=-1}^1 (\alpha_{j,l} u_{j+l})$, where $\alpha_{j,-1} = 1/h_j$, $\alpha_{j,1} = 1/h_{j+1}$ and $\alpha_{j,0} = -\alpha_{j,-1} - \alpha_{j,1}$. However, taking into account that the supports for both functions $\phi_j(x)$ and functions $\psi_j(x)$ lay on the intervals $[x_{j-1}, x_{j+1}]$, as well as that for every point in the interval $[0, 1]$ it is verified that $\sum_{k=0}^{N+1} \phi'_k(x) = 0$, then, the only possible choices of G and B for function $G(u_{j-1}, u_j, u_{j+1}, h_j, h_{j+1})$ to verify Property (iv) are

$$G(u_{j-1}, u_j, u_{j+1}, h_j, h_{j+1}) = \sum_{i=j-1}^{j+1} u_i \int_0^1 \phi_i(x) \left(\sum_{k=j-1}^{j+1} u_k \phi'_k(x) \right) \psi_j(x) dx,$$

$$B(u_{j-1}, u_j, u_{j+1}, h_j, h_{j+1}) = \sum_{i=j-1}^{j+1} u_i \int_0^1 \phi_i(x) \psi_j(x) dx.$$

Property (iii) might not hold, when G is as previously stated. As a matter of fact,

$$\frac{\partial G}{\partial y_{-1}}(u_{j-1}, u_j, u_{j+1}, h_j, h_{j+1}) = \frac{u_j - 2u_{j-1}}{h_j} (b_1 - a_1)(u_j, h_j) - \frac{u_j}{h_j} a_1(u_j, h_j),$$

$$\begin{aligned} \frac{\partial G}{\partial y_1}(u_{j-1}, u_j, u_{j+1}, h_j, h_{j+1}) = \\ \frac{2u_{j+1} - u_j}{h_{j+1}}(b_2 - a_2)(u_j, h_{j+1}) + \frac{u_j}{h_{j+1}}a_2(u_j, h_{j+1}), \end{aligned}$$

where functions a_1 , a_2 , b_1 and b_2 are the same which were defined in (2.7) and the parameter β takes the value $sh/2\varepsilon$. Then, if we suppose that u_j cancels, but that both u_{j-1} and u_{j+1} do not, and if we consider the limits for a_1 , a_2 , b_1 and b_2 as stated in (2.7), we have that

$$\begin{aligned} \frac{\partial G}{\partial y_{-1}}(u_{j-1}, u_j, u_{j+1}, h_j, h_{j+1}) &= \frac{-u_{j-1}}{3}, \\ \frac{\partial G}{\partial y_1}(u_{j-1}, u_j, u_{j+1}, h_j, h_{j+1}) &= \frac{u_{j+1}}{3}. \end{aligned}$$

Furthermore, if both u_{j-1} and u_{j+1} have the same sign, it is evident that both expressions cannot be simultaneously negative and this renders impossible the fact that scheme (1.6) verifies Property (iii). \checkmark

In order to be able to assure the existence of solution we reproduce now the next result, whose proof can be found in [9].

Proposition 3.3. *Let T be an operator defined from \mathbb{R}^N onto \mathbb{R}^N such that it verifies Properties (i) and (iii) from Lemma 3.1 and let us suppose that there exist vectors $\bar{\mathbf{u}}$ and $\underline{\mathbf{u}}$ such that*

$$T(\underline{\mathbf{u}}) \leq f \leq T(\bar{\mathbf{u}}) \quad (3.3)$$

—where these vector inequalities are understood as N component-wise—. Then, system $Tu = f$ has a solution in \mathbb{R}^N .

