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Maximal Virtual Schottky Groups: Explicit Constructions

Grupos de Schottky virtuales maximales: construcciones explícitas

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Abstract. A Schottky group of rank g is a purely loxodromic Kleinian group, with non-empty region of discontinuity, isomorphic to the free group of rank g.

A virtual Schottky group is a Kleinian group K containing a Schottky group Γ as a finite index subgroup. In this case, let g be the rank of Γ. The group K is an elementary Kleinian group if and only if $g \in \{0, 1\}$. Moreover, for each $g \in \{0, 1\}$ and for every integer $n \geq 2$, it is possible to find K and Γ as above for which the index of Γ in K is n. If $g \geq 2$, then the index of Γ in K is at most $12(g-1)$.

If K contains a Schottky subgroup of rank $g \geq 2$ and index $12(g-1)$, then K is called a maximal virtual Schottky group. We provide explicit examples of maximal virtual Schottky groups and corresponding explicit Schottky normal subgroups of rank $g \geq 2$ of lowest rank and index $12(g-1)$. Every maximal Schottky extension Schottky group is quasiconformally conjugate to one of these explicit examples.

Schottky space of rank g, denoted by $\mathcal{S}_g$, is a finite dimensional complex manifold that parametrizes quasiconformal deformations of Schottky groups of rank g. If $g \geq 2$, then $\mathcal{S}_g$ has dimension $3(g-1)$. Each virtual Schottky group, containing a Schottky group of rank g as a finite index subgroup, produces a sublocus in $\mathcal{S}_g$, called a Schottky strata. The maximal virtual Schottky groups produce the maximal Schottky strata. As a consequence of the results, we see that the maximal Schottky strata is the disjoint union of properly embedded quasiconformal deformation spaces of maximal virtual Schottky groups.

Key words and phrases. Schottky groups, Kleinian groups, Automorphisms, Riemann surfaces.

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RESUMEN. Un grupo de Schottky de rango g es un grupo Kleiniano puramente loxodrómico, con región de discontinuidad no vacía, e isomorfo al grupo libre de rango g.

Un grupo de Schottky virtual es un grupo Kleiniano K que contiene un grupo de Schottky Γ como subgrupo de índice finito. En tal caso, sea g el rango de Γ. El grupo K es un grupo Kleiniano elemental si y sólo si $g \in \{0,1\}$. Más aún, para cada $g \in \{0,1\}$ y para cada entero $n \geq 2$, es posible construir Γ and K de manera que Γ tenga índice n en K. Si $g \geq 2$, entonces el índice de Γ en K es a lo más $12(g-1)$.

Si K contiene un subgrupo de Schottky de rango $g \geq 2$ e índice $12(g-1)$, entonces K es llamado un grupo de Schottky virtual maximal. Proveemos ejemplos explícitos de grupos de Schottky virtuales maximales y correspondientes subgrupos de Schottky normales de rango $g \geq 2$ e índice $12(g-1)$. Todo grupo de Schottky virtual maximal es cuasiconformemente conjugado a uno de estos ejemplos.

El espacio de Schottky de rango g, denotado por $\mathcal{S}_g$, es una variedad compleja finito dimensional que parametriza las deformaciones cuasiconformes de grupos de Schottky de rango g. Si $g \geq 2$, entonces $\mathcal{S}_g$ tiene dimensión $3(g-1)$. Cada grupo de Schottky virtual, conteniendo un grupo de Schottky de rango g como subgrupo de índice finito, produce un subconjunto en $\mathcal{S}_g$, llamado un estrato de Schottky. Los grupos de Schottky virtuales maximales producen el estrato de Schottky maximal. Como consecuencia de los resultados obtenidos, se obtiene que el estrato de Schottky maximal es la unión disjunta de incrustaciones de espacios de deformación cuasiconforme de grupos de Schottky virtuales maximales.

Palabras y frases clave. Grupos de Schottky, grupos Kleinianos, automorfismos, superficies de Riemann.

1. Introduction

A Kleinian group, isomorphic to a free group of rank g, with non-empty region of discontinuity and containing no parabolic transformation is called a *Schottky group of rank g*. The lowest regular planar covers of closed Riemann surfaces of genus g are exactly the ones with Deck group being a Schottky group of rank g [15]. A *virtual Schottky group* is a Kleinian group containing a Schottky group as a finite index subgroup; in particular, it contains a Schottky group as a finite index normal subgroup.

Let S be a closed Riemann surface of genus g and let $P : \Omega \to S$ be a regular planar cover of S whose Deck group is a Schottky group Γ. Let $H < \mathrm{Aut}(S)$ be a finite group, where $\mathrm{Aut}(S)$ denotes the full group of conformal automorphisms of S. We say that H *lifts* with respect to the previous cover if for every $h \in H$ there is a Möbius transformation $\widehat{h}$ so that $\widehat{h}(\Omega) = \Omega$ and $P \circ \widehat{h} = h \circ P$. If $g \geq 2$, necessary and sufficient conditions for the group H to lift to a suitable regular planar cover of S, whose Deck group is a Schottky group, is the existence of a collection of pairwise disjoint simple loops on S with the properties that (i) the collection is invariant under the action of H and (ii) the complement of these loops consists of planar surfaces [12]. In particular, this obligates to H to have order at most $12(g-1)$ [11, 21]. Note that Hurwitz's bound for $\mathrm{Aut}(S)$ is $84(g-1)$, so there are examples of groups $H < \mathrm{Aut}(S)$ which cannot lift with respect to any regular planar cover whose Deck group is a Schottky group.

