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Commensurator Subgroups of Surface Groups

Subgrupos comensuradores del grupo fundamental de superfícies

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Abstract. Let M be a surface, and let H be a subgroup of $\pi_1 M$. In this paper we study the commensurator subgroup $C_{\pi_1 M}(H)$ of $\pi_1 M$, and we extend a result of L. Paris and D. Rolfsen [7], when H is a geometric subgroup of $\pi_1 M$. We also give an application of commensurator subgroups to group representation theory. Finally, by considering certain closed curves on the Klein bottle, we apply a classification of these curves to self-intersection Nielsen theory.

Key words and phrases. Commensurator, Fundamental group, Surface.

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Resumen. Sean M una superfície y H un subgrupo de $\pi_1 M$. En este artículo estudiamos los subgrupos conmensuradores $C_{\pi_1 M}(H)$ de $\pi_1 M$, y extendemos un resultado obtenido por L. Paris y D. Rolfsen en [7], cuando H es un subgrupo geométrico de $\pi_1 M$. También daremos una aplicación de estos subgrupos conmensuradores a la teoría de representaciones de grupos. Finalmente, considerando ciertas curvas cerradas en la botella de Klein, aplicaremos una clasificación de estas curvas a la Teoría de Nielsen de auto-intersección.

Palabras y frases clave. Comensurador, grupo fundamental, superfície.

1. Introduction

Commensurators play an important role in representation theory, especially in the study of induced representations. Let G be a group, and let H be a subgroup of G. The set of elements $g \in G$ such that $gHg^{-1} \cap H$ has finite index in both gHg^{-1} and H is a subgroup of G, called the *commensurator subgroup* of H in G, and denoted by $C_G(H)$.

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G. W. Mackey [5] pointed out an important connection between these subgroups and unitary group representations. For example, if a subgroup H of G is its own commensurator, then any finite dimensional irreducible representation of H induces an irreducible representation of G. D. Rolfsen calculated commensurator subgroups of classical braid groups [8], and applied Mackey's result to them. Some years later, L. Paris and D. Rolfsen calculated commensurator subgroups of geometric subgroups of surface braid groups [7].

Let M be a surface. It is known that the one-string braid group of M is the fundamental group of M. In [7], the commensurator subgroup, $C_{\pi_1 M}(H)$, is completely described when H is a geometric subgroup of $\pi_1 M$. We calculate $C_{\pi_1 M}(H)$ for any subgroup H of the fundamental group of a non-large compact surface. Also, we show that if H is a finitely-generated subgroup of $\pi_1 M$ for any surface M, then $\left[C_{\pi_1 M}(H) : C_{\pi_1 \widetilde{M}}(H)\right]$ is finite, where $\widetilde{M}$ is a finite covering space of M and H is a geometric subgroup of $\pi_1 \widetilde{M}$. Using this, we try to improve the description of $C_{\pi_1 M}(H)$ for any finitely-generated subgroup of $\pi_1 M$ and any surface M.

This paper is based on [7], which was also the main article of the author's M.Sc dissertation [6], and is organized as follows. In Section 2, we classify compact large surfaces with help of a classification of compact abelian surfaces [4, Theorem 4.3]. We study the commensurator subgroup $C_{\pi_1 M}(H)$ of the fundamental group of a surface M in Section 3. To highlight the interest of commensurator subgroups, in Section 4, we give an application to group representation theory. Finally, in Section 5, we study closed curves on the Klein bottle. We classify certain curves called *special curves*, and we apply this to self-intersection Nielsen theory for closed curves on the Klein bottle.

2. Large Surfaces

In this section, we recall the classification of large compact surfaces, which we shall use to study algebraic properties of their fundamental group. In particular, we focus on the center and commensurator subgroups of $\pi_1 M$, when M is a compact surface.

As in [4], we say that a surface is *Abelian* (resp. *non Abelian*) if its fundamental group is Abelian (resp. non Abelian). Also, we have the following classification of compact Abelian surfaces:

Proposition 1. *[4, Theorem 4.3] Let M be a compact Abelian surface.*

(1) If $\partial M = \phi$, then M is the sphere S^2, the torus T^2, or the real projective plane P^2.