Proposition 3.4. *Under the assumptions of Lemma 3.1, scheme (2.6) presents a solution for every $f \in \mathbb{R}^N$.*

Proof. It suffices to apply Proposition 3.3 to the operator $\bar{\mathcal{L}}_B$ from (2.6). Therefore, in order to find two vectors $\bar{\mathbf{u}}$ and $\underline{\mathbf{u}}$ which satisfy inequalities (3.3), we define:

$$m = \frac{1}{3\mu} \max_{1 \leq j \leq N} |B(0, 0, 0, h_j, h_{j+1}) - f_j| = \frac{1}{3\mu} \max_{1 \leq j \leq N} |f_j|,$$

where μ is the constant from Property (v) in Lemma 3.1. We now take $\underline{\mathbf{u}}$ to be the vector $(-m, \dots, -m)^T$. Then, as Properties (ii) and (iv) of Lemma 3.1 hold, each component j of the vector $\bar{\mathcal{L}}_B(\underline{\mathbf{u}})$ takes the value

$$B(-m, -m, -m, h_j, h_{j+1}).$$

If we apply Property (v) three times to the latter number, one for every partial derivative of B , is to say, for $l = -1, 0, 1$, and we also take into account

the definition of m , we then obtain the sequence of inequalities

$$B(-m, -m, -m, h_j, h_{j+1}) \leq B(0, 0, 0, h_j, h_{j+1}) - 3\mu m \leq f_j, \quad \forall j \in \{1, \dots, N\},$$

and, as a consequence, $\bar{\mathcal{L}}_B(\underline{u}) \leq f$. Finally, if we choose \bar{u} to be the vector $(m, \dots, m)^T$, using the same reasoning as before, we attain the conclusion that $f \leq \bar{\mathcal{L}}_B(\bar{u})$. \checkmark

Once completed the proof for the existence of solution for (2.6) –which means that operator $\bar{\mathcal{L}}_B$ is onto–, we shall now study its uniqueness.

Theorem 3.5. *Under the assumptions of Proposition 3.4, system (2.6) presents a unique solution for every f in \mathbb{R}^N .*

Proof. It suffices to note that for every real u_j and every h_j and h_{j+1}

$$\begin{aligned} & |(a_1 - \varphi_1)(u_j, h_{j+1}) + (a_2 - \varphi_2)(u_j, h_j)| \\ & \quad - |(\varphi_1 + b_1 - a_1)(u_j, h_j)| - |(\varphi_2 + b_2 - a_2)(u_j, h_{j+1})| \end{aligned}$$

is bigger than 0, which is equivalent to

$$\left| -\varepsilon\alpha_{j,0} + \frac{\partial G}{\partial y_0} + \frac{\partial B}{\partial y_0} \right| > \sum_{l \in \{-1, 1\}} \left| -\varepsilon\alpha_{j,l} + \frac{\partial G}{\partial y_l} + \frac{\partial B}{\partial y_l} \right|.$$

The last inequality means that, for every h_j and h_{j+1} and every vector $u \in \mathbb{R}^N$, the jacobian matrix of system (2.6) is strictly diagonally dominant. This implies that this matrix is invertible (see, for example, [7], p. 70). Now, by making use of the inverse function theorem we can assure the existence of neighborhoods of u and f where the operator $\bar{\mathcal{L}}_B$ is homeomorphism, in particular, one-to-one. As Proposition 3.4 assures the operator to be onto, the previous reasoning can hold for every $f \in \mathbb{R}^N$, and then injectivity does not limit to a mere neighborhood, but holds for all the domain \mathbb{R}^N . This enables us to assure the existence and uniqueness of solution for whichever vector $f \in \mathbb{R}^N$. \checkmark

4. Grid adaptation

A problem defined on the interval $I = [A, B]$ and belonging to the type (1.1) is to be solved by a multigrid method. This involves the choice of different grids, the finest of which will be the set $R_l = \{A, x_1^l, \dots, x_{N_l}^l, B\}$. The first possible strategy is to take R_l a uniform grid and afterwards modify it, according to an established adaption criterion which suits the considered problem. This criterion must pay attention to the variation of the approximate function and its derivatives, by relocating the grid points in the regions where the highest variations take place. In fact, it is preferable to adapt the finest grid instead of the coarsest one, since this choice allows to adjust better the width of each

