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Common fixed point theorems for compatible and weakly compatible mappings

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Abstract. Results on common fixed points for pairs of single and multivalued mappings on a complete metric space are examined. Our work establishes a common fixed point theorem for a pair of generalized contraction self-maps and a pair of set-valued mappings.

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1. Introduction

There have been several extensions of known results on fixed points of single valued mappings to fixed points of multivalued mappings, i.e., of mappings which take points of a metric space (X, d) into closed and bounded subsets of X. On the other hand, Khan [4] has established fixed point theorems for self-maps of a complete metric space by altering the distance between points by means of a continuous and strictly increasing function $\phi : [0, +\infty) \to [0, +\infty)$ such that

$$(H) : \qquad \phi(t) = 0 \quad \text{iff} \quad t = 0.$$

Following this technique, for example, Rashwan and Sadeek [7] established the following theorem.

Theorem 1.1. *Let T, S be self-maps of a complete metric space (X, d) and ϕ be a continuous and strictly increasing function: $[0, +\infty) \to [0, +\infty)$ satisfying (H). Furthermore, let a, b, and c be three decreasing functions of $\mathbb{R}$ into $[0, 1)$ such that*

$$a(t) + 2b(t) + c(t) < 1$$

for all $t > 0$. Suppose that T and S satisfy

$$\phi(d(Tx, Sy)) \leq a(d(x, y))\phi(d(x, y)) + b(d(x, y))[\phi(d(x, Tx)) + \phi(d(y, Sy))] + c(d(x, y)) \min\{\phi(d(x, Sy)), \phi(d(y, Tx))\} \quad (1.1)$$

for all $x, y \in X$, $x \neq y$. Then T and S have a unique common fixed point.

In this note we obtain a common fixed point result, by using the notion of compatibility between a set-valued mapping and a single-valued mapping due to Jungck [3], for a pair (I, J) of generalized contraction self-maps of a complete metric space (X, d) and a pair (S, T) of set-valued mappings on x satisfying (see Section 2 for the meaning of the terms).

$$\begin{aligned}\phi(d(Tx, Sy)) \leq\ & a(d(Ix, Jy))\phi(d(Ix, Jy)) \\ & + b(d(Ix, Jy))\big[\phi(\delta(Ix, Tx)) + \phi(\delta(Jy, Sy))\big] \\ & + c(d(Ix, Jy)) \min\big\{\phi(D(Ix, Sy)), \phi(D(Jy, Tx))\big\}, \quad (1.2)\end{aligned}$$

where a, b, and c are continuous functions of $[0, +\infty)$ into $[0, 1)$ such that

$$a(t) + 2b(t) + c(t) < 1,\ t > 0, \quad (1.3)$$

and $\phi : [0, +\infty) \to [0, +\infty)$ is a continuous and increasing function which satisfies (H).

2. Definitions and Preliminaries

Let (X, d) be a metric space. Then, following Fhisher [1] and Nadler [6], we define

$$B(X) = \{A \mid A \text{ is a nonempty bounded subset of } X\}.$$

$$D(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}. \quad (2.1)$$

$$\text{If } A = \{a\}, \text{we write } D(\{a\}, B) = d(a, B) = d(B, a).$$

$$H(A, B) = \max\{\sup\{d(a, B) \mid a \in A\}, \sup\{d(b, A) \mid b \in B\}\}.$$

$$\delta(A, B) = \sup\{d(a, b) \mid a \in A, b \in B\}. \quad (2.2)$$

It is known, for example (Kuratowski [5]), that $CB(X)$, the set of closed subsets of X in $B(X)$, is a metric space with distance function H.

Definition 2.1. A sequence (A_n) of subset of X is said to be convergent to a subset A of X if

(i) For every $a \in A$, there is a sequence (a_n) in X, $a_n \in A_n$ for $n = 0, 1, 2, \ldots$, which converges to a.
(ii) Given $\varepsilon > 0$, there exists a positive integer N such that $A_n \subseteq A_\varepsilon$ for every $n \geq N$, where $A_\varepsilon = \bigcup_{x \in A} B(x, \varepsilon)$ and $B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$.