If $H < \mathrm{Aut}(S)$ lifts with respect to a regular planar cover $P : \Omega \to S$, whose Deck group is a Schottky group Γ, then the lifted Möbius transformations generate a Kleinian group K containing Γ as a finite index normal subgroup so that $K/\Gamma = H$; in particular, K is a virtual Schottky group. Conversely, if K is a virtual Schottky group, Γ is a Schottky group of rank g, which is a normal subgroup of finite index in K, Ω is the region of discontinuity of K (which is the same as for Γ) and $S = \Omega/\Gamma$, then the group $H = K/\Gamma < \mathrm{Aut}(S)$ lifts with respect to the regular planar cover $P : \Omega \to S$ whose Deck group is Γ. It follows that if a virtual Schottky group K contains a Schottky group Γ of genus $g \geq 2$ as a finite index subgroup, then the index of Γ in K is at most $12(g-1)$. We say that K is a maximal virtual Schottky group if we may chose a Schottky subgroup Γ with the maximal index $12(g-1)$.

A decomposition structure theorem for maximal Schottky extension groups was provided in [10] (see Theorem 2). In this paper we provide explicit constructions, in terms of the Klein-Maskit's combination theorems, of the maximal virtual Schottky groups.

A *marked Schottky group of rank* g is a pair $(\Gamma, (A_1, \ldots, A_g))$, where Γ is a Schottky group of rank g and $A_1,\ldots, A_g$ is a set of generators of Γ. Two such marked Schottky groups, say $(\Gamma_1, (A_1, \ldots, A_g))$ and $(\Gamma_2, (B_1, \ldots, B_g))$, are equivalent if there is a Möbius transformation T so that $TA_jT^{-1} = B_j$, for every $j \in \{1, \ldots, g\}$. The space $\mathcal{S}_g$, that parameterizes marked Schottky groups of rank g, is called the *Schottky space of rank* g. This space can be identified with the quasiconformal deformation space of any Schottky group of rank g (see [5, 6, 13, 20] and Section 2 for a more precise definition). Schottky space of rank g is a complex manifold of dimension $3(g-1)$ for $g \geq 2$ (dimension 1 for $g = 1$ and a point if $g = 0$) and it is an intermediate (non-regular) cover of moduli space of genus g.

Schottky strata $\mathcal{E}_g \subset \mathcal{S}_g$ is defined by those classes of marked Schottky groups which are non-trivial normal subgroups of finite index of some virtual Schottky group. The sublocus of $\mathcal{E}_g$ for which the virtual Schottky group can be chosen to be with index $12(g-1)$ is the *maximal Schottky strata* $\mathcal{ME}_g$. Schottky strata is the union of some properly embedded quasiconformal deformation spaces of virtual Schottky groups (called the irreducible components of the Schottky strata). The configuration of Schottky strata is not known, for instance, it is not known how the irreducible components intersect and what are the possible intersections. The main obstruction to this problem is the fact that no every subgroup of conformal automorphisms of a closed Riemann surface needs to lift with respect to a suitable regular planar cover whose Deck group is a Schottky group. Corollary 2 provides a partial answer to this; it says that maximal Schottky strata consists of pairwise disjoint irreducible components, each one being a copy of a quasiconformal deformation space of a maximal Schottky virtual group. A general study of the irreducible components of the Schottky strata will pursued elsewhere.

Schottky space of rank g can also be seen as the spaces that parameterizes (marked) complete geometrically finite hyperbolic structures, with injectivity radius bounded away from zero, on the interior of a handlebody of genus g; we talk of a *Schottky structure* on the corresponding handlebody. In this setting, Schottky strata corresponds to those structures with extra isometries. Two irreducible components of the Schottky strata intersect if there is a handlebody with two groups of isometries, each group providing one of the components.

This paper is organized as follows. In Section 2 we recall some basic definitions (not already stated in the introduction) and standard results we will need in the rest of this paper. In Section 3 we describe the decomposition theorem of maximal virtual Schottky groups (Theorem 2), define the (maximal) Schottky strata and provide the structure of such locus in Schottky space (Corollary 2). We also provide, in terms of Schottky structures on handlebodies, a description of (maximal) Schottky strata. In Section 4 we provide the explicit constructions of maximal virtual Schottky groups.

2. Preliminaries

2.1 In what follows, $U < V$ (respectively, $U \lhd V$) means that U is a subgroup (respectively, normal subgroup) of V, $[V : U]$ denotes the index of U in V, and if R is a Riemann surface, then $Aut(R)$ denotes its full group of conformal automorphisms. If $\widehat{\mathbb{C}}$ denotes the Riemann sphere, then it is well known that $Aut(\widehat{\mathbb{C}}) = \mathbb{M}$; the group of Möbius transformations.

2.2 An orientation-preserving homeomorphism $W : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, with local L^2 derivatives $\partial_{\overline{z}}W$ and $\partial_z W$ is called a *quasiconformal homeomorphism* of the Riemann sphere.

2.3 A group $K < \mathbb{M}$ is said to act *discontinuously* at the point $p \in \widehat{\mathbb{C}}$ if: (i) the K-stabilizer $K_p = \{k \in K : k(p) = p\}$ is finite and (ii) there is an open set $U \subset \widehat{\mathbb{C}}$, $p \in U$, so that, for every $k \in K - K_p$, it holds that $k(U) \cap U = \varnothing$.

2.4 A *Kleinian group* is a discrete subgroup K of $\mathbb{M}$ and its *region of discontinuity* is the open subset $\Omega(K)$ of $\widehat{\mathbb{C}}$ of points on which it acts discontinuously. Observe that $\Omega(K)$ might be empty. The complement $\Lambda(K) = \widehat{\mathbb{C}} - \Omega(K)$ is the *limit set* of K.

Let $K_1 < K_2 < \mathbb{M}$, where $[K_2 : K_1] < \infty$; then K_1 is a Kleinian group if and only if K_2 is a Kleinian groups; moreover they have the same region of discontinuity. Generalities on Kleinian groups can be seen, for instance, in the books [18, 19].

2.5 Two Kleinian groups, say K_1 and K_2 are said to be *topologically equivalent* (respectively, *quasiconformally equivalent*) if there is an orientation-preserving homeomorphism (respectively, quasiconformal homeomorphism) $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $K_2 = fK_1f^{-1}$.