subinterval, as well as to locate the boundary layers much faster. Then, in every multigrid step, once the finest grid R_l is fixed, we also fix R_d for every $l-1 \geq d \geq 0$ as the grid whose points are $\{A, x_1^d, \dots, x_{N_d}^d, B\}$, where for every index $i \in \{1, \dots, N_d\}$, we have that $x_i^d = x_{2i}^{d+1}$. The first set of grids is chosen as R_0^0, \dots, R_l^0 , for every i , $x_i^{0,d} = ih_d$ and $h_d = (B-A)/(N_d+1)$. Then we apply a multigrid iteration and check whether an adaption process is to be performed or not. The adaption criterion handled in this work has been one used in [3]. It aims to distribute a certain positive weight function $m(x)$ in a uniform way among the different subintervals generated by R_l . The function $m(x)$ to be used has been selected from [2]. It considers both the tangent and the curvature for the approximate solution obtained at each iteration and its expression is

$$m(x) = \max \left\{ \frac{1}{I1} \left| \frac{du}{dx} \right|, \frac{1}{I2} \left| \frac{d^2u}{dx^2} \right| \right\},$$

where $I1 = \int_A^B \left| \frac{du}{dx} \right| dx$ and $I2 = \int_A^B \left| \frac{d^2u}{dx^2} \right| dx$. For simplicity we shall denote the finest grid R_l^j as $R^j = \{x_0^j, x_1^j, \dots, x_N^j, x_{N+1}^j\}$, where $x_0^j = A$ and $x_{N+1}^j = B$, so that we have $h_i^j = x_{i+1}^j - x_i^j$, for $0 \leq i \leq N$. We also consider $u_i^j = u(x_i^j)$, for every $0 \leq i \leq N+1$, or rather we take u_i^j as the approximation given by the last multigrid iteration. Then, the adaption method consists then of the following steps:

Step 1. Approximate calculus of the integrals $I1$ and $I2$ as

$$IA1 = \sum_{i=1}^{N+1} \left| h_{i-1}^j \frac{du}{dx}(x_i^j) \right|, \quad IA2 = \sum_{i=1}^{N+1} \left| h_{i-1}^j \frac{d^2u}{dx^2}(x_i^j) \right|,$$

where, in order to approximate the derivatives of u on the points x_i^j , we use the following difference formulae:

$$\begin{aligned} \frac{du}{dx}(x_i^j) &= \frac{u_i^j - u_{i-1}^j}{h_{i-1}^j}, \quad \forall i \in \{1, \dots, N+1\}, \\ \frac{d^2u}{dx^2}(x_i^j) &= \frac{2[h_{i-1}^j(u_{i+1}^j - u_i^j) - h_i^j(u_i^j - u_{i-1}^j)]}{h_{i-1}^j h_i^j (h_{i-1}^j + h_i^j)}, \quad \forall i \in \{1, \dots, N\}, \\ \frac{d^2u}{dx^2}(x_{N+1}^j) &= \frac{d^2u}{dx^2}(x_N^j). \end{aligned}$$

Step 2. Calculus of the integral of the function $m(x)$ on the interval I . We start with the value $S_0 = 0$ and we construct the following sequence S_i :

$$S_i = S_{i-1} + h_{i-1}^j \max \left\{ \frac{1}{IA1} \left| \frac{du}{dx}(x_i^j) \right|, \frac{1}{IA2} \left| \frac{d^2u}{dx^2}(x_i^j) \right| \right\},$$

for all $i \in \{1, \dots, N+1\}$.

Step 3. Uniform distribution on each subinterval of the function $m(x)$. Let δ be the mean value of the integral of $m(x)$, S_{N+1} , calculated in the previous step, that means, $\delta = \frac{S_{N+1}}{N+1}$.

Step 3A. If it happens that $\max_{1 \leq i \leq N+1} \left| \frac{S_i - S_{i-1}}{\delta} - 1 \right| < \tau$ for a given tolerance τ , then we consider $m(x)$ to be well distributed and no further refinement is developed. If this is not the case, we proceed with a new grid adaption.