We shall make frequent use of the following lemmas:

Lemma 2.1. *If (A_n) and (B_n) are sequences in $B(X)$ converging to A and B in $B(X)$, respectively, then the sequence $(\delta(A_n, B_n))$ converges to $\delta(A, B)$.*

Lemma 2.2. *Let (A_n) be a sequence in $B(X)$ and y be a point of X such that $\delta(A_n, y) \to 0$. Then, the sequence (A_n) converges to the set $\{y\}$ in $B(X)$.*

Lemma 2.3. *Let (A_n) be a sequence of nonempty subsets of X and let $a \in X$ be such that $\lim_{n \to +\infty} A_n = \{a\}$. If the self-map I on X is continuous, then $\{Ia\}$ is the limit of the sequence (IA_n).*

For a proof of Lemma 2.3, see [2].

Definition 2.2. The mappings $T : X \to B(X)$ and $I : X \to X$ are said to be *weakly commuting* on X if $ITx \in B(X)$ and

$$\delta(ITx, TIx) \leq \max\big\{\delta(Ix, Tx), \delta(Tx, Tx)\big\}, \quad x \in X.$$

Two commuting mappings T and I ($TIx = ITx, x \in X$) are clearly weakly commuting. The converse is not true in general.

Definition 2.3. The mappings $T : X \to B(X)$ and $I : X \to X$ are *weakly compatible* if they commute at their coincidence points (a point $a \in X$ is a coincidence point of I and T if $Ta = \{Ia\}$).

Definition 2.4. The mappings $T : X \to B(X)$ and $I : X \to X$ are compatible if the following holds: For any sequence (x_n) in X such that $ITx_n \in B(X)$, $Tx_n \to \{t\}$ and $Ix_n \to t$ for some t in X, it follows that $\delta(TIx_n, ITx_n) \to 0$.

Remark 2.1. It is immediate that two compatible mappings T and I are weakly compatible (if a is a coincidence point of T and I, it suffices to consider the constant sequence $x_n = a, \quad n \in \mathbb{N}$).

Two weakly commuting mappings are compatible, but the converse is false, as it is shown in the following example.

Example 2.1. Let $X = [0, +\infty)$ with the Euclidean distance, $Ix = x^2 + 2x$, and $Tx = [0, x^2]$ for all $x \in X$. Then I and T are compatible but not weakly commuting. In fact, for $x = 1$ we have

$$\delta(IT1, TI1) = 9 > 3 = \max\{\delta(I1, T1), \mathrm{diam}(IT1)\},$$

and thus $TI1 = [0, 9] \neq [0, 3] = IT1$.

3. Main Result

In the next theorem we prove the existence of a unique common fixed point for a pair of multi-valued mappings (T, S) and a pair of self-maps (I, J).

Theorem 3.1. *Let (X, d) be a complete metric space and I, J be functions from X into itself. Let $T, S : X \to B(X)$ be set-valued mappings such that*

$$Tx \subseteq JX \quad \text{and} \quad Sx \subseteq IX \tag{3.1}$$

for all $x \in X$. Let ϕ be an increasing and continuous function of $[0, +\infty)$ into $[0, +\infty)$ satisfying (H) and

$$\begin{aligned}\phi(\delta(Tx, Sy)) \leq a(d(Ix, Jy))\phi(d(Ix, Jy))\\ + (d(Ix, Jy))\big[\phi(\delta(Ix, Tx)) + \phi(\delta(Jy, Sy))\big]\\ + c(d(Ix, Jy))\min\big\{\phi(D(Ix, Sy)), \phi(D(Jy, Tx))\big\}\end{aligned} \tag{3.2}$$

for all x, y $x \neq y$, in X, where $a, b, c : [0, +\infty)$ into $[0, 1)$ are continuous functions satisfying (1.3). Suppose in addition that either

(I) *T and I are compatible, I is continuous and S, J are weakly compatible,*

or

(II) *S and J are compatible, J is continuous and T, I are weakly compatible.*

Then I, J, T and S have a unique common fixed point a: $Ta = Sa = \{Ia\} = \{Ja\} = \{a\}$.