2.6 An *elementary group* is a Kleinian group whose limit set is finite (it is known that its cardinality is at most two); otherwise, we say that it is a *non-elementary Kleinian group.* A *function group* is a finitely generated Kleinian group K for which there is a connected component of $\Omega(K)$ which is invariant under K. Next we list some examples of non-elementary function groups. A *quasifuchsian group* is a function group whose limit set is a Jordan curve (so each of the two components of its region of discontinuity is invariant). A *totally degenerate group* is a non-elementary finitely generated Kleinian group whose region of discontinuity is connected and simply-connected. These groups are the basic groups in the construction of function groups from the Klein-Maskit combination theorems [18].

Theorem 1 (Decomposition theorem of function groups [17])**.** *Any function group can be constructed from a finite collection of elementary groups, quasifuchsian groups and totally degenerate groups by a finite number of applications of the Klein-Maskit combination theorems.*

2.7 If Γ is a Schottky group of rank $g > 0$ and $A_1, \dots, A_g$ is any set of generators of Γ, then there is a collection of pairwise disjoint simple loops, say $C_1, C'_1, \dots, C_g, C'_g$, all of them bounding a common domain of connectivity $2g$, say $\mathcal{D}$, so that, for each $j \in \{1, \dots, g\}$, it holds that $A_j(C_j) = C'_j$ and $A_j(\mathcal{D}) \cap \mathcal{D} = \varnothing$ [8, 16]. It is known that the limit set of a Schottky group is totally disconnected (this fact can be seen from the previous geometric picture; the collection of Γ translates of all the loops C_j and C'_j separates different limit points). In particular, a Schottky group is a function group (as its region of discontinuity is connected). It is also clear from this geometric picture that any two Schottky groups of the same rank are quasiconformally equivalent.

2.8 A virtual Schottky group K is elementary if and only if it contains, as finite index subgroup, a Schottky group Γ of rank $g \in \{0, 1\}$. In this case, for any integer $n \geq 2$ there are examples of pairs K and Γ, where Γ is a Schottky group of rank g and of index n in K. In fact, if $g = 1$, then we may consider any Schottky group $K = \langle A \rangle$ and let $\Gamma = \langle A^n \rangle$. If $g = 0$, then, as a Schottky group of rank $g = 0$ is the trivial group, then it is enough to set $K = \left\langle A(z) = e^{2\pi i/n} z \right\rangle \cong \mathbb{Z}_n$.

2.9 If K is a non-elementary virtual Schottky group, then it contains a Schottky group Γ of rank $g \geq 2$ as a finite index normal subgroup. As both, K and Γ, have the same limit set, and the limit set of a Schottky group is a totally disconnected set, K is a function group with totally disconnected limit set. It follows, from this and Maskit's decomposition theorem of function groups, the following decomposition result.

Theorem 2 (Decomposition theorem of virtual Schottky groups)**.** *Any virtual Schottky group can be constructed from a finite collection of finite groups of Möbius transformations and cyclic groups generated by loxodromic transformations by a finite number of applications of the Klein-Maskit combination theorems.*

2.10 Generalities on quasiconformal maps can be found, for instance, in [3, 2] and on quasiconformal deformation spaces of Kleinian grops in [5, 6, 13, 20]. We proceed to recall the basic properties we need in this paper. Let K be a finitely generated Kleinian group, with region of discontinuity $\Omega \neq \varnothing$ and limit set $\Lambda = \widehat{\mathbb{C}} - \Omega$. Let $L^\infty(K)$ be the Banach space consisting of measurable functions $\mu : \Omega \to \mathbb{C}$ with $\|\mu\|_\infty = \text{ess sup}_{z\in\Omega}|\mu(z)| < \infty$ such that, for every $k \in K$ and for almost every $z \in \Omega$, it holds that $\mu(k(z))\overline{k'(z)} = \mu(z)k'(z)$ (one extends μ to be zero in Λ). A *Beltrami differential* for K is an element of the unit ball $L^\infty_1(K)$ of the Banach space $L^\infty(K)$. As a consequence of results of Ahlfors-Bers [4], for every $\mu \in L^\infty_1(K)$ there is a quasiconformal homeomorphism (or μ-*quasiconformal homeomorphism*) $W : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ satisfying the *Beltrami equation* $\partial_{\overline{z}}W(z) = \mu(z)\partial_z W(z)$, for almost every $z \in \Omega$. Any other μ-quasiconformal homeomorphism is of the form AW, where A is a Möbius transformation. The above μ-quasiconformal homeomorphism W (or AW) is called a *quasiconformal deformation of* K. If we fix three different values $a, b, c \in \widehat{\mathbb{C}}$, as a Möbius transformation that fixes them is just the identity and the Möbius group acts triply-transitive on $\widehat{\mathbb{C}}$, there is one and unique μ-quasiconformal homeomorphism W normalized by the condition that $W(a) = a$, $W(b) = b$ and $W(c) = c$.

If W is a μ-quasiconformal deformation of the Kleinian group K, then, for every $k \in K$ it holds that WkW^{-1} is again a Möbius transformation; so WKW^{-1} is a finitely generated Kleinian group, quasiconformally equivalent to K. In this way, we have an isomorphism $\phi_W : K \to WKW^{-1}$ defined by $\phi_W(k) = WkW^{-1}$. As noted, a different μ-quasiconformal deformation is of the form $W_2 = AW_1$, for a suitable Möbius transformation A. It follows that $\phi_{W_2}(k) = A\phi_{W_1}(k)A^{-1}$, that is, the corresponding isomorphisms are conjugate in the Möbius group.

Two Beltrami differentials $\mu_1, \mu_2 \in L^\infty_1(K)$ are said to be *equivalent* if, for given quasiconformal homeomorphisms W_1 and W_2 (where W_j is a μ_j-quasiconformal homeomorphism), there is a Möbius transformation A so that AW_1 and W_2 coincide on the limit set Λ. We denote by $[\mu]$ the equivalence class of $\mu \in L^\infty_1(K)$. Note that, if K is non-elementary, that is, Λ is infinite, this definition is equivalent to saying that $\phi_{W_2}(k) = A\phi_{W_1}(k)A^{-1}$, that is, the corresponding isomorphisms are conjugate in the Möbius group. If K is elementary, these two definitions are not longer equivalent. In this paper we will be restricted to the case of non-elementary Kleinian groups, so we may work with any of the two definitions.

