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Operational calculi for Kontorovich-Lebedev and Mehler-Fock transforms on distributions with compact support

B. J. González
E. R. Negrín*
Universidad de La Laguna, Islas Canarias, ESPAÑA

Abstract. Operational calculi are analized for Kontorovich-Lebedev and Mehler-Fock transforms on distributions with compact supports.

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1. **Introduction**

The purpose of this paper is to exhibit operational calculi on distributions with compact support for two of the most useful index transforms: Kontorovich-Lebedev ($\mathfrak{KL}$) and Mehler-Fock ($\mathfrak{MF}$). Their corresponding inversion formulae are the key to these calculi. Starting with their inversion formulae and using a variant of the Banach-Steinhaus theorem for barreled spaces, we obtain in both cases a distribution which solves a differential equations with constants coefficients which involves certain operators related with each of the transforms.

Specifically, one solves distributional equations of type $P\left(A_t'\right) f = g$, where, for the Kontorovich-Lebedev case, P denotes any polynomial with no zeros in

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the interval $(-\infty, 0)$, g denotes any distribution with compact support on the interval $(0, \infty)$, and A_t' denotes the formal adjoint of the differential operator

$$A_t \equiv t^2 D_t + t D_t - t^2, \tag{1.1}$$

and for the Mehler-Fock case, P denotes any polynomial with no zeros in the interval $\left(-\infty, -\left(n+\frac{1}{2}\right)^2\right)$, g denotes any distribution with compact support on the interval $(1, \infty)$ and A_t' denotes the formal adjoint of the differential operator

$$A_t \equiv \left(t^2-1\right)^{-n/2} D_t \left(t^2-1\right)^{n+1} D_t \left(t^2-1\right)^{-n/2}, \tag{1.2}$$

$n \in \mathbb{N}$, fixed.

An outstanding result, basic to our purposes, is an equivalence of the usual topology with topologies arising from the aforementioned operators, on the space of infinitely differentiable functions on the interval I, where $I = (0, \infty)$ for the Kontorovich-Lebedev case and $I = (1, \infty)$ for the Mehler-Fock case. This equivalence of topologies provides certain operational rules for the respective index transforms which allow to obtain the distribution solution as a limit of specific distributions connected with the corresponding inversion formulae.

Related work on operational calculi for index transforms have been carried out in [6], [8], [9] and [10] among others.

2. Operational calculus: the Kontorovich-Lebedev case

In [12], the Kontorovich-Lebedev transform, whose kernel $K_{i\tau}(t)$, $t \in I = (0, \infty)$, $\tau > 0$, is the Macdonald function, and which acts on the space of distributions with compact support on the interval $(0, \infty)$, has been studied in detail. In particular, an inversion formula is established, which is basic for our purposes. The precise result is:

If $f \in \mathcal{E}'(I)$, and for $\tau > 0$ we set

$$F(\tau) = (\mathfrak{KL}[f])(\tau) = \langle f(t), K_{i\tau}(t) \rangle, \tag{2.1}$$

then, for every $\phi \in \mathcal{D}(I)$,

$$\langle f, \phi \rangle = \lim_{T \to \infty} \left\langle \frac{2}{\pi^2 y} \int_0^T F(\tau) K_{i\tau}(y) \tau \sinh \pi\tau d\tau, \phi(y) \right\rangle, \tag{2.2}$$

the limit being in the sense of $\mathcal{D}'(I)$.

In order to establish the main result of this section we need the following two lemmas:

Lemma 2.1. *For each compact subset $K \subset I$ and each $k \in \mathbb{N} \cup \{0\}$, let $\gamma_{k,K}$ be the seminorm on $\mathcal{E}(I)$ given by*

$$\gamma_{k,K}(\phi) = \sup_{t \in K} \left| A_t^k \phi(t) \right|, \qquad \phi \in \mathcal{E}(I),$$

where A_t is the differential operator given by (1.1) and A_t^k is its k^{th} iteration. Then $\{\gamma_{k,K}\}$ generates a topology on $\mathcal{E}(I)$ which agrees with the usual topology of this space.