Proof. Let $x_0 \in X$, be given. By (3.1) one can choose a point x_1 in X such that $Jx_1 \in Tx_0 = Y_1$, and a point x_2 in X such that $Ix_2 \in Sx_1 = Y_2$. Continuing this way, we define by induction a sequence (x_n) in X such that

$$Jx_{2n+1} \in Tx_{2n} = Y_{2n+1}, \quad Ix_{2n+2} \in Sx_{2n+1} = Y_{2n+2}. \tag{3.3}$$

For simplicity, we set

$$\delta_n = \delta(Y_n, Y_{n+1}), \quad n = 0, 1, 2, \dots \tag{3.4}$$

It follows from (3.2) that for $n = 0, 1, 2, \dots$

$$\phi(\delta_{2n+1}) = \phi(\delta(Y_{2n+1}, Y_{2n+2})) = \phi(\delta(Tx_{2n}, Sx_{2n+1})) \leq A_1 + A_2 + A_3,$$

where

$$\begin{aligned}A_1 &= a(d(Ix_{2n}, Jx_{2n+1}))\phi(d(Ix_{2n}, Jx_{2n+1})) \leq a(\delta_{2n})\phi(\delta_{2n}),\\ A_2 &= b(d(Ix_{2n}, Jx_{2n+1}))\big[\phi(\delta(Ix_{2n}, Tx_{2n})) + \phi(\delta(Jx_{2n+1}, Sx_{2n+1}))\big]\\ &\leq b(\delta_{2n})\big[\phi(\delta_{2n}) + \phi(\delta_{2n+1})\big],\\ A_3 &= c(d(Ix_{2n}, Jx_{2n+1}))\min\big\{\phi(D(Ix_{2n}, Sx_{2n+1})), \phi(D(Jx_{2n+1}, Tx_{2n}))\big\}.\end{aligned}$$

Since $Jx_{2n+1} \in Tx_{2n}$ then $A_3 = 0$, which implies that

$$\phi(\delta_{2n+1}) \leq a(\delta_{2n})\phi(\delta_{2n}) + b(\delta_{2n})\big[\phi(\delta_{2n}) + \phi(\delta_{2n+1})\big], \tag{3.5}$$

so that, taking (1.3) into account,

$$\phi(\delta_{2n+1}) \leq \frac{a(\delta_{2n}) + b(\delta_{2n})}{1 - b(\delta_{2n})}\phi(\delta_{2n}) < \phi(\delta_{2n}). \tag{3.6}$$

Similarly, we have

$$\phi(\delta_{2n+2}) \leq \frac{a(\delta_{2n+1}) + b(\delta_{2n+1})}{1 - b(\delta_{2n+1})}\phi(\delta_{2n+1}) < \phi(\delta_{2n+1}). \tag{3.7}$$

Since ϕ is increasing, (δ_n) is a decreasing sequence. Put $\delta = \lim_{n\to+\infty}\delta_n$. Then $\delta = 0$. In fact, from (3.6) and (3.7),

$$\phi(\delta) \leq \phi(\delta_n) \leq \frac{a(\delta_n) + b(\delta_n)}{1 - b(\delta_n)}\phi(\delta_{n-1}) \tag{3.8}$$

for all n, and letting $n \to +\infty$ in (3.8) yields

$$\phi(\delta) \leq \frac{a(\delta) + b(\delta)}{1 - b(\delta)}\phi(\delta) \tag{3.9}$$

which, in view of (1.3), gives $\phi(\delta) = 0$. Hence, $\delta = 0$.