The space of equivalence classes of Beltrami differentials for K is called the *quasiconformal deformation space* of K and it will be denoted by $\mathcal{Q}(K)$. It is known that $\mathcal{Q}(K)$ is a finite dimensional complex manifold, in fact, holomorphically equivalent to a domain in some $\mathbb{C}^n$ [13].

2.11 Let K and $\widehat{K}$ be finitely generated quasiconformally equivalent Kleinian groups and let $W_0 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiconformal homeomorphism so that $W_0 K W_0^{-1} = \widehat{K}$. Let $\mu_0 \in L_1^\infty(K)$ be a Beltrami differential associated to W_0, that is, $\partial_{\overline{z}} W_0(z) = \mu_0(z)\partial_z W_0(z)$, for almost every $z \in \Omega$. For each $\mu \in L_1^\infty(\widehat{K})$, we consider a quasiconformal homeomorphism $W_\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ associated to μ. As

$$W_\mu W_0 K (W_\mu W_0)^{-1} = W_\mu \widehat{K} W_\mu^{-1},$$

there is a natural biholomorphism $F : \mathcal{Q}(\widehat{K}) \to \mathcal{Q}(K)$, where $F([\mu])$ is the equivalence class of a Beltrami differential for $W_\mu W_0$. In this way, we may identify $\mathcal{Q}(\widehat{K})$ and $\mathcal{Q}(K)$; in this identification the origin $[0] \in \mathcal{Q}(\widehat{K})$ corresponds to $[\mu_0] \in \mathcal{Q}(K)$.

2.12 Let K be a finitely generated non-elementary Kleinian group and let Γ be a finite index subgroup of K. In this situation, both Γ and K have the same limit set Λ, which is infinite. Let us fix three different limit points, say $a, b, c \in \Lambda$. Each Beltrami differential for K provides, by restriction, a Beltrami differential for Γ, that is, we may assume $L_1^\infty(K) \subset L_1^\infty(\Gamma)$. It follows that there is a natural holomorphic map $\phi : \mathcal{Q}(K) \to \mathcal{Q}(\Gamma)$. This map is a complex analytic embedding. In fact, for $j \in \{1, 2\}$, let μ_j be a Beltrami differential for K and let $W_j : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a μ_j-quasiconformal homeomorphism. We may assume that they are normalized as $W_j(a) = a$, $W_j(b) = b$ and $W_j(c) = c$. With this normalization, these two Beltrami differentials are equivalent with respect to K (respectively, with respect to Γ) if and only if the restrictions of W_1 and W_2 to Λ are equal, so we are done. We may think of the image $\phi(\mathcal{Q}(K))$ as the complex submanifold of $\mathcal{Q}(\Gamma)$ consisting of those classes of Beltrami differentials of Γ which are also Beltrami differentials for the bigger group K.

2.13 As any two Schottky groups of the same rank, say Γ_1 and Γ_2, are quasiconformally equivalent, the corresponding quasiconformal deformation spaces $\mathcal{Q}(\Gamma_1)$ and $\mathcal{Q}(\Gamma_2)$ are holomorphically equivalent. Let us fix a marked Schottky group of rank g, say $\big(\Gamma, (A_1, \dots, A_g)\big)$. If $\big(\Gamma_1, (B_1, \dots, B_g)\big)$ is another marked Schottky group, then, as a consequence of the geometric picture of Schottky groups (see Section 2.7) we may find a quasiconformal homeomorphism $W : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ so that $W\Gamma W^{-1} = \Gamma_1$ and $WA_jW^{-1} = B_j$, for every $j \in \{1, \dots, g\}$. In this way, the quasiconformal deformation of Γ can be holomorphically identified with the Schottky space of rank g; we say that $\mathcal{Q}(\Gamma)$ is a model for $\mathcal{S}_g$. This also asserts that $\mathcal{S}_g$ is a complex manifold of dimension $3(g-1)$ if $g \geq 2$.

3. Maximal Virtual Schottky Extension Groups

3.1 Theorem 2 provides a general decomposition result for virtual Schottky groups. In the case of maximal virtual Schottky groups a more explicit decomposition theorem is provided in [10]. In this paper we provide explicit constructions, in terms of the Klein-Maskit's combination theorems, of the maximal virtual Schottky groups.

Theorem 3 (Decomposition theorem of maximal virtual Schottky groups [10])**.** *We denote by D_r the dihedral group of order $2r$, by $\mathcal{A}_r$ the alternating group in r letters and by $\mathfrak{S}_4$ the symmetric group in 4 letters.*

(a) Each maximal virtual Schottky groups can be constructed, using the first Klein-Maskit combination theorem, as the free product of two finite Kleinian groups, say K_1 and K_2, amalgamated over a finite cyclic group $K_0 = K_1 \cap K_2$, where K_1, K_2 and K_0 are as described below.