(2) If $\partial M \neq \phi$, then M is the Möbius strip $S^1 \widehat{\times} I$, or the annulus $S^1 \times I$.

Let $S^1 \widehat{\times} S^1$ denote the Klein bottle. The following definition, as well as the statement of Proposition 2, can be found in [7]. A compact surface M is called *large* if

$$M \neq S^2, \quad P^2, \quad D^2, \quad T^2, \quad S^1 \times I, \quad S^1 \widehat{\times} I, \quad S^1 \widehat{\times} S^1.$$

The following proposition gives us an algebraic characterization for these surfaces. We give a proof using the classification of compact Abelian surfaces.

Proposition 2. *[7] Let M be a compact surface, and let $q \in M$. The surface M is large if and only if its fundamental group $\pi_1(M, q)$ has no Abelian subgroup of finite index.*

Proof. Let M be a large compact surface, and suppose that there exists an Abelian subgroup G of $\pi_1(M, q)$ of finite index k, $k \in \mathbb{N}$. Then there exists a covering $p : M_G \longrightarrow M$ such that if $p(\widetilde{q}) = q$, for some $\widetilde{q} \in M_G$, then $\pi_1(M_G, \widetilde{q})$ is isomorphic to G. Since M is a compact surface and $[\pi_1(M, q) : G]$ is finite, M_G is compact, $\pi_1(M_G, \widetilde{q})$ is Abelian, and hence M_G must be one of the following Abelian surfaces:

$$S^2, \quad P^2, \quad T^2, \quad D^2, \quad S^1 \times I, \quad \text{or} \quad S^1 \widehat{\times} I.$$

The Riemann-Hurwitz formula yields:

$$\chi(M_G) = k \cdot \chi(M).$$

If $M_G = S^2$ then $\chi(M_G) = 2$, and hence either $\chi(M) = 1$ and $k = 2$, or $\chi(M) = 2$ and $k = 1$. In the first case, M is the disc or the projective plane, while in the second case, M is the sphere.

In the cases $M_G = P^2, D^2$, we have $\chi(M_G) = 1$ and it follows that $\chi(M) = 1 = k$. Thus M is the projective plane P^2, or the disc D^2, since M is a compact surface with Abelian fundamental group, and $\chi(M) = 1$.

Now, if $M_G = T^2, S^1 \times I$, or $S^1 \widehat{\times} I$, then $\chi(M_G) = 0$ and therefore $\chi(M) = 0$ since $k \in \mathbb{N}$. Hence M must be one of the following surfaces:

$$T^2, \quad S^1 \times I, \quad S^1 \widehat{\times} I, \quad \text{or} \quad S^1 \widehat{\times} S^1.$$

In either case, M is not a large compact surface, which is a contradiction. Thus $\pi_1(M, q)$ has no Abelian subgroup of finite index.

Conversely, suppose that $\pi_1(M, q)$ has no Abelian subgroup of finite index. Then M cannot be an Abelian surface, and hence $M \neq S^2, P^2, T^2, D^2, S^1 \times I, S^1 \widehat{\times} I$. Thus, it suffices to show that the Klein bottle $S^1 \widehat{\times} S^1$ has an Abelian subgroup of finite index. Consider the fundamental group, $\pi_1(S^1 \widehat{\times} S^1, q)$, of the Klein bottle given by the following presentation

$$\pi_1(S^1 \widehat{\times} S^1, q) = \langle a, b \mid abab^{-1} = 1 \rangle. \tag{1}$$

The equation $b^s a^r = a^{(-1)^s r} b^s$, with $s, r \in \mathbb{Z}$ in $\pi_1(S^1 \widehat{\times} S^1, q)$ implies that the subgroup generated by $\{a, b^2\}$ is Abelian of index two in $\pi_1(S^1 \widehat{\times} S^1, q)$. Consequently,

$$M \neq S^2, \quad P^2, \quad D^2, \quad T^2, \quad S^1 \times I, \quad S^1 \widehat{\times} I, \quad \text{or} \quad S^1 \widehat{\times} S^1. \qquad ☑$$

By [4, Theorem 4.4], the only surfaces whose fundamental group has non-trivial center are $P^2, T^2, S^1 \times I, S^1 \widehat{\times} I, S^1 \widehat{\times} S^1$. In particular, if M is a large compact surface then the center, $Z(\pi_1 M)$, of $\pi_1 M$ is trivial.