Let y_n be an arbitrary point in Y_n for $n = 0, 1, 2, \ldots$ We claim that (y_n) is a Cauchy sequence. Since

$$\lim_n d(y_n, y_{n+1}) \leq \lim_n \delta(Y_n, Y_{n+1}) = 0,$$

it is sufficient to show that (y_{2n}) is a Cauchy sequence. We proceed by contradiction. Thus, assume there exists $\varepsilon > 0$ such that for each even integer $2k$, $k = 0, 1, 2, \ldots$, even integers $2m(k)$ and $2n(k)$ with $2k \leq 2n(k) \leq 2m(k)$ can be found for which

$$d(Y_{2m(k)}, Y_{2n(k)}) > \varepsilon. \tag{3.10}$$

For each integer k, fix $2n(k)$ and let $2m(k)$ be the least even integer exceeding $2n(k)$ and satisfying (3.10). Then

$$\delta(Y_{2m(k)-2}, Y_{2n(k)}) \leq \varepsilon, \quad \delta(Y_{2m(k)}, Y_{2n(k)}) > \varepsilon.$$

Hence, for each even integer $2k$ we have, by the triangle inequality,

$$\varepsilon < \delta(Y_{2m(k)}, Y_{2n(k)}) \leq \delta(Y_{2n(k)}, Y_{2m(k)-2}) + \delta_{2m(k)-2} + \delta_{2m(k)-1}.$$

Letting $k \to +\infty$, we obtain

$$\lim_{k\to+\infty} \delta(Y_{2m(k)}, Y_{2n(k)}) = \varepsilon. \tag{3.11}$$

Moreover, by the triangle inequality we also have

$$\begin{aligned} -\delta_{2m(k)} - \delta_{2n(k)} + \delta(Y_{2m(k)}, Y_{2n(k)}) &\leq \delta(Y_{2n(k)+1}, Y_{2m(k)+1}) \\ &\leq \delta_{2m(k)} + \delta_{2n(k)} + \delta(Y_{2m(k)}, Y_{2n(k)}), \end{aligned}$$

and therefore

$$\delta(Y_{2m(k)+1}, Y_{2n(k)+1}) \to \varepsilon \tag{3.12}$$

when $k \to +\infty$. The same argument shows that

$$\begin{aligned}\delta(Y_{2m(k)+1}, Y_{2n(k)+1}) - \delta_{2n(k)} &\leq \delta(Y_{2m(k)+1}, Y_{2n(k)}) \\ &\leq \delta(Y_{2m(k)}, Y_{2n(k)}) + \delta_{2m(k)} \\ &\leq \delta_{2m(k)} + \delta(Y_{2m(k)}, Y_{2n(k)}),\end{aligned}$$