1. *$K_1 = \langle A, B : A^3 = B^2 = (BA)^2 = 1\rangle \cong D_3$, $K_2 = \langle B, C : B^2 = C^2 = (CB)^2 = 1\rangle \cong D_2$ and $K_0 = \langle B\rangle \cong \mathbb{Z}_2$, where C preserves a simple loop around one of the fixed points of B, with both fixed points of C on such a loop. In this case $K = \langle A, B, C\rangle \cong D_2 *_{\mathbb{Z}_2} D_3$ and we say that K is of type (1).*
2. *$K_1 = \langle A, B : A^3 = B^2 = (BA)^3 = 1\rangle \cong \mathcal{A}_4$, $K_2 = \langle A, C : A^3 = C^2 = (CB)^2 = 1\rangle \cong \mathbb{D}_3$ and $K_0 = \langle A\rangle \cong \mathbb{Z}_3$, where C preserves a simple loop around one of the fixed points of A, with both fixed points of C on such a loop. In this case $K = \langle A, B, C\rangle \cong D_3 *_{\mathbb{Z}_3} \mathcal{A}_3$ and we say that K is of type (2).*
3. *$K_1 = \langle A, B : A^4 = B^2 = (BA)^3 = 1\rangle \cong \mathfrak{S}_4$, $K_2 = \langle A, C : A^4 = C^2 = (CB)^2 = 1\rangle \cong \mathbb{D}_4$ and $K_0 = \langle A\rangle \cong \mathbb{Z}_4$, where C preserves a simple loop around one of the fixed points of A, with both fixed points of C on such a loop. In this case $K = \langle A, B, C\rangle \cong D_4 *_{\mathbb{Z}_4} \mathfrak{S}_4$ and we say that K is of type (3).*
4. *$K_1 = \langle A, B : A^5 = B^2 = (BA)^3 = 1\rangle \cong \mathcal{A}_5$, $K_2 = \langle A, C : A^5 = C^2 = (CB)^2 = 1\rangle \cong \mathbb{D}_5$ and $K_0 = \langle A\rangle \cong \mathbb{Z}_5$, where C preserves a simple loop around one of the fixed points of A, with both fixed points of C on such a loop. In this case $K = \langle A, B, C\rangle \cong D_5 *_{\mathbb{Z}_5} \mathcal{A}_5$ and we say that K is of type (4).*

(b) Each of the above constructed groups is a maximal virtual Schottky group.

(c) Two maximal virtual Schottky groups of different type are non-isomorphic as abstract groups.

(d) Two maximal virtual Schottky groups are algebraically isomorphic if and only if they are topologically (and also quasiconformally) equivalent if and only if they are of the same type.

The following rigidity property holds at the level of maximal virtual Schottky groups.

Corollary 1 ([10])**.** *If two maximal virtual Schottky groups K_1 and K_2 contain a common Schottky group Γ of rank $g \geq 2$ as a normal subgroup of index $12(g-1)$, then $K_1 = K_2$.*

Proof. As Γ is of finite index and is a normal subgroup of K_j (for $j = 1, 2$), it follows that Γ has finite index and is a normal subgroup of $K = \langle K_1, K_2 \rangle$. It follows that, if $K_1 \neq K_2$, then K is a virtual Schottky group containing the Schottky group Γ as a normal subgroup of index greater than $12(g-1)$, a contradiction. ☑

The proof of Theorem 3 done in [10] was obtained as follows. Let K be a maximal virtual Schottky group and let Γ be a Schottky group of rank $g \geq 2$ which is a finite index normal subgroup of K and of index $12(g-1)$. Let Ω be the region of discontinuity of K (the same as for Γ). We first consider a certain minimal collection $\mathcal{F}$ of pairwise disjoint simple loops on $S = \Omega/\Gamma$ which is invariant under the action of $H = K/\Gamma$, so that $S - \mathcal{F}$ consists of planar surfaces and with the property that every loop in $\mathcal{F}$ lifts to simple loops on Ω. Secondly, we consider the collection $\mathcal{G}$ of simple loops in Ω obtained by the lifting of all the loops in $\mathcal{F}$. Finally, we proceed with a careful study of the K-stabilizers of each of the loops in $\mathcal{G}$ and of each of the connected components of $\Omega - \mathcal{G}$.

In Section 4 we provide explicit constructions of maximal virtual Schottky groups and corresponding explicit Schottky normal subgroups of rank $g \geq 2$ of lowest rank and index $12(g-1)$. It can be seen directly from Theorem 3, that every maximal Schottky extension Schottky group is quasiconformally conjugate to one of these explicit examples. In this way, the results of this paper give a more constructive approach to Theorem 3. These constructions may also be of help in understanding the Klein-Maskit combinations theorems in an explicit way.

3.2 Let us fix a Schottky group Γ of rank $g \geq 2$ and let Λ be its limit set. We consider $\mathcal{Q}(\Gamma)$ as a fixed model for the Schottky space of rank g. Let us also fix three different limit points $a, b, c \in \Lambda$. Next, we proceed to describe the Schottky strata in this fixed model.

If $\mu \in L_1^\infty(\Gamma)$ and $W_\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is the μ-quasiconformal homeomorphism normalized by $W_\mu(a) = a$, $W_\mu(b) = b$ and $W_\mu(c) = c$, then $\Gamma^\mu = W_\mu \Gamma W_\mu^{-1}$ is a Schottky group of rank g. If $\nu, \mu \in L_1^\infty(\Gamma)$ are equivalent, then Γ^ν and Γ^μ are conjugate by a suitable Möbius transformation. In particular, if Γ^μ is a finite index normal subgroup of some virtual Schottky group, then Γ^ν is also a finite index normal subgroup of some other virtual Schottky group (with the same index). This observation permits to give the following description of the Schottky strata.

The Schottky extension strata $\mathcal{E}_g \subset \mathcal{S}_g$ in the model $\mathcal{Q}(\Gamma)$ is defined as the sublocus of points $[\mu] \in \mathcal{Q}(\Gamma)$ for which Γ^μ is contained (strictly) as a finite index normal subgroup in some virtual Schottky group. Similarly, the *maximal Schottky extension strata* $\mathcal{ME}_g \subset \mathcal{S}_g$ is defined as the sublocus of those points $[\mu] \in \mathcal{Q}(\Gamma)$ for which Γ^μ is contained as a normal subgroup of index $12(g-1)$ in some virtual Schottky group.