Proof. The expression for A_t^k given in [3, Formula (2.16), p. 73] yields that any sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathcal{E}(I)$ which tends to zero for the usual topology on $\mathcal{E}(I)$ also tends to zero for the topology generated by the family of seminorms $\{\gamma_{k,K}\}$.

Conversely, let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence on $\mathcal{E}(I)$ which tends to zero with respect to the topology generated by $\{\gamma_{k,K}\}$. It is clear that ϕ_n and $A_t\phi_n$ tend to zero as $n \to \infty$, uniformly in each compact subset $K \subset I$. Moreover, for (1.1),

$$A_t\phi_n(t) + t^2\phi_n(t) = t^2 D_t^2 \phi_n(t) + tD_t\phi_n(t), \tag{2.3}$$

and the left hand side of (2.3) also tends to zero as $n \to \infty$, uniformly on each compact subset $K \subset I$.

Now, the right-hand side of (2.3) is $tD_t[tD_t\phi_n(t)]$,and thus, $D_t[tD_t\phi_n(t)]$ tends to zero as $n \to \infty$, uniformly on each compact subset $K \subset I$.

Since, for any $a \in I$, $a \notin K$,

$$\int_a^t D_x[xD_x\phi_n(x)]\,dx = tD_t\phi_n(t) - aD_t\phi_n(a), \tag{2.4}$$

it follows that $tD_t\phi_n(t) - aD_t\phi_n(a)$ tends to zero as $n \to \infty$, uniformly for t on the compact subset $K \subset I$. Dividing by t and integrating once again, one has

$$\int_a^t \left[D_x\phi_n(x) - \frac{a}{x}D_x\phi_n(a)\right]dx = \phi_n(t) - \phi_n(a) - aD_t\phi_n(a)(\ln t - \ln a),$$

and thus, noting that $\ln t - \ln a$ is bounded away from zero for all $t \in K$, we see that $D_t\phi_n(a) \longrightarrow 0$ as $n \to \infty$. Consequently, $D_t\phi_n$ tends to zero as $n \to \infty$, uniformly on each compact subset $K \subset I$. From (2.3), the same conclusion holds for $D_t^2\phi_n$.

Now assume by induction that, for $0 \le m \le 2k-2$, $D_t^m\phi_n$ tends to zero as $n \to \infty$, uniformly on each compact subset $K \subset I$. From the equality

[3, Formula (2.16), p. 73],

$$A_t^k \phi_n(t) = \sum_{j=0}^{2k} t^j P_j^k(t) D_t^j \phi_n(t),$$

where the P_j^k are polynomials such that $P_{2k}^k(t) = 1$ and $P_{2k-1}^k(t) = k(2k-1)$. Then

$$\begin{aligned} A_t^k \phi_n(t) - \sum_{j=0}^{2k-2} t^j P_j^k(t) D_t^j \phi_n(t) &= t^{2k} D_t^{2k} \phi_n(t) - t^{2k-1} k(2k-1) D_t^{2k-1} \phi_n(t) \\ &= t^{k(3-2k)} D_t \left[t^{k(2k-1)} D_t^{2k-1} \phi_n(t) \right], \end{aligned}$$

which, arguing as for the case $k = 1$, yields that $D_t^{2k-1}\phi_n$ and $D_t^{2k}\phi_n$ tend to zero as $n \to \infty$, uniformly in each compact subset $K \subset I$.

Finally, taking into account that the topologies on $\mathcal{E}(I)$ for both families of semi-norms, the usual and the $\gamma_{k,K}$'s, are metrizable, the conclusion follows. ☑

The next assertion establishes the asymptotic behaviour of the function F in (2.1).