Step 3B. Considering that the first and last grid points A and B are fixed, this is to say, $x_0^{j+1} = x_0^j$ and $x_{N+1}^{j+1} = x_{N+1}^j$, the following operations are to be performed to generate a new grid:

Step 3B1. We initialize the values $i = 1$, $k = 1$ and $D = \delta$.

Step 3B2. If $D > S_k$, then we reset $k = k + 1$ and compare the resulting S_k with D .

Step 3B3. Otherwise, if $D \leq S_k$, we construct the new grid point as

$$x_i^{j+1} = x_{k-1}^j + \frac{D - S_{k-1}}{S_k - S_{k-1}} h_{k-1}^j.$$

It must be noted that, if $D = S_k$, then $x_i^{j+1} = x_k^j$, this is to say, x_k^j remains in the new grid, although perhaps related to another index.

Finally, if $i < N$, we go back to *Step 3B2* and replace i by $i + 1$ and reset $D = i\delta$. Otherwise, if $i = N$, it means that the construction process of a new grid has come to an end. We apply a new multigrid iteration to obtain a new approximate solution of u and then we return to Step 1 to check if, according to the most recent approximate solution, the last grid obtained is optimal or not. It must be noted that the values of function u must be renewed in *Step 3B3* by interpolation of the previous points, except for the case when in *Step 3B* we have that $i\delta = S_k$; then, we just take u_i^{j+1} to be u_k^j , for it happens that $x_i^{j+1} = x_k^j$.

5. Numerical experiments

In this section we treat some problems related to the operator \mathcal{L}_B defined by (1.5), for whose resolution we use and compare Nijima's scheme, the adjoint method scheme deduced in this paper and the direct Petrov-Galerkin discretization 1.6. As all these schemes are not linear, a Newton-multigrid method may be performed to solve them (cf. [4]). The Newton-multigrid method is the iterative method:

$$\begin{cases} u_l^0, \\ L_B^n(u_l^{n+1} - u_l^n) = -\mathcal{L}_B^n(u_l^n), \end{cases} \quad \forall n \geq 0, \quad (5.1)$$

where $\mathcal{L}_B^n(u_l^n)$ is the outcome of applying the operator \mathcal{L}_B to the n -th iteration of the method, u_l^n , and where L_B^n is the jacobian matrix evaluated at this last

iteration, is to say, $L_B(u_l^n)$. Every n -th iteration is obtained by solving the system (5.1) with a multigrid method. It all results in a process which consists of two steps:

Step 1. Take an initial approximation u_l^0 .

Step 2. For every $n \geq 0$, take v_l as $u_l^{n+1} - u_l^n$. Then, solve system $L_l^n(v_l) = -\mathcal{L}_l^n(u_l^n)$, using a multigrid method and taking the first estimate for this process, $v_l^0 = 0$. As the solution v_l is expected to be close to 0, the use of a nested grid iteration (cf. [4]) does not seem to be required.

In our computations we considered three problems of the same type

$$\begin{cases} -\varepsilon u'' + uu' + u = 0, & \forall x \in [0, 1], \\ u(0) = u_0; & u(1) = u_{N+1}, \end{cases}$$

but with different boundary conditions:

PROBLEM 1. $u_0 = 0$, $u_{N+1} = 1$;

PROBLEM 2. $u_0 = 1/10$, $u_{N+1} = -2$;

PROBLEM 3. $u_0 = -1$, $u_{N+1} = 1/2$.

In order to solve these problems we used Nijjima's scheme, the direct finite element method and the adjoint method. When performing the last two methods, some multigrid iterations using the grid adaption method previously described were first carried out in order to locate the boundary layer. Once an suitable grid was reached, it was kept fixed and new multigrid iterations were effected until $\max_{1 \leq i \leq 0} |u_i^{n+1} - u_i^n|$ was smaller than a given tolerance τ_e . As for Nijjima's scheme, no grid adaption is possible, so that a uniform grid was taken. However, if there exists a boundary layer in the solution Nijjima's method turns out inadequate, as a highest number of unknowns would be required, so that the results could faithfully reflect the multiscale character of the solution.