3. Geometric Elements and Commensurators in $\pi_1 M$

We now extend [7, Theorem 3.1], which describes the commensurator subgroup $C_G(H)$ of the fundamental group of a surface M, when H is a geometric subgroup.

3.1. Geometric Elements and Subsurfaces

Let M be a surface, and let α be an element of $\pi_1 M$. The element α is called *geometric* if it can be represented by a simple closed curve in M. In general, the elements of $\pi_1 M$ are non-geometric. For instance, when M is the torus, a non-trivial power of a non-trivial element of its fundamental group cannot be represented by a simple closed curve, hence is non-geometric.

A *subsurface* N of a surface M is the closure of an open subset of M. For simplicity, we suppose that every boundary component of N is either a boundary component of M, or lies in the interior of M. Let M be a (compact) surface, and let N be a subsurface of M such that no component of $\overline{M \setminus N}$ is a disc. Then the map induced by the inclusion, $\psi : \pi_1 N \longrightarrow \pi_1 M$, is injective [7]. In this case, we can think of $\pi_1 N$ as a subgroup of $\pi_1 M$. Subgroups obtained in this way are called *geometric subgroups*, or simply *geometric*.

Let N be a subsurface of a connected surface M. We call N a *Möbius collar* in M, if N is a cylinder $S^1 \times I$ and $\overline{M \setminus N}$ has two components, N_1, N_2, one of which, N_1 say, is a Möbius strip. Then $M_0 = N \cup N_1$ will be called the *Möbius strip collared by N in M*.

The following proposition yields information about infinite cyclic geometric subgroups.

Proposition 3. *[9] An infinite cyclic subgroup of $\pi_1 M$ is geometric if and only if it has a geometric generator.*

Proof. Suppose that H is an infinite cyclic geometric subgroup of $\pi_1 M$. Then there exists a subsurface $N \subseteq M$ such that no connected component of $\overline{M \setminus N}$ is a disc, and the homomorphism $\psi : \pi_1 N \longrightarrow \pi_1 M$ is injective on H. We can choose a generator α of $\pi_1 N$ to be geometric. Since $\psi(\alpha)$ is a generator of H and ψ is induced by inclusion, one of the generators of H is geometric.

The converse is clear by taking a tubular neighborhood of a simple closed curve that represents a geometric generator. ☑

We can easily find models of simple closed curves for representatives of the classes $a^m b^n \in \pi_1(S^1 \widehat{\times} S^1, p)$, a, b being given by Equation 1, when $n = 0$ and $m = \pm 1$, or $m = 0$ and $n = \pm 2$, or $n = \pm 1$ and m is arbitrary. In [3] it is shown that these are the only possibilities for m, n that give rise to classes of homotopic closed curves in $\pi_1(S^1 \widehat{\times} S^1, p)$ possessing a simple closed curve as a representative. Thus we have a characterization of simple closed curves on the Klein bottle.

Proposition 4. *Let γ be a closed curve on the Klein bottle such that $[\gamma] = a^m b^n \in \pi_1(S^1 \widehat{\times} S^1, p)$. Then γ is a simple closed curve if and only if one of the following conditions on m and n holds:*

(1) $n = 0$ *and* $m = \pm 1$,

(2) $m = 0$ *and* $n = \pm 2$,

(3) $n = \pm 1$ *and m is arbitrary.*

3.2. Commensurator Subgroups

Let G be a group, and let H be a subgroup of G. The set of elements $g \in G$ such that $gHg^{-1} \cap H$ has finite index in both gHg^{-1} and H is a subgroup of G, called the *commensurator subgroup* of H in G, and denoted by $C_G(H)$. Also, we denote the centralizer of H in G, and the normalizer of H in G by $Z_G(H)$ and $N_G(H)$, respectively. It is easy to see that

$$Z_G(H) \subseteq N_G(H) \subseteq C_G(H).$$

Clearly, if H is a finite order subgroup of G, or if H is a normal subgroup of G, or if G is an Abelian group, then $C_G(H) = G$. From this, it follows that if $H = Z(G)$, the center of G, or if $H = [G, G]$, the commutator subgroup of G, then $C_G(H) = G$. In particular, we have $Z_G(Z(G)) = N_G(Z(G)) = C_G(Z(G)) = G$. It is very useful to understand commensurator subgroups.