Lemma 2.2. *Let f be in $\mathcal{E}'(I)$, and let F be defined by (2.1). Then one has*

$$F(\tau) = O(1), \qquad \tau \to 0^+, \tag{2.5}$$

and

$$F(\tau) = O\left(\tau^r e^{-\pi\tau/2}\right), \qquad \tau \to \infty, \tag{2.6}$$

for some nonnegative integer r.

Proof. According to Lemma 2.1 above, we may consider the space $\mathcal{E}(I)$ equipped with the topology arising from the family of seminorms $\gamma_{k,K}$. From [7, Proposition 2, p. 97], there exist $C > 0$ and a nonnegative integer p, both depending on f, such that

$$|F(\tau)| = |\langle f(t), K_{i\tau}(t)\rangle| \le C \max_{0\le k\le p} \max_{t\in K} \left|A_t^k K_{i\tau}(t)\right| = C \max_{0\le k\le p} \max_{t\in K} \left|\tau^{2k} K_{i\tau}(t)\right|.$$

Now taking into account that for each fixed $t \in (0, \infty)$ one has

$$K_{i\tau}(t) = O(1), \qquad \tau \to 0^+, \tag{2.7}$$

as follows from the integral representation ([2, Formula (21), p. 82])

$$K_{i\tau}(t) = \int_0^\infty e^{-t\cosh u} \cos \tau u \, du,$$

and that for each fixed $t \in (0, \infty)$,

$$K_{i\tau}(t) = O\left(e^{-\pi\tau/2}\left(\tau^2 - t^2\right)^{-1/4}\right), \qquad \tau \to \infty, \tag{2.8}$$

(see [2, Formula (19), p. 88]), the conclusion follows since t ranging on a compact subset K of I. ☑

We observe that for $f \in \mathcal{E}'(I)$, it is straightforward that

$$\left(\mathfrak{KL}\left[\left(A_t'\right)^k f\right]\right)(\tau) = (-1)^k \tau^{2k} \left(\mathfrak{KL}[f]\right)(\tau),$$

for all $k \in \mathbb{N} \cup \{0\}$ and $\tau > 0$, where A_t' denotes the formal adjoint of the differential operator A_t in (1.1).

Next, we establish the main result of this section.

Theorem 1. *Assume $g \in \mathcal{E}'(I)$ and let P be a polynomial with no zeros in the interval $(-\infty, 0)$. Then, the distribution f in $\mathcal{D}'(I)$ defined for any $\phi \in \mathcal{D}(I)$ by*

$$\langle f, \phi \rangle = \lim_{T\to\infty} \left\langle \frac{2}{\pi^2 y} \int_0^T \frac{G(\tau)}{P(-\tau^2)} K_{i\tau}(y)\, \tau \sinh \pi\tau d\tau, \phi(y) \right\rangle, \tag{2.9}$$

where G denotes the Kontorovich-Lebedev transform (2.1) of g, satisfies the operational equation

$$P\left(A_t'\right) f = g. \tag{2.10}$$

Proof. To show the existence of the limit in (2.9), we use the variant of the Banach-Steinhaus theorem in [7, Corollary of Proposition 5, p. 216]. To do so, we take a polynomial Q of degree $r+1$, with no zeros in $(-\infty, 0)$. We have

$$\begin{aligned} &\left\langle \frac{2}{\pi^2 y} \int_0^T \frac{G(\tau)}{P(-\tau^2)} K_{i\tau}(y)\, \tau \sinh \pi\tau d\tau, \phi(y) \right\rangle \\ &= \left\langle \frac{2}{\pi^2} Q\left(A_y'\right) \int_0^T \frac{G(\tau)}{P(-\tau^2)\, Q(-\tau^2)} \frac{K_{i\tau}(y)}{y} \tau \sinh \pi\tau d\tau, \phi(y) \right\rangle \\ &= \left\langle \frac{2}{\pi^2} \int_0^T \frac{G(\tau)}{P(-\tau^2)\, Q(-\tau^2)} \frac{K_{i\tau}(y)}{y} \tau \sinh \pi\tau d\tau, Q\left(A_y\right) \phi(y) \right\rangle. \end{aligned} \tag{2.11}$$