	EX=2	EX=3	EX=4	EX=5	EX	EX=9
Nijjima	6	7	↗	↗	↗	↗
Adjoint method	3+5	7+4	6+14	9+5	6+4	9+6
Newton (Quadratic)	3+4	7+4	4+5	8+4	7+4	8+5
Newton (Exponential)	3+4	6+4	4+4	7+4	8+4	11+3
Newton (Hughes-Brooks)	3+4	5+4	6+4	6+4	7+4	18+4
Newton (Galerkin)	4+3	7+3	6+4	9+3	↑	↑

TABLE 1

Problem 1 was proposed in [14], while the rest have been solved in [11], using Nijjima's scheme. The solutions for the selected problems represent different

	EX=2	EX=3	EX=4	EX=5	EX	EX=9
Nijjima	7	8	12	12	12	12
Adjoint method	1+6	3+6	4+5	5+5	6+6	8+6
Newton (Quadratic)	5+3	6+3	7+4	8+3	8+4	11+4
Newton (Exponential)	5+3	8+3	9+3	7+3	6+4	8+4
Newton (Hughes-Brooks)	5+3	6+4	6+4	6+4	8+3	11+3
Newton (Galerkin)	7+3	9+14	↑	↑	↑	↑

TABLE 2

	EX=2	EX=3	EX=4	EX=5	EX	EX=9
Nijjima	6	7	↗	↗	↗	↗
Adjoint method	2+6	3+8	4+8	4+25	5+18	7+22
Newton (Quadratic)	3+4	6+5	4+↗	8+↗	7+↗	10+5
Newton (Exponential)	3+4	5+4	6+3	6+4	7+4	11+4
Newton (Hughes-Brooks)	3+5	4+13	5+12	6+9	8+8	12+8
Newton (Galerkin)	3+4	7+4	↑	↑	↑	↑

TABLE 3

models of boundary layers depending on the choice of u_0 and u_{N+1} . Thus, Problem 1 presents a boundary layer close to 1, while Problem 2 gives rise to a boundary layer on the left of the interval and Problem 3 presents two boundary layers at both ends of the interval. In Figures 1, 2 and 3 the numerical solutions corresponding to each problem are depicted in relationship to the parameter EXP from $\varepsilon = 10^{-\text{EXP}}$. To obtain them, the adjoint method discretization with exponential functions was used. We present as well three different tables related to each considered problem. We specify there the number of iterations necessary to attain an error between two consecutive iterations smaller than $\tau_e = 10^{-8}$ for the three different possibilities of discretization considered in this paper. When applying them, we chose a Kaczmarz smoothing process. This is an outstandingly working smoothing method for non-symmetric matrices, which was studied in [4], among others. For the Petrov-Galerkin discretization the test functions were chosen from a rank of quadratic, exponential and linear ones, those latter functions continuous (the Galerkin case) or discontinuous (the Hughes-Brooks case). We remind that the adjoint method, as indicated in Section 2, was derived using exponential test functions. We also indicate that for every discretization case 3 different grids were built, the coarsest one owning 11 grid points and the finest one 47 grid points apart from the ends of the interval.

The first digit of each entry in the tables indicates the number of grids which must be constructed by the different methods in order to properly adjust the