Proposition 5. *[7] Let G be a group, and let H, F be subgroups of G such that $F \leq H$. If F has finite index in H, then*

$$C_G(H) = C_G(F).$$

Proof. Suppose that $g \in C_G(F)$. Thus $F_1 = gFg^{-1} \cap F$ has finite index in F and in gFg^{-1}. Set $H_1 = gHg^{-1} \cap H$. We need to prove that H_1 has finite index in H and in gHg^{-1}. It is clear that $[H : F_1]$ is finite. Also, we have $[H : H_1][H_1 : F_1] = [H : F_1]$, and so $[H : H_1]$ is also finite. In a similar manner one proves that $[gHg^{-1} : H_1]$ is finite, and thus $g \in C_G(H)$. The converse inclusion is proved similarly. ☑

We are now going to relate short exact sequences to certain commensurator subgroups.

Lemma 1. *[7, Lemma 5.5] Consider the following short exact sequence*

$$1 \longrightarrow G_1 \longrightarrow G_2 \xrightarrow{\phi} G_3 \longrightarrow 1.$$

Suppose that $G_1 \leq G_2$, $H_2 \leq G_2$, $H_3 = \phi(H_2)$ *and* $H_1 = H_2 \cap G_1$. *Then*

$$\phi\big(C_{G_2}(H_2)\big) \subseteq C_{G_3}(H_3) \quad \textit{and} \quad C_{G_2}(H_2) \cap G_1 \subseteq C_{G_1}(H_1).$$

Note that the second inclusion holds under the weaker hypothesis that $G_1 \longrightarrow G_2$ is injective, $G_1 \leq G_2$ and $H_2 \leq G_2$.

The following theorem describes the commensurator of geometric subgroups of the fundamental group of a surface M.

Theorem 1. *[7, Theorem 3.1] Let* M *be a connected surface, and let* N *be a subsurface of* M *such that no connected component of* $\overline{M \setminus N}$ *is a disc. Let* $P_0 \in N$, *and set* $\pi_1 M = \pi_1(M, P_0)$ *and* $\pi_1 N = \pi_1(N, P_0)$.

(1) If M *is not large, or if* $\pi_1 N = \{1\}$, *then* $C_{\pi_1 M}(\pi_1 N) = \pi_1 M$.

(2) If M *is large,* $\pi_1 N \neq \{1\}$ *and* N *is not a Möbius collar in* M, *then*

$$C_{\pi_1 M}(\pi_1 N) = \pi_1 N.$$

(3) If M *is large and* N *is a Möbius collar in* M, *then*

$$C_{\pi_1 M}(\pi_1 N) = \pi_1 M_0,$$

where M_0 *is the Möbius strip collared by* N *in* M.

The following theorem, together with the characterization of closed curves given in Proposition 4, not only provides a proof for Theorem 1(1), but also generalizes it to arbitrary subgroups of the fundamental group of non-large compact surfaces.

Theorem 2. *Let* M *be a non-large compact surface.*

(1) Let M *be a surface different from the Klein bottle, and let* $H \leq \pi_1 M$. *Then* $C_{\pi_1 M}(H) = \pi_1 M$.

(2) Let M *be the Klein bottle, let* $p \in S^1 \widehat{\times} S^1$, *and let* $H \leq \pi_1\big(S^1 \widehat{\times} S^1, p\big)$. *Then*

$$C_{\pi_1(S^1 \widehat{\times} S^1, p)}(H) = \pi_1\big(S^1 \widehat{\times} S^1, p\big),$$

unless $H = \langle a^m b^n \rangle$ *where* $m \neq 0, n \neq 0$ *and* n *is even. In this exceptional case,* $C_{\pi_1(S^1 \widehat{\times} S^1, p)}\big(\langle a^m b^n \rangle\big)$ *is the free Abelian subgroup of* $\pi_1\big(S^1 \widehat{\times} S^1, p\big)$ *of rank 2 generated by* $\{a, b^2\}$.