Now, we may assume that the support of ϕ is contained in the interval $[a, b] \subset (0, \infty)$. Thus, (2.11) can be written as

$$\frac{2}{\pi^2} \int_0^T \frac{G(\tau)\, \tau \sinh \pi\tau}{P(-\tau^2)\, Q(-\tau^2)} \int_a^b \frac{K_{i\tau}(y)\, Q\left(A_y\right) \phi(y)}{y} dy d\tau. \tag{2.12}$$

From (2.5) and (2.6), and taking into account (2.7) and (2.8), it follows that, for some suitable positive constants C, D, E, T_1 and T_2, (2.12) is bounded above by

$$C\int_0^{T_1}\left|\frac{\tau^2}{P\left(-\tau^2\right)Q\left(-\tau^2\right)}\right|d\tau + D\int_{T_1}^{T_2}\left|\frac{G\left(\tau\right)\tau\sinh\pi\tau}{P\left(-\tau^2\right)Q\left(-\tau^2\right)}\right|d\tau$$
$$+E\int_{T_1}^{T}\left|\frac{\tau^{2r+1}e^{-\pi\tau/2}e^{\pi\tau}}{P\left(-\tau^2\right)Q\left(-\tau^2\right)}\tau^{-1/2}e^{-\pi\tau/2}\right|d\tau.$$

Clearly, these integrals are bounded. Therefore, the limit in (2.9) exists for all $\phi\in\mathcal{D}\left(I\right)$, which proves that $f\in\mathcal{D}'\left(I\right)$.

In order to prove that f satisfies equation (2.10), observe that, from the inversion formula (2.2), it follows for all $\phi\in\mathcal{D}\left(I\right)$, that

$$\begin{aligned}\left\langle P\left(A'_y\right)f,\phi\right\rangle &= \left\langle f,P\left(A_y\right)\phi\right\rangle\\ &= \lim_{T\to\infty}\left\langle\frac{2}{\pi^2 y}\int_0^T\frac{G\left(\tau\right)}{P\left(-\tau^2\right)}K_{i\tau}\left(y\right)\tau\sinh\pi\tau d\tau, P\left(A_y\right)\phi\left(y\right)\right\rangle\\ &= \lim_{T\to\infty}\left\langle\frac{2}{\pi^2}P\left(A'_y\right)\int_0^T\frac{G\left(\tau\right)}{P\left(-\tau^2\right)}\frac{K_{i\tau}\left(y\right)}{y}\tau\sinh\pi\tau d\tau,\phi\left(y\right)\right\rangle\\ &= \lim_{T\to\infty}\left\langle\frac{2}{\pi^2 y}\int_0^T G\left(\tau\right)K_{i\tau}\left(y\right)\tau\sinh\pi\tau d\tau,\phi\left(y\right)\right\rangle = \left\langle g,\phi\right\rangle. \quad ☑\end{aligned}$$

3. Operational calculus: the Mehler-Fock case

The Mehler-Fock transform of order n, whose kernel $P^{-n}_{-\frac{1}{2}+i\tau}\left(t\right)$, $t\in I=(1,\infty)$, $\tau>0$, is the associated Legendre function of the first kind and order $n\in\mathbb{N}$, and which acts on the space of distributions with compact support on the interval $(1,\infty)$, is examined in detail in [5]. In particular, an inversion formula there established will be of utmost importance to our purposes. To be precise, the result needed is [5, Theorem 4.1]:

If $f\in\mathcal{E}'\left(I\right)$, $\tau>0$, $n\in\mathbb{N}$, *and we set*

$$F\left(\tau\right)=\left(\mathfrak{MF}\left[f\right]\right)\left(\tau\right)=\left\langle f\left(t\right),P^{-n}_{-\frac{1}{2}+i\tau}\left(t\right)\right\rangle, \tag{3.1}$$

then

$$\left\langle f,\phi\right\rangle = \lim_{T\to\infty}\left\langle\frac{1}{\pi}\int_0^T\tau\sinh\pi\tau\Gamma\Big(n+\frac{1}{2}+i\tau\Big)\Gamma\Big(n+\frac{1}{2}-i\tau\Big)\times P^{-n}_{-\frac{1}{2}+i\tau}(t)F(\tau)d\tau,\phi(t)\right\rangle \tag{3.2}$$

for all $\phi \in \mathcal{D}(I)$, the limit being in the sense of $\mathcal{E}'(I)$.

The following result will be also useful in this section.

Lemma 3.1. *For each compact subset $K \subset I$ and $k \in \mathbb{N} \cup \{0\}$, let $\gamma_{k,K}$ be the seminorm in $\mathcal{E}(I)$ given by*

$$\gamma_{k,K}(\phi) = \sup_{t \in K} \left| A_t^k \phi(t) \right|, \qquad \phi \in \mathcal{E}(I),$$

where A_t is the differential operator given by (1.2). Then $\{\gamma_{k,K}\}$ generates a topology on $\mathcal{E}(I)$ which agrees with its usual topology.

Proof. The proof is parallel to that of Lemma 2.1. Observe in this case that, for each $\phi \in \mathcal{E}(I)$,

$$A_t\phi(t) + \left[n(n+1) + \frac{n^2}{t^2-1}\right]\phi(t) = \left(t^2-1\right) D_t^2\phi(t) + 2tD_t\phi(t), \tag{3.3}$$

and that the right hand side of (3.3) is $D_t\left[\left(t^2-1\right) D\phi(t)\right]$.

Now let $\{\phi_m\}_{m\in\mathbb{N}}$ be a sequence in $\mathcal{E}(I)$ which tends to zero with respect to the topology generated by $\{\gamma_{k,K}\}$,that for any $a \in I$, and $a \notin K$,

$$\int_a^t D_x\left[\left(x^2-1\right) D_x\phi_m(x)\right] dx = \left(t^2-1\right) D_t\phi_m(t) - \left(a^2-1\right) D_t\phi_m(a), \tag{3.4}$$

tends to zero uniformly on K.

Arguing as in Lemma 2.1, both

$$D_t\phi_m(t) - \frac{a^2-1}{t^2-1} D_t\phi_m(a)$$

and

$$\int_a^t \left[D_x\phi_m(x) - \frac{a^2-1}{x^2-1} D_x\phi_m(a)\right] dx$$
$$= \phi_m(t) - \phi_m(a) - \left(a^2-1\right) D_t\phi_m(a) \int_a^t \frac{1}{x^2-1} dx$$

tend to zero uniformly on K. Thus, $D_t\phi_m(a) \longrightarrow 0$ as $m \to \infty$.

Now, from (3.4) it follows that $D_t\phi_m \longrightarrow 0$ as $m \to \infty$, and from (3.3) that $D_t^2\phi_m \longrightarrow 0$ as $m \to \infty$, both uniformly on K.

For the general case, one proceeds by induction on k. To do so, we make use of the equality (see [4, p. 121])

$$A_t^k = \sum_{j=0}^{2k} \left(t^2-1\right)^{j-k} p_{j,k}(t) D_t^j$$

where the $p_{j,k}$ are polynomials such that the $p_{2k,k}(t) = 1$ and $p_{2k-1,k}(t) = 2k^2t$.