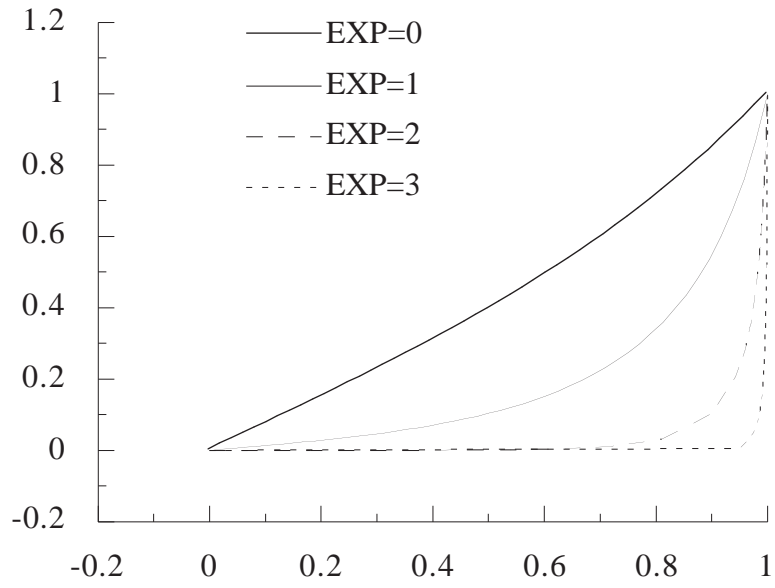


FIGURE 1

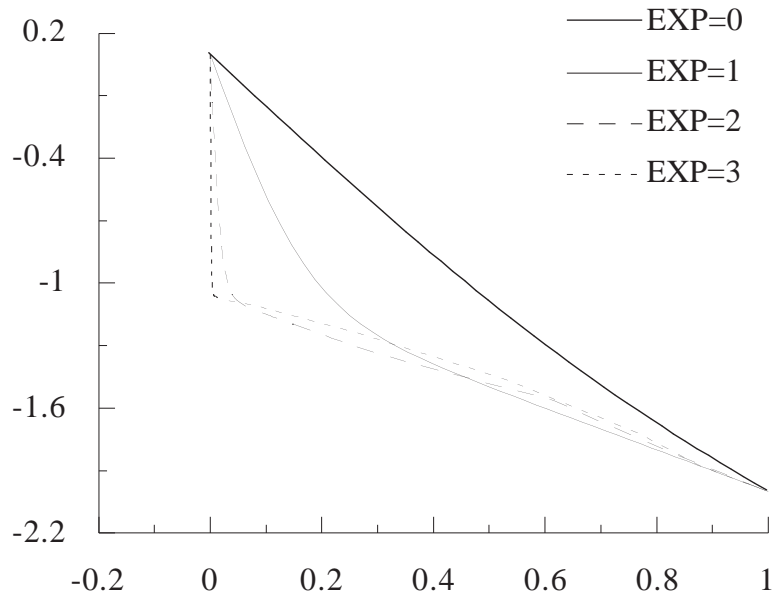


FIGURE 2

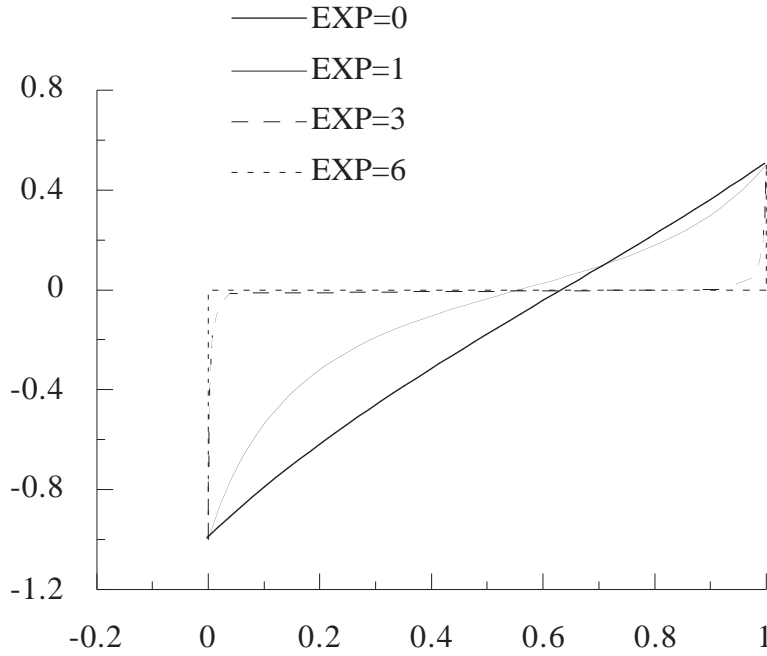


FIGURE 3

shape of the solution. For example, 7 means that 6 grid adaptations took place and 1 that no grid adaption was issued and the original uniform grid remained in the following multigrid iterations. The second digit stands for the number of iterations executed once a fixed grid has been reached. So, if 7+4 is to be read, that means that 7 iterations with changing grids were executed until a definite grid was attained and afterwards other 4 iterations were carried out until the error diminished below the before-mentioned tolerance. If there states \uparrow , it means that the corresponding method applied to the chosen problem for the indicated value of ε diverged. On the other hand, if \nearrow is to be found, it means that the method yielded an oscillating solution, close to the real one but incapable of reducing the error under the given tolerance 10^{-8} . For Nijima's method only one digit stands in the table, as this method is only applicable to uniform grids. Logically when the exponent is bigger than 2 no trace of the boundary layer can be found at the 47 grid points of the finest grid (that is, no fitted shape to the true solution can be expected), so that the method cannot actually be compared to the others even in the case of Problem 2, when it presents convergence. In general, we conclude that, even if both the adjoint method scheme and the direct finite element discretization require a similar number of iterations to reach convergence, the first one yields a definitive grid much faster. This suggests that a combination of both methods, using first the

adjoint scheme to obtain an optimal grid and, after that, a Petrov-Galerkin discretization might be used in order to accelerate the convergence. This will be the focus of a subsequent work.

References

- [1] L. ABRAHAMSSON AND S. OSHER, *Monotone difference schemes for singular perturbation problems*, SIAM J. Numer. Anal. **19** (1982), 979-92.