Proof.

(1) In this case, M is an Abelian surface. The result is immediate since $\pi_1 M$ is Abelian.

(2) The subgroups of the fundamental group of the Klein bottle are either trivial, free of rank one, free Abelian of rank two, or non-Abelian of rank two. In the last two cases, the subgroups are of finite index in $\pi_1\big(S^1 \widehat{\times} S^1, p\big)$. If H is a free subgroup of $\pi_1\big(S^1 \widehat{\times} S^1, p\big)$ of rank two, it follows from Proposition 5 that $C_{\pi_1(S^1 \widehat{\times} S^1, p)}(H) = \pi_1\big(S^1 \widehat{\times} S^1, p\big)$.

We now analyze the free subgroups of rank one of $\pi_1\big(S^1 \widehat{\times} S^1, p\big)$. Take $a^m b^n \in \pi_1\big(S^1 \widehat{\times} S^1, p\big)$. Recall that every element of $\pi_1\big(S^1 \widehat{\times} S^1, p\big)$ is of the form $a^r b^s$ with $r, s \in \mathbb{Z}$. Denote $f = a^r b^s a^m b^n b^{-s} a^{-r}$, and note that $a^r b^s \cdot \langle a^m b^n \rangle \cdot b^{-s} a^{-r} = \langle f \rangle$.

If n is odd then we have $f = a^{2r + (-1)^s m} b^n$ whose square is exactly b^{2n}. Furthermore $(a^m b^n)^2 = b^{2n}$, since n is odd. Thus $\langle f \rangle \cap \langle a^m b^n \rangle \neq \{1\}$ for all $r, s \in \mathbb{Z}$. Since $\langle a^m b^n \rangle$ is an infinite cyclic subgroup, $\big[\langle a^m b^n \rangle : \langle f \rangle \cap \langle a^m b^n \rangle\big]$ is finite for all $r, s \in \mathbb{Z}$.

Similarly, $\big[\langle f \rangle : \langle f \rangle \cap \langle a^m b^n \rangle\big]$ is finite for all $r, s \in \mathbb{Z}$, and so $C_{\pi_1(S^1 \widehat{\times} S^1, p)}\big(\langle a^m b^n \rangle\big) = \pi_1\big(S^1 \widehat{\times} S^1, p\big)$.

If $m = 0$ or $n = 0$, we clearly have $C_{\pi_1(S^1 \widehat{\times} S^1, p)}\big(\langle a^m b^n \rangle\big) = \pi_1\big(S^1 \widehat{\times} S^1, p\big)$.

Now suppose that $m \neq 0, n \neq 0$ and n is even. Then we have $f = a^{(-1)^s m} b^n$. If s is even, then $f = a^m b^n$ and consequently $a^r b^s \in C_{\pi_1(S^1 \widehat{\times} S^1, p)}\big(\langle a^m b^n \rangle\big)$. If s is odd, then $f = a^{-m} b^n$. In this case, since $n \neq 0$, we have $\langle a^{-m} b^n \rangle \cap \langle a^m b^n \rangle = \{1\}$, and so $a^r b^s \notin C_{\pi_1(S^1 \widehat{\times} S^1, p)}\big(\langle a^m b^n \rangle\big)$. Hence if $m \neq 0, n \neq 0$ and n is even, it follows that the commensurator $C_{\pi_1(S^1 \widehat{\times} S^1, p)}\big(\langle a^m b^n \rangle\big)$ is equal to the group $\big\{a^r b^s \in \pi_1\big(S^1 \widehat{\times} S^1, p\big) \mid s \text{ is even}\big\}$, which is exactly $\big\langle a, b^2 \mid ab^2 = b^2 a\big\rangle$. ☑

Note that if m, n satisfy the conditions of Proposition 4, then $a^m b^n \in \pi_1\big(S^1 \widehat{\times} S^1, p\big)$ is a geometric element and $C_{\pi_1(S^1 \widehat{\times} S^1, p)}\big(\langle a^m b^n \rangle\big) = \pi_1\big(S^1 \widehat{\times} S^1, p\big)$. However, there are non-geometric elements $a^m b^n \in \pi_1\big(S^1 \widehat{\times} S^1, p\big)$ for which the commensurator $C_{\pi_1(S^1 \widehat{\times} S^1, p)}\big(\langle a^m b^n \rangle\big)$ is equal to $\pi_1\big(S^1 \widehat{\times} S^1, p\big)$, for instance, when n is odd, with $|n| \neq 1$.