Thus, for all $\phi \in \mathcal{E}(I)$,

$$\begin{aligned} A_t^k \phi(t) - \sum_{j=0}^{2k-2} \left(t^2-1\right)^{j-k} p_{j,k}(t) D_t^j \phi(t) \\ &= \left(t^2-1\right)^k D_t^{2k}\phi(t) + \left(t^2-1\right)^{k-1} 2k^2 t D_t^{2k-1}\phi(t) \\ &= \left(t^2-1\right)^{k(1-k)} D_t\left[\left(t^2-1\right)^{k^2} D_t^{2k-1}\phi(t)\right]. \end{aligned}$$

Therefore, the same argument as for the case $k=1$ yields the conclusion for all k. ☑

Arguing as in Lemma 2.2, it is proved in [5, Theorem 3.2] that for every $f \in \mathcal{E}'(I)$ there exists a nonnegative integer r such that

$$F(\tau) = O(1), \qquad \tau \to 1^+, \tag{3.5}$$

and

$$F(\tau) = O\left(\tau^{2r}\right), \qquad \tau \to \infty. \tag{3.6}$$

Furthermore, for all $f \in \mathcal{E}'(I)$,

$$\left(\mathfrak{M}\mathfrak{F}[A_t'^k f]\right)(\tau) = \left[-\left(n+\frac{1}{2}\right)^2 - \tau^2\right]^k \left(\mathfrak{M}\mathfrak{F}[f]\right)(\tau), \tag{3.7}$$

holds for all $k \in \mathbb{N} \cup \{0\}$, $\tau > 0$ and $n \in \mathbb{N}$, where A_t' denotes the formal adjoint of the differential operator A_t given by (1.2) (see [5, Proposition 3.1]).

Next we establish the main result of this section.

Theorem 2. *Assume $g \in \mathcal{E}'(I)$ and let P be a polynomial with no zeros in the interval $\left(-\infty, -\left(n+\frac{1}{2}\right)^2\right)$, where $n \in \mathbb{N}$ is fixed. Then, the distribution f in $\mathcal{D}'(I)$ given for any $\phi \in \mathcal{D}(I)$ by*

$$\begin{aligned} \langle f, \phi \rangle = \lim_{T\to\infty} \Bigg\langle \frac{1}{\pi} \int_0^T \tau \sinh \pi\tau \Gamma\left(n+\frac{1}{2}+i\tau\right) \Gamma\left(n+\frac{1}{2}-i\tau\right) \\ \times P_{-\frac{1}{2}+i\tau}^{-n}(t) \frac{G(\tau)}{P\left(-\left(n+\frac{1}{2}\right)^2 - \tau^2\right)} d\tau, \phi(t) \Bigg\rangle, \end{aligned} \tag{3.8}$$

G denoting the Mehler-Fock transform (3.1) of g, satisfies the operational equation

$$P\left(A_t'\right) f = g \tag{3.9}$$

Proof. In order to prove that f is in fact in $\mathcal{D}'(I)$ we resort again to the variant of the Banach-Steinhaus theorem considered in [7, Corollary of Proposition 5, p. 216] and prove that the limit in (3.8) exists for all $\phi \in \mathcal{D}(I)$. To do so, we take a polynomial Q of degree $r+n+1$, with no zeros in $\left(-\infty, -\left(n+\frac{1}{2}\right)^2\right)$. Now,

$$\begin{aligned}
&\left\langle \frac{1}{\pi}\int_0^T \tau\sinh\pi\tau\Gamma(n+\frac{1}{2}+i\tau)\Gamma(n+\frac{1}{2}-i\tau)P^{-n}_{-\frac{1}{2}+i\tau}(t)\right.\\
&\qquad\qquad \left.\times\frac{G(\tau)}{P(-(n+\frac{1}{2})^2-\tau^2)}d\tau, \phi(t)\right\rangle\\
&= \left\langle Q(A'_t)\frac{1}{\pi}\int_0^T \tau\sinh\pi\tau\Gamma(n+\frac{1}{2}+i\tau)\Gamma(n+\frac{1}{2}-i\tau)P^{-n}_{-\frac{1}{2}+i\tau}(t)\right.\\
&\qquad\qquad \left.\times\frac{G(\tau)}{P(-(n+\frac{1}{2})^2-\tau^2)Q(-(n+\frac{1}{2})^2-\tau^2)}d\tau, \phi(t)\right\rangle\\
&= \left\langle \frac{1}{\pi}\int_0^T \tau\sinh\pi\tau\Gamma(n+\frac{1}{2}+i\tau)\Gamma(n+\frac{1}{2}-i\tau)P^{-n}_{-\frac{1}{2}+i\tau}(t)\right.\\
&\qquad \left.\times\frac{G(\tau)}{P(-(n+\frac{1}{2})^2-\tau^2)Q(-(n+\frac{1}{2})^2-\tau^2)}d\tau, Q(A_t)\phi(t)\right\rangle.
\end{aligned}\tag{3.10}$$