Corollary 2. *The Schottky extension strata, in Schottky space of rank $g \geq 2$, is a union of quasiconformal deformations spaces (embedded ones) of virtual Schottky groups (different from Schottky groups of rank g). Moreover, $\mathcal{ME}_g$ is the disjoint union of such embeddings.*

Proof. Let us fix a Schottky group Γ of rank $g \geq 2$ and let Λ its limit set. We consider $\mathcal{Q}(\Gamma)$ as a fixed model for the Schottky space of rank g. Let us also fix three different limit points $a, b, c \in \Lambda$. For each $\mu \in L_1^\infty(\Gamma)$ we consider the normalized μ-quasiconformal homeomorphism $W_\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ ($W_\mu(a) = a$, $W_\mu(b) = b$ and $W_\mu(c) = c$) and the Schottky group $\Gamma^\mu = W_\mu \Gamma W_\mu^{-1}$. For each $p \in \mathcal{S}_g$, we fix some $\mu \in L_1^\infty(\Gamma)$ with $[\mu] = p$, and set $\Gamma_p = \Gamma^\mu$. With this fixed objects, we have the corresponding (maximal) Schottky strata in $\mathcal{Q}(\Gamma)$ as seen previously.

If $[\mu] \in \mathcal{E}_g$, then there is natural holomorphic embedding $\phi : \mathcal{Q}(K^\mu) \to \mathcal{S}_g$ so that $\phi([0]) = [\mu]$. Clearly, $\phi(\mathcal{Q}(K^\mu)) \subset \mathcal{E}_g$.

As a consequence of Corollary 1, if $[\mu] \in \mathcal{ME}_g$ and K_1 and K_2 are maximal Schottky extension groups so that $\Gamma_{[\mu]} \lhd K_j$ and $[K_j : \Gamma_{[\mu]}] = 12(g-1)$, then $K_1 = K_2$. We denote by $K_{[\mu]}$ such a maximal Schottky extension group. This provides the disjoint condition, for the maximal Schottky strata. ☑

3.3 In terms of hyperbolic structures on handlebodies, Schottky strata is the locus of points (in Teichmüller space of a handlebody) corresponding to those admitting non-trivial symmetries.

Let M be a handlebody of genus g, M^0 be its interior and let S its boundary. Each complete hyperbolic structure on M^0 is provided by a Kleinian group isomorphic to a free group and, conversely, every Kleinian group isomorphic to a free group of rank g provides a complete hyperbolic structure on M^0; this is a consequence of the Marden conjecture (or tame ends conjecture), recently proved by Agol [1] and Calegari-Gabai [7]. Schottky groups are exactly those Kleinian groups producing complete geometrically finite hyperbolic structures on M^0 with injectivity radius bounded away from zero (in this case, $S = \Omega/\Gamma$ is the conformal boundary).

A marking of M is a pair (N_Γ, f), where $N_\Gamma = \mathbb{H}^3/\Gamma$, Γ a Schottky group of rank g, and $f : M \to N_\Gamma$ is an orientation preserving diffeomorphism. Two markings (N_1, f_1) and (N_2, f_2) are said to be *Teichmüller equivalent* if there is a conformal diffeomorphism $h : N_1 \to N_2$ so that $f_2^{-1} \circ h \circ f_1$ is isotopic to the identity. The Teichmüller space $\mathcal{T}(M)$ is the set of equivalence classes of markings of M. There is a natural identification of $\mathcal{T}(M)$ with $\mathcal{S}_g$; seen as the space that parametrizes (marked) complete hyperbolic structures, with injectivity radius bounded away from zero, on M^0.

Theorem 4. *Let Γ_0 be a Schottky group of rank g. Then $\mathcal{T}(M)$ can be naturally identified with $\mathcal{Q}(\Gamma_0)$.*

Proof. Let $\pi_{\Gamma_0} : \mathbb{H}^3 \cup \Omega(\Gamma_0) \to M_0$ be a universal covering with Γ_0 as Deck group. It is not difficult to see that $\mathcal{T}(M)$ and $\mathcal{T}(M_0)$ can be identified by a homeomorphism.

Let (N_Γ, f) be a marking of M_0 and let $\pi_\Gamma : \mathbb{H}^3 \cup \Omega(\Gamma) \to N_\Gamma$ be a universal covering with Γ as Deck group. We may lift the diffeomorphism f to a diffeomorphism $\widehat{f} : \mathbb{H}^3 \cup \Omega(\Gamma_0) \to \mathbb{H}^3 \cup \Omega(\Gamma)$ satisfying that, for every $k \in \Gamma_0$, $\widehat{f} \circ k = \theta(k) \circ \widehat{f}$, where $\theta : \Gamma_0 \to \Gamma$ is an isomorphism of groups. The restriction $\widehat{f} : \Omega(\Gamma_0) \to \Omega(\Gamma)$ is a quasiconformal diffeomorphism. It follows from Marden's isomorphism theorem [14] that we may assume it to be a quasiconformal homeomorphism of the Riemann sphere that conjugates Γ_0 onto Γ.

Conversely, again as a consequence of Marden's isomorphism theorem [14], each quasiconformal diffeomorphism $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ so that $h\Gamma_0 h^{-1} = \Gamma$, extends to an orientation preserving diffeomorphism $\widehat{h} : \mathbb{H}^3 \cup \Omega(\Gamma_0) \to \mathbb{H}^3 \cup \Omega(\Gamma)$ keeping the conjugacy property. It follows that $\widehat{h}$ induces a marking of M_0 making the above two process inverse of each other. ☑

The *modular group* of M is $\mathrm{Mod}^+(M) = \mathrm{Diff}^+(M)/\mathrm{Diff}_0(M)$, where $\mathrm{Diff}^+(M)$ is the group of orientation preserving diffeomorphisms of M and $\mathrm{Diff}_0(M)$ its normal subgroup of diffeomorphisms isotopic to the identity. Earle [9] proved that $\mathrm{Mod}^+(M)$ is isomorphic to the group of outer automorphisms of the free group of rank g.

An element $[h] \in \mathrm{Mod}^+(M)$ acts on $\mathcal{T}(M)$ by the following rule: $[h]([N, f]) = [N, fh^{-1}]$. The *moduli space* of M is defined by $\mathcal{M}(M) = \mathcal{T}(M)/\mathrm{Mod}^+(M)$.