The following theorem shows that the subgroups of the fundamental group of a surface are "almost" geometric, i.e., they are geometric in some finite covering space of the surface.

Theorem 3. *[9] Let M be a surface, let H be a finitely-generated subgroup of $\pi_1 M$, and let $g \in \pi_1 M \setminus H$. Then there exists a finite covering $\widetilde{M}$ of M such that $\pi_1 \widetilde{M}$ contains H but not g and H is geometric in $\widetilde{M}$.*

We use Theorem 3 to prove that the commensurator of an arbitrary subgroup of the fundamental group of a surface M has a subgroup of finite index that can be calculated using Theorem 1.

Theorem 4. *Let M be a compact surface (or compact with a finite number of points removed), and let H be a finitely-generated subgroup of $\pi_1 M$. Then $\left[C_{\pi_1 M}(H) : C_{\pi_1 \widetilde{M}}(H)\right]$ is finite, where $\widetilde{M}$ is a finite covering space of M and H is geometric in $\pi_1 \widetilde{M}$.*

Proof. By Theorem 3, there exists a finite covering space $\widetilde{M}$ of M such that $H \leq \pi_1 \widetilde{M}$ and H is geometric in $\widetilde{M}$. We can suppose that $\pi_1 \widetilde{M} \leq \pi_1 M$. Clearly $C_{\pi_1 \widetilde{M}}(H) \leq C_{\pi_1 M}(H) \cap \pi_1 \widetilde{M}$.

On the other hand, Lemma 1 implies that $C_{\pi_1 M}(H) \cap \pi_1 \widetilde{M} \leq C_{\pi_1 \widetilde{M}}(H)$, and thus $C_{\pi_1 M}(H) \cap \pi_1 \widetilde{M} = C_{\pi_1 \widetilde{M}}(H)$. Since $\left[\pi_1 M : \pi_1 \widetilde{M}\right]$ is finite, $\left[C_{\pi_1 M}(H) : C_{\pi_1 \widetilde{M}}(H)\right]$ is finite. ☑

***Remark* 1.**

(1) The commensurator subgroup $C_{\pi_1 \widetilde{M}}(H)$ is completely determined by Theorem 1.

(2) Theorem 4 implies that $C_{\pi_1 M}(H)$ is the disjoint union $C_{\pi_1 \widetilde{M}}(H) \cup t_1 C_{\pi_1 \widetilde{M}}(H) \cup \cdots \cup t_l C_{\pi_1 \widetilde{M}}(H)$, where $T = \{1, t_1, \ldots, t_l\}$ is a left transversal of $C_{\pi_1 \widetilde{M}}(H)$ in $C_{\pi_1 M}(H)$.

4. An Application to Group Representation Theory

In [5] G. W. Mackey made an important connection between commensurator subgroups and unitary representation theory. In [8], we can see how results related to commensurator subgroups of classical braid groups can be applied to representation theory using this connection. We apply some results of Section 3 to unitary representation theory. Recommended references for this topic are [2, 5].

Consider a discrete group G with subgroup G_0. Given a (unitary) representation ρ of G_0, there is a well-defined induced representation, $Ind_{G_0}^{G}(\rho)$, of G. In particular, if ρ is the trivial representation, λ_{G/G_0} is the left regular representation of G in $l^2(G/G_0)$. As in [2, Theorem 2.1], we have the following result of Mackey:

the representation λ_{G/G_0} is irreducible if and only if $C_G(G_0) = G_0$. When $C_G(G_0) = G_0$, $Ind_{G_0}^{G}(\rho)$ is irreducible for any unitary finite-dimensional irreducible representation ρ of G_0.

The following theorem is an immediate consequence of Theorem 1.

Theorem 5. *Let M be a connected surface, and let N be a subsurface of M such that no component of $\overline{M \setminus N}$ is a disc.*

(1) If M is large, $\pi_1 N \neq \{1\}$, and N is not a Möbius collar in M, then $\lambda_{\pi_1 M/\pi_1 N}$ is irreducible. Furthermore, $Ind_{\pi_1 N}^{\pi_1 M}(\rho)$ is irreducible for any unitary finite-dimensional irreducible representation ρ of $\pi_1 N$.