Again, we assume that the support of ϕ is contained in the interval $[a,b] \subset (1,\infty)$. Thus, expression (3.10) can be written as

$$\begin{aligned}
&\frac{1}{\pi}\int_0^T \frac{\tau\sinh\pi\tau\Gamma\left(n+\frac{1}{2}+i\tau\right)\Gamma\left(n+\frac{1}{2}-i\tau\right)G(\tau)}{P\left(-\left(n+\frac{1}{2}\right)^2-\tau^2\right)Q\left(-\left(n+\frac{1}{2}\right)^2-\tau^2\right)}\\
&\qquad\times\int_a^b P^{-n}_{-\frac{1}{2}+i\tau}(t)\,Q(A_t)\,\phi(t)dtd\tau.
\end{aligned}$$

Now observe that there exists a constant M such that

$$\begin{aligned}
&\int_0^T \left|\frac{\tau\sinh\pi\tau\Gamma\left(n+\frac{1}{2}+i\tau\right)\Gamma\left(n+\frac{1}{2}-i\tau\right)G(\tau)}{P\left(-\left(n+\frac{1}{2}\right)^2-\tau^2\right)Q\left(-\left(n+\frac{1}{2}\right)^2-\tau^2\right)}\right|\\
&\qquad\times\int_a^b \left|P^{-n}_{-\frac{1}{2}+i\tau}(t)\,Q(A_t)\,\phi(t)\right|dtd\tau\\
&\le M\int_0^T \left|\frac{\tau\sinh\pi\tau\Gamma\left(n+\frac{1}{2}+i\tau\right)\Gamma\left(n+\frac{1}{2}-i\tau\right)G(\tau)}{P\left(-\left(n+\frac{1}{2}\right)^2-\tau^2\right)Q\left(-\left(n+\frac{1}{2}\right)^2-\tau^2\right)}\right|\\
&\qquad\times\int_{\arg\cosh a}^{\arg\cosh b} \left|P^{-n}_{-\frac{1}{2}+i\tau}(\cosh t)\sinh t\right|dtd\tau,
\end{aligned}\tag{3.11}$$

and that from (3.5), (3.6) and the following facts

$$P^{-n}_{-\frac{1}{2}+i\tau}\left(\cosh t\right)=O\left(1\right),\quad \text{as } \tau\to 0^{+},\quad \text{for all } t\in(0,\infty),$$

([1, Formula 3.7(6), p. 155] and [5, (2.3)]),

$$P^{-n}_{-\frac{1}{2}+i\tau}\left(\cosh t\right)=O\left(\tau^{-n-\frac{1}{2}}\frac{e^{t/2}}{\left(e^{2t}-1\right)^{1/2}}\right),\quad \text{as} \tau\to+\infty,\quad \text{for all } t\in(0,\infty),$$

[11, Formula (24), p. 231], and

$$\tau\sinh\pi\tau\Gamma\left(n+\frac{1}{2}+i\tau\right)\Gamma\left(n+\frac{1}{2}-i\tau\right)=O\left(\tau^{2}\right),\qquad \text{as } \tau\to 0^{+},$$

$$\tau\sinh\pi\tau\Gamma\left(n+\frac{1}{2}+i\tau\right)\Gamma\left(n+\frac{1}{2}-i\tau\right)=O\left(\tau^{2n+1}\right),\qquad \text{as } \tau\to+\infty,$$