so that also

$$\delta(Y_{2m(k)+1}, Y_{2n(k)}) \to \varepsilon. \tag{3.13}$$

On the other hand, by assumption(3.2),

$$\begin{aligned}\phi(\delta(Y_{2m(k)+2}, Y_{2n(k)+1}) &= \phi(\delta(Sx_{2m(k)+1}, Tx_{2n(k)})) \\ &\leq B_1 + B_2 + B_3 \\ &\leq C_1 + C_2 + C_3,\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
B_1 &= a(d(Ix_{2n(k)}, Jx_{2m(k)+1}))\phi(d(Ix_{2n(k)}, Jx_{2m(k)+1})). \\
B_2 &= b(d(Ix_{2n(k)}, Jx_{2m(k)+1}))\big[\phi(\delta(Ix_{2n(k)}, Tx_{2n(k)})) \\
&\qquad + \phi(\delta(Jx_{2m(k)+1}, Sx_{2m(k)+1}))\big]. \\
B_3 &= c(d(Ix_{2n(k)}, Jx_{2m(k)+1}))\min\big\{\phi(D(Ix_{2n(k)}, Sx_{2m(k)+1})), \\
&\qquad \phi(D(Jx_{2m(k)+1}, Tx_{2n(k)}))\big\}. \\
C_1 &= a(\delta(Y_{2m(k)}, Y_{2n(k)}) - \delta_{2m(k)})\phi(\delta(Y_{2m(k)}, Y_{2n(k)}) + \delta_{2m(k)}). \\
C_2 &= b(\delta(Y_{2m(k)}, Y_{2n(k)}) - \delta_{2m(k)})\big[\phi(\delta_{2n(k)}) + \phi(\delta_{2m(k)+1})\big]. \\
C_3 &= c(\delta(Y_{2m(k)}, Y_{2n(k)} - \delta_{2m(k)}))\min\big\{\phi(\delta(Y_{2m(k)}, Y_{2n(k)}) + \delta_{2m(k)} \\
&\qquad + \delta_{2m(k)+1}, \phi(\delta(Y_{2m(k)+1}, Y_{2n(k)}))\big\}.
\end{aligned}$$

Thus, from (3.11), (3.12) and (3.13), and letting $k \to +\infty$ in (3.14), we obtain

$$\phi(\varepsilon) \leq a(\varepsilon)\phi(\varepsilon) + c(\varepsilon)\phi(\varepsilon) < \phi(\varepsilon)$$

which is a contradiction. This proves our claim.

Since (X, d) is complete, the sequence (y_n) converges in X. Hence, the sequences (Ix_{2n}), (Jx_{2n+1}) constructed in (3.3) converge to one and the same $a \in X$. Furthermore, the sequences of sets (Tx_{2n}) and (Sx_{2n+1}) converge to the singleton $\{a\}$.

Now suppose that (I) is satisfied. Then $I^2x_{2n} \to Ia$ and $ITx_{2n} \to Ia$, which, since T and I are compatible, implies that $TIx_{2n} \to Ia$.

Now we wish to show that a is a common fixed point of I, J, T and S.

(i) a is a fixed point of I. Indeed, we have

$$\begin{aligned}\phi(\delta(TIx_{2n}, Sx_{2n+1})) &\le a(d(I^2x_{2n}, Jx_{2n+1}))\phi(d(I^2x_{2n}, Jx_{2n+1})) \\ &+ b(d(I^2x_{2n}, Jx_{2n+1}))\big[\phi(\delta(I^2x_{2n}, TIx_{2n})) + \phi(\delta(Jx_{2n+1}, Sx_{2n+1}))\big] \\ &+ c(d(I^2x_{2n}, Jx_{2n+1}))\min\big\{\phi(D(I^2x_{2n}, Sx_{2n+1})), \phi(D(Jx_{2n+1}, TIx_{2n}))\big\}.\end{aligned} \tag{3.15}$$

Letting $n \to +\infty$ yields

$$\begin{aligned}\phi(d(Ia,a)) &\le a(d(Ia,a))\phi(d(Ia,a)) + b(d(Ia,a))\big[\phi(d(Ia,Ia)) + \phi(d(a,a))\big] \\ &+ c(d(Ia,a))\min\big\{\phi(d(Ia,a)), \phi(d(Ia,a))\big\} \\ &= \big[a(d(Ia,a)) + c(d(Ia,a))\big]\phi(d(Ia,a)).\end{aligned}$$

Hence, $Ia = a$.

(ii) a is a fixed point of T. Indeed,

$$\begin{aligned}\phi(\delta(Ta, Sx_{2n+1})) &\le a(d(Ia, Jx_{2n+1}))\phi(d(Ia, Jx_{2n+1})) \\ &+ b(d(Ia, Jx_{2n+1}))\big[\phi(\delta(Ia, Ta)) + \phi(\delta(Jx_{2n+1}, Sx_{2n+1}))\big] \\ &+ c(d(Ia, Jx_{2n+1}))\min\big\{\phi(D(Ia, Sx_{2n+1})), \phi(D(Jx_{2n+1}, Ta))\big\},\end{aligned}$$

and letting $n \to +\infty$, gives

$$\phi(d(Ta,a)) \le \big[a(d(a,a)) + b(d(a,a)) + c(d(a,a))\big]\phi(d(Ia,a)) = 0.$$

Hence, $Ta = \{a\}$.