- [2] B. AUPOIX AND D. ARNAL, *CLIC: Calcul des couches limites compressibles*, O.N.E.R.A., Centre d'Études et de Recherches de Toulouse, 1988.
- [3] C. CLAVERO, *Métodos en diferencias sobre mallas irregulares para la integración de problemas de contorno singularmente perturbados*, Doctoral Thesis, Universidad de Zaragoza, 1989.
- [4] W. HACKBUSCH, *Multigrid methods and applications*, Springer Verlag, Berlín, 1985.
- [5] P. HEMKER, *A numerical study of stiff two-point boundary problems*, Doctoral Thesis, Stichting Mathematisch Centrum, Amsterdam, 1977.
- [6] M. K. KADALBAJOO AND Y. N. REDDY, *Initial-value technique for a class of nonlinear singular perturbation problems*, J. Optimaz. Th. and Appl. **52** (1987), 243-54
- [7] P. LASCAUX AND R. THÉODOR, *Analyse numérique matricielle appliquée a l'art de l'ingénieur*, Ed. Masson, Paris, 1987.
- [8] F. LISBONA, *Análisis de una clase de esquemas en diferencias para problemas singularmente perturbados no lineales*, Internal Communication, Departamento de Matemática Aplicada, Universidad de Zaragoza, 1990.
- [9] J. LORENTZ, *Nonlinear singular perturbation problems and the Engquist-Osher difference scheme*, Report n. 8115, University of Nijmegen, 1981.
- [10] J. F. MAÏTRE AND F. MUSY, *Multigrid methods: convergence theory in a variational framework*, SIAM J. Numer. Anal. **21** (1984), 657-71.
- [11] K. NIJIMA, *On a difference scheme of exponential type for a nonlinear singular perturbation problem*, Num. Math. **46** (1985), 521-39.
- [12] L. RÁNDEZ, *Resolución numérica de problemas de contorno en EDOs por técnicas de tiro múltiple*, Doctoral Thesis, Universidad de Zaragoza, Zaragoza 1991.
- [13] F. USÓN, *Métodos multimalla para discretizaciones por elementos finitos de tipo Petrov-Galerkin*, Doctoral Thesis, Universidad de Zaragoza, Zaragoza 1992.
- [14] F. USÓN, F. PÉTRIZ AND F. LISBONA, *Discretización tipo Petrov-Galerkin para un problema no lineal singularmente perturbado. Algoritmo multimalla*, Actas XIV Jornadas Hispano-Lusas de Matemáticas, Puerto de la Cruz (1989), 1447-51.

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