The moduli space of M can be identified to the space of unmarked Schottky groups of rank g, that is, the space of conjugacy classes (in $\mathbb{M}$) of Schottky groups of rank g.

The natural projection $\pi : \mathcal{T}(M) \to \mathcal{M}(M)$ fails to be a covering map exactly at those points in $\mathcal{T}(M)$ with no-trivial stabilizer in $\mathrm{Mod}^+(M)$. These points correspond exactly to those Schottky groups of rank g so that there is a virtual Schottky extension group K containing Γ as a finite index normal subgroup of index bigger than one. In this way, the Schottky strata $\mathcal{E}_g$ is exactly the locus of critical points of π.

4. Explicit Construction of Maximal Virtual Schottky Groups

In this section we provide explicit examples of maximal virtual Schottky groups. We also construct explicitly, in each case, a Schottky group of some rank $g \geq 2$ as a normal subgroup of index $12(g-1)$. In these examples we find the examples with low values of g. We believe these are the lowest possible ranks. The constructions are explicit applications of the Klein-Maskit combination theorems.

Case (1) Let

$$H_1 = \left\langle X(z) = e^{2\pi i/3} z,\ Z(z) = \frac{1}{z} \right\rangle \cong D_3.$$

Choose a point $p_0 \in \left(1, 2+\sqrt{3}\right)$ (for instance, $p_0 = 3$) and let Σ the circle through the point p_0 and orthogonal to the unit circle. If Y is the elliptic involution with fixed points being p_0 and $1/p_0$, then

$$K = \left\langle X, Y, Z : X^3 = Y^2 = Z^2 = (ZX)^2 = (ZY)^2 = 1 \right\rangle \cong D_3 *_{\mathbb{Z}_2} \mathbb{Z}_2^2,$$

where

$$\begin{aligned} D_3 &= \left\langle X, Z : X^3 = Z^2 = (ZY)^2 = 1 \right\rangle \\ \mathbb{Z}_2^2 &= \left\langle Y, Z : Y^2 = Z^2 = (ZY)^2 = 1 \right\rangle \\ \mathbb{Z}_2 &= \left\langle Z \right\rangle. \end{aligned}$$

A fundamental domain for K is provided in Figure 1. We consider the following circles

$$\Sigma_1 = X(\Sigma), \quad \Sigma_1' = Y(\Sigma_1), \quad \Sigma_2 = X^{-1}(\Sigma), \quad \Sigma_2' = Y(\Sigma_2).$$

The circle Σ_1 is invariant under the involution XYX^{-1} and the circle Σ_2 is invariant under the involution $X^{-1}YX$. Let Q be the common domain bounded by the circles Σ_1, Σ_2, Σ_1' and Σ_2'.

Set $A_1 = YXYX^{-1}$ and $A_2 = YX^{-1}YX$. Then clearly, $A_1(\Sigma_1) = \Sigma_1'$, $A_2(\Sigma_2) = \Sigma_2'$ and $A_1(Q) \cap Q = A_2(Q) \cap Q = \varnothing$. It follows that $G =$

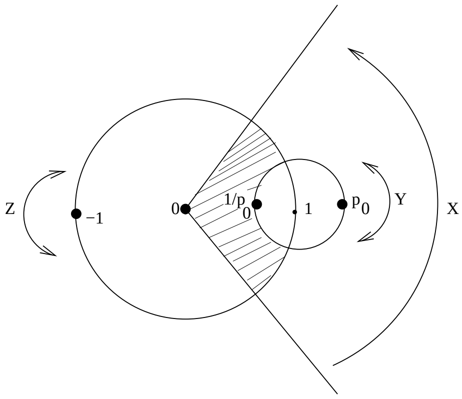

FIGURE 1.

$\langle YXYX^{-1}, YX^{-1}YX \rangle$ is a classical Schottky group of rank $g = 2$ and of index 12 in K. Direct computations permit to see that G is in fact a normal subgroup of K.

Case (2) In this case, we consider

$$H_1 = \left\langle X(z) = e^{2\pi i/3}z,\ Z(z) = \frac{z + \sqrt{3} - 1}{\left(1 + \sqrt{3}\right) z - 1} \right\rangle \cong \mathcal{A}_4.$$

Set $q_0 = \left(1 - \sqrt{3}\right) / \left(1 + \sqrt{3}\right)$ and choose a point $p_0 \in (0, -q_0)$. In this case we take as Σ the circle with center at 0 and radius p_0. If Y is the elliptic involution with fixed points being $\pm p_0$, then

$$K = \langle X, Y, Z : X^3 = Y^2 = Z^2 = (ZX)^3 = (XY)^2 = 1 \rangle \cong D_3 *_{\mathbb{Z}_2} \mathbb{Z}_2^2,$$

where

$$D_3 = \langle X, Y : X^3 = Y^2 = (XY)^2 = 1 \rangle$$
$$\mathcal{A}_4 = \langle X, Z : X^3 = Z^2 = (XZ)^3 = 1 \rangle$$
$$\mathbb{Z}_3 = \langle X \rangle.$$

A fundamental domain can be seen in Figure 2. We consider the following circles

$$\Sigma_1 = Z(\Sigma), \quad \Sigma_1' = Y(\Sigma_1), \quad \Sigma_2 = X^{-1}(\Sigma_1),$$
$$\Sigma_2' = Y(\Sigma_2), \quad \Sigma_3 = X(\Sigma_1), \quad \Sigma_3' = Y(\Sigma_3).$$

The circle Σ_1 is invariant under the involution ZYZ, the circle Σ_2 is invariant under the involution $X^{-1}ZYZX$ and the circle Σ_3 is invariant

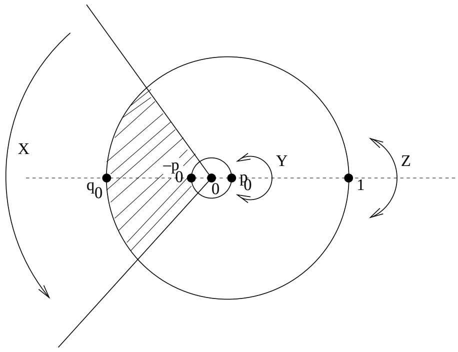

FIGURE 2.

under the involution $XZYZX^{-1}$. Let Q be the common domain bounded by the circles Σ_1, Σ_2, Σ_3, Σ_1', Σ_2' and Σ_3'.