(2) If M and N are not as in item (1), then $\lambda_{\pi_1 M/\pi_1 N}$ is reducible.

The following theorem is an immediate consequence of Theorem 2.

Theorem 6. *Let M be a non-large compact surface. Then, for any subgroup H of $\pi_1 M$, $\lambda_{\pi_1 M/H}$ is reducible.*

Let $M, \widetilde{M}$ and H be as in Theorem 4. Let S be a subsurface of $\widetilde{M}$ related to H. The following theorem is an immediate consequence of Theorem 1 and Theorem 4.

Theorem 7. *If $\widetilde{M}$ and S do not satisfy the conditions of Theorem 1(2), or if the number of sheets of the finite covering space $\widetilde{M}$ is greater than 1, then $\lambda_{\pi_1 M/H}$ is reducible.*

5. Nielsen Number of Self-Intersection Points of Closed Curves in the Klein Bottle

By analysing closed curves and commensurator subgroups of the Klein bottle, we give an application to self-intersection Nielsen theory. This application is completely inspired by the paper of S. Bogatyi, E. Kudryavtseva and H. Zieschang [1].

5.1. Basic Concepts

We start by introducing the basic concepts for the rest of this section. We say that a closed curve on a surface is *orientation-preserving* if the local orientation of the surface is preserved under the continuous transfer of the orientation along the curve. Otherwise it is called *orientation-reversing*. We assume in this whole subsection that $\gamma : S^1 \longrightarrow M$ is a closed curve on the surface M.

A *self-intersection point* of γ is a pair (v_1, v_2), with $v_1, v_2 \in S^1$, $v_1 \neq v_2$, such that $\gamma(v_1) = \gamma(v_2)$. The *minimal number of self-intersection points*, $MI[\gamma]$, is defined to be the minimal number of self-intersection points of all curves γ' such that $\gamma' \simeq \gamma$.

Self-intersection points (v_1, v_2) and (v_1', v_2') are called *Nielsen-equivalent self-intersection points* if there exist paths $\alpha_i : [0,1] \longrightarrow S^1$, $i = 1, 2$, satisfying

$$\alpha_i(0) = v_i, \qquad \alpha_i(1) = v_i', \qquad \gamma \circ \alpha_1 \simeq_\partial \gamma \circ \alpha_2.$$

A self-intersection point (v_1, v_2) is *trivial* or *equivalent to zero*, if there exists a path $\alpha : [0,1] \longrightarrow S^1$ satisfying

$$\alpha(0) = v_1, \qquad \alpha(1) = v_2, \qquad \gamma \circ \alpha \simeq_\partial 0.$$

Trivial self-intersection points form a Nielsen class called the *trivial Nielsen class*. Such a class may be empty.

The index of an isolated self-intersection point, and of a class of self-intersection points is defined to be the index of the corresponding coincidence point or of the coincidence class, respectively, of the mappings $\tilde{\gamma}_1, \tilde{\gamma}_2 : S^1 \times S^1 \longrightarrow M$, $\tilde{\gamma}_i(t_1, t_2) = \gamma(t_i)$, for $i = 1, 2$.

A Nielsen class of self-intersection points is called *essential* if its index does not vanish; otherwise it is called *inessential*. The *Nielsen number of self-intersection points*, $NI[\gamma]$, is the number of essential classes. Clearly, $MI[\gamma] \geq NI[\gamma]$. If the numbers $MI[\gamma]$ and $NI[\gamma]$ coincide, we say that the *Wecken property holds for the intersection problem*. We refer the reader to [1] for more details on these topics.

5.2. Special Curves

We now study a family of so-called *special curves* on the Klein bottle. The definition of special curve on a surface is general [1]. A closed curve $\gamma : S^1 \longrightarrow M$ on a surface M is called *special* if γ is orientation-preserving, non-contractible, and homotopic to a proper power of a closed curve γ_0 on M.