(iii) Since $Tx \subseteq JX$ for all $x \in X$, there is a point $b \in X$ such that

$$Ta = \{a\} = \{Jb\}. \tag{3.16}$$

We show that b is a coincidence point for J and S. Indeed, by (3.2) we have

$$\begin{aligned}\phi(\delta(Ta, Sb)) &\le a(d(a, Jb))\phi(d(a, Jb)) + b(d(a, Jb))\big[\phi(\delta(a, Ta)) + \phi(\delta(Jb, Sb))\big] \\ &+ c(d(a, Jb))\min\big\{\phi(D(a, Sb)), \phi(D(Jb, Ta))\big\} \\ &= b(0)\phi(\delta(Jb, Sb)), r\end{aligned}$$

the last equality being a consequence of (3.16). Thus

$$Sb = \{a\} = Ta = \{Jb\}, \tag{3.17}$$

and b is as claimed.

Since J and S are weakly compatible, we deduce that

$$JSb = SJb = Sa = \{Ja\}. \tag{3.18}$$

Also, $\phi(d(a, Ja)) = \phi(d(Ta, Sa))$ and (3.2), together with $Ia = a$, $Ta = \{a\}$, (3.16) and (3.17), ensures that $d(Ta, Sa) = 0$. This implies that $\{a\} = \{Ja\} = Sa$, and the proof of existence of a common fixed point is complete under assumption (I). The proof under assumption (II) is entirely similar. Since uniqueness follows at once from (3.2), the proof of the theorem is complete. ☑

Remark 3.1. It follows from Remark 2.1, that the result of the above theorem holds if T and I (or J and S) are assumed to be weakly commuting.

Corollary 3.1. *Let (X,d) be a complete metric space and let $T,S : X \to B(X)$ be set-valued mappings such that*

$$\phi(\delta(Tx,Sy)) \leq a(d(x,y))\phi(d(x,y)) + b(d(x,y))\big[\phi(\delta(x,Tx)) + \phi(\delta(y,Sy))\big] + c(d(x,y))\min\big\{\phi(D(x,Sy)), \phi(D(y,Tx))\big\} \tag{3.19}$$

for all x, y, $x \neq y$, in X, where $\phi : [0,+\infty) \to [0,+\infty)$ is an increasing and continuous function which satisfies (H), and $a, b, c : [0,+\infty) \to [0,1)$ are as in Theorem 3.1. Then T and S have a unique common fixed point a: $Ta = Sa = \{a\}$.

Proof. It suffices to consider $I = J = id_X$, the identity map of X, and apply Theorem 3.1. ☑

Remark 3.2. If we suppose that I, J, T and S are as in Theorem 3.1, but with the condition

$$\phi(\delta(Tx,Sy)) \leq a(d(Ix,Jy))\phi(d(Ix,Jy)) + b(d(Ix,Jy))\big[\phi(\delta(Ix,Tx)) + \phi(\delta(Jy,Sy))\big] + c(d(Ix,Jy))\left[\frac{\phi(D(Ix,Sy)) + \phi(D(Jy,Tx))}{2}\right]$$

replacing (3.2), and if ϕ satisfies, in addition to the hypothesis of Theorem 3.1, the condition

$$\phi(2t) \leq 2\phi(t), \quad t \geq 0,$$

then we can prove similarly that I, J, T and S have a unique common fixed point a:

$$\{Ia\} = \{Ja\} = Ta = Sa = \{a\}.$$

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