Set $A_1 = (YZ)^2$, $A_2 = YX^{-1}ZYZX$ and $A_3 = YXZYZX^{-1}$. Then clearly, $A_1(\Sigma_1) = \Sigma_1'$, $A_2(\Sigma_2) = \Sigma_2'$, $A_3(\Sigma_3) = \Sigma_3'$ and $A_1(Q) \cap Q = A_2(Q) \cap Q = A_3(Q) \cap Q = \varnothing$. It follows that $G = \langle (YZ)^2, YX^{-1}ZYZX, YXZYZX^{-1} \rangle$ is a classical Schottky group of rank $g = 3$ and of index 24 in K. Direct computations permit to see that G is in fact a normal subgroup of K.

Case (3) In this case, we consider

$$H_1 = \left\langle X(z) = e^{2\pi i/5}z, \quad Z(z) = \frac{2z + \sqrt{10 - 2\sqrt{5}} - 2}{\left(\sqrt{10 - 2\sqrt{5}} + 2\right)z - 2} \right\rangle \cong \mathcal{A}_5.$$

Set $q_0 = \left(2 - \sqrt{10 - 2\sqrt{5}}\,\right) / \left(2 + \sqrt{10 - 2\sqrt{5}}\,\right)$ and choose a point $p_0 \in (0, -q_0)$. In this case we take as Σ the circle with center at 0 and radius p_0. If Y is the elliptic involution with fixed points being $\pm p_0$, then

$$K = \langle X, Y, Z : X^5 = Y^2 = Z^2 = (ZX)^3 = (XY)^2 = 1 \rangle \cong D_3 *_{\mathbb{Z}_2} \mathbb{Z}_2^2,$$

where

$$\begin{aligned} D_5 &= \langle X, Y : X^5 = Y^2 = (XY)^2 = 1 \rangle \\ \mathcal{A}_5 &= \langle X, Z : X^5 = Z^2 = (XZ)^3 = 1 \rangle \\ \mathbb{Z}_5 &= \langle X \rangle. \end{aligned}$$

A fundamental domain can be seen in Figure 2. Let us consider the circles $\Sigma_1, \ldots, \Sigma_{11}$ (all of them different from Σ) we obtain by following the orbit of Σ under the action of H_1. For each $j = 1, \ldots, 11$, take $T_j \in H_1$ so that

$\Sigma_j = T_j(\Sigma)$. Clearly, Σ_j is invariant under the involution $T_j Y T_j^{-1}$. We set $\Sigma'_j = Y(\Sigma_j)$ and $A_j = Y T_j Y T_j^{-1}$. Then, $A_j(\Sigma_j) = \Sigma'_j$ and $A_j(Q) \cap Q = \varnothing$, for every j, where Q is the common domain bounded by all the circles Σ_1, Σ'_1,..., Σ_{11}, Σ'_{11}. It follows that $G = \langle A_1, \ldots, A_{11} \rangle = \langle\langle (ZY)^2 \rangle\rangle$ is a Schottky group of rank 11 and index 120. It can be seen that G is also normal in K.

Case (4) In this case, we consider

$$H_1 = \left\langle X(z) = e^{\pi i/2} z,\ Z(z) = \frac{z\sqrt{2} + \sqrt{6+2\sqrt{2}} - \sqrt{2}}{\left(\sqrt{6+2\sqrt{2}} + \sqrt{2}\right) z - \sqrt{2}} \right\rangle \cong \mathfrak{S}_4.$$

Set $q_0 = \left(\sqrt{2} - \sqrt{6+2\sqrt{2}}\right) / \left(\sqrt{2} + \sqrt{6+2\sqrt{2}}\right)$ and choose a point $p_0 \in (0, q_0)$. In this case we take as Σ the circle with center at 0 and radius p_0. If Y is the elliptic involution with fixed points being $\pm p_0$, then

$$K = \left\langle X, Y, Z : X^4 = Y^2 = Z^2 = (ZX)^3 = (XY)^2 = 1 \right\rangle \cong D_3 *_{\mathbb{Z}_2} \mathbb{Z}_2^2,$$

where

$$\begin{aligned} D_4 &= \left\langle X, Y : X^4 = Y^2 = (XY)^2 = 1 \right\rangle \\ \mathfrak{S}_4 &= \left\langle X, Z : X^4 = Z^2 = (XZ)^3 = 1 \right\rangle \\ \mathbb{Z}_4 &= \left\langle X \right\rangle. \end{aligned}$$

A fundamental domain can be seen in Figure 2. Let us consider the circles Σ_1,..., Σ_5 (all of them different from Σ) we obtain by following the orbit of Σ under the action of H_1. For each $j = 1, \ldots, 5$, take $T_j \in H_1$ so that $\Sigma_j = T_j(\Sigma)$. Clearly, Σ_j is invariant under the involution $T_j Y T_j^{-1}$. We set $\Sigma'_j = Y(\Sigma_j)$ and $A_j = Y T_j Y T_j^{-1}$. Then, $A_j(\Sigma_j) = \Sigma'_j$ and $A_j(Q) \cap Q = \varnothing$, for every j, where Q is the common domain bounded by all the circles Σ_1, Σ'_1,..., Σ_5, Σ'_5. It follows that

$$\begin{aligned} &G = \langle A_1, \ldots, A_5 \rangle = \\ &\left\langle (YZ)^2, YXZYZX^{-1}, YX^2ZYZX^2, YX^{-1}ZYZX, YZX^2ZYZX^2Z \right\rangle = \\ &\qquad \langle\langle (ZY)^2 \rangle\rangle \end{aligned}$$

is a Schottky group of rank 5 and index 48. It can be seen that G is also normal in K.

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