(see [1, 1.18(6), p. 47]), there exist suitable positive constants C, D, E, T_1 and T_2 such that the right hand side of (3.11) is bounded above by

$$C\int_0^{T_1}\left|\frac{\tau^2}{P\left(-\left(n+\frac{1}{2}\right)^2-\tau^2\right)Q\left(-\left(n+\frac{1}{2}\right)^2-\tau^2\right)}\right|d\tau$$

$$+D\int_{T_1}^{T_2}\left|\frac{\tau\sinh\pi\tau\Gamma\left(n+\frac{1}{2}+i\tau\right)\Gamma\left(n+\frac{1}{2}-i\tau\right)G(\tau)}{P\left(-\left(n+\frac{1}{2}\right)^2-\tau^2\right)Q\left(-\left(n+\frac{1}{2}\right)^2-\tau^2\right)}\right|$$

$$\times\int_{\arg\cosh a}^{\arg\cosh b}\left|P^{-n}_{-\frac{1}{2}+i\tau}\left(\cosh t\right)\sinh t\right|dt\,d\tau$$

$$+E\int_{T_1}^{T}\left|\frac{\tau^{2r+n+\frac{1}{2}}}{P\left(-\left(n+\frac{1}{2}\right)^2-\tau^2\right)Q\left(-\left(n+\frac{1}{2}\right)^2-\tau^2\right)}\right|d\tau.$$

Clearly the integrals are bounded. Therefore, the limit in (3.8) exists for all $\phi \in \mathcal{D}\left(I\right)$, and thus $f\in\mathcal{D}'\left(I\right)$.

In order to prove that f satisfies equation (3.9) just observe that from the inversion formula (3.2) it follows, for all $\phi \in \mathcal{D}(I)$ that

$$\begin{aligned}
&\left\langle P(A_t')f, \phi\right\rangle = \left\langle f, P\left(A_t\right)\phi\right\rangle = \\
&= \lim_{T\to\infty}\left\langle \frac{1}{\pi}\int_0^T \tau\sinh\pi\tau\Gamma\left(n+\frac{1}{2}+i\tau\right)\Gamma\left(n+\frac{1}{2}-i\tau\right)P^{-n}_{-\frac{1}{2}+i\tau}(t)\right. \\
&\qquad\qquad \left.\times\frac{G(\tau)}{P\left(-\left(n+\frac{1}{2}\right)^2-\tau^2\right)}d\tau, P\left(A_t\right)\phi(t)\right\rangle \\
&= \lim_{T\to\infty}\left\langle P\left(A_t'\right)\frac{1}{\pi}\int_0^T \tau\sinh\pi\tau\Gamma\left(n+\frac{1}{2}+i\tau\right)\Gamma\left(n+\frac{1}{2}-i\tau\right)P^{-n}_{-\frac{1}{2}+i\tau}(t)\right. \\
&\qquad\qquad \left.\times\frac{G(\tau)}{P\left(-\left(n+\frac{1}{2}\right)^2-\tau^2\right)}d\tau, \phi(t)\right\rangle \\
&= \lim_{T\to\infty}\left\langle \frac{1}{\pi}\int_0^T \tau\sinh\pi\tau\Gamma\left(n+\frac{1}{2}+i\tau\right)\Gamma\left(n+\frac{1}{2}-i\tau\right)P^{-n}_{-\frac{1}{2}+i\tau}(t)\right. \\
&\qquad\qquad \left.\times G(\tau)d\tau, \phi(t)\right\rangle \\
&= \langle g, \phi\rangle, \qquad ☑
\end{aligned}$$

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Departamento de Análisis Matemático
Universidad de La Laguna
38271 Canary Islands, ESPAÑA (SPAIN)
e-mail: bjglez@ull.es
e-mail: enegrin@ull.es