Let γ be a closed curve (non-contractible) on $S^1 \widehat{\times} S^1$ such that $[\gamma] = a^m b^n \in \pi_1(S^1 \widehat{\times} S^1, p)$.

If n is odd, then γ is orientation-reversing, and hence it is not special. Now, suppose that n is even. In this case, $a^m b^n$ is orientation-preserving. If $gcd(m,n) = 1$, then $a^m b^n$ cannot be a proper power of another element in $\pi_1(S^1 \widehat{\times} S^1, p)$, thus γ cannot be special. Suppose that $gcd(m,n) = t$ is different from 1. If n/t is even, then $a^m b^n = \left(a^{\frac{m}{t}} b^{\frac{n}{t}}\right)^t$, and consequently γ is special. If n/t is odd with $t \neq 2$ (hence t is even), we have $\left(a^{\frac{2m}{t}} b^{\frac{2n}{t}}\right)^{\frac{t}{2}} = a^m b^n$, hence γ is special. Finally, if n/t is odd and $t = 2$ we cannot write $a^m b^n$ as a proper power of another element in $\pi_1\left(S^1 \widehat{\times} S^1, p\right)$.

By the above analysis, we deduce the following classification of special curves on the Klein bottle.

Proposition 6. *Let γ be a non-contractible closed curve on the Klein bottle such that $[\gamma] = a^m b^n \in \pi_1\left(S^1 \widehat{\times} S^1, p\right)$. Then γ is special on $S^1 \widehat{\times} S^1$ if and only if one of the following conditions on m and n holds:*

(1) $gcd(m,n) = t \neq 1$ and n/t is even,

(2) $gcd(m,n) = t \neq 2$, n is even, and n/t is odd.

5.3. An Application

We now use the characterization of closed curves on the Klein bottle, given by Proposition 6, in order to calculate the Nielsen number of self-intersection points of closed curves on $S^1 \widehat{\times} S^1$. For this we need the following proposition.

Proposition 7. *[1] Let M be a surface, not necessarily compact, and let $\gamma : S^1 \longrightarrow M$ be a closed curve.*

(1) A curve γ which is not special has the Wecken property for the self-intersection problem:

$$MI(\gamma) = NI(\gamma).$$

In this case, if $\gamma \simeq \gamma_0^k$, with $k \in \mathbb{Z}$, $|k| \geq 1$, and the curve γ_0 is not homotopic to a proper power of any closed curve on M, then

$$MI(\gamma) = NI(\gamma) = k^2 \cdot NI(\gamma_0) + |k| - 1.$$

(2) Let γ be a special curve. Then γ is homotopic to a proper power γ_0^k, with $k \in \mathbb{Z}$, $|k| > 1$, where the curve γ_0 is orientation-preserving or k is even, and γ_0 is not homotopic to a proper power of any closed curve on M. Set $k' = k$ if γ_0 is orientation-preserving, and $k' = k/2$ if γ_0 is orientation-reversing. Then

$$MI(\gamma) = k^2 \cdot NI(\gamma_0) + 2(|k'| - 1),$$
$$NI(\gamma) = k^2 \cdot NI(\gamma_0).$$

The following theorem follows easily by applying the classification of special closed curves given by Proposition 6, and Proposition 4 to Proposition 7.

Theorem 8. *Let γ be a closed curve non-contractible in $S^1 \widehat{\times} S^1$. Choose a point p in γ, and let $[\gamma] = a^m b^n \in \pi_1\big(S^1 \widehat{\times} S^1, p\big)$ and $t = gcd(m,n)$.*

(1) If n is odd, then $NI(\gamma) = n - 1$,

(2) If n is even, n/t is odd and $t \neq 2$, then $NI(\gamma) = \big(\frac{t}{2}\big)^2 \big(\frac{2n}{t} - 1\big)$,

(3) If n is even, $m, n \neq 0$ and $t = 1$, then $NI(\gamma) \neq 0$,

(4) If n and n/t are even and $t \neq 1$, then $NI(\gamma) = t^2 NI(\gamma_0)$, where $[\gamma_0] = a^{\frac{m}{t}} b^{\frac{n}{t}}$,

(5) If $n = 0$ and $m = \pm 1$, or if $m = 0$ and $n = \pm 2$, then $NI(\gamma) = 0$.

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