PORTUGALIAE MATHEMATICA
Vol. 63 Fasc. 1 – 2006
Nova Série

REAL GELFAND–MAZUR ALGEBRAS

Olga Panova *

Recommended by A.F. dos Santos

Abstract: Several classes of real Gelfand–Mazur algebras are described. Conditions, when the trace $M \cap B$ of a closed maximal left (right) ideal M of a real topological algebra A would be a maximal ideal in a subalgebra B of the center of A are given.

1 – Introduction

1. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers, A a *topological algebra* over $\mathbb{K}$ with associative separately continuous multiplication (in short, topological algebra) and $m(A)$ the set of all closed regular (or modular) two-sided ideals of A, which are maximal as left or right ideals. If the quotient algebra A/M (in the quotient topology) is topologically isomorphic to $\mathbb{K}$ for each $M \in m(A)$, then A is called a *Gelfand–Mazur algebra* (see [1], [2], [3] or [4]). Herewith, A is a *real Gelfand–Mazur algebra* if $\mathbb{K} = \mathbb{R}$ and is a *complex Gelfand–Mazur algebra* if $\mathbb{K} = \mathbb{C}$.

Moreover, a unital topological algebra A is a *Q-algebra* if the set $\mathrm{Inv}A$ of all invertible elements of A is open in A; is a *Waelbroeck algebra* or a *topological algebra with continuous inverse* if A is a Q-algebra in which the inversion $a \to a^{-1}$ is continuous in $\mathrm{Inv}A$; is a *Fréchet algebra* if the underlying linear topological space

Received: January 26, 2004; *Revised*: June 1, 2005.
AMS Subject Classification: Primary 46H05; Secondary 46H20.
Keywords: real topological algebra; real Gelfand–Mazur algebra; real exponentially galbed algebra; real locally pseudoconvex algebra; real locally A-pseudoconvex algebra; real locally m-pseudoconvex algebra; traces of maximal ideals.
* Research in part supported by Estonian Science Foundation grant 6205.

of A is complete and metrizable; is a *topological algebra with bounded elements* if every $a \in A$ is bounded in A that is, there is a nonzero complex number λ_a such that the set

$$\left\{\left(\frac{a}{\lambda_a}\right)^n : \ n \in \mathbb{N}\right\}$$

is bounded in A; is an *exponentially galbed algebra* if its underlying topological vector space is an *exponentially galbed space* that is, for each neighbourhood O of zero in A there exists another neighbourhood U of zero in A such that

$$\left\{\sum_{k=0}^{n} \frac{a_k}{2^k} : \ a_0, ..., a_n \in U\right\} \subset O$$

for each $n \in \mathbb{N}$ and is a *locally pseudoconvex algebra* if A has a base $\mathcal{B} = \{U_\alpha : \alpha \in \mathcal{A}\}$ of neighbourhoods of zero consisting of *balanced* (i.e. $\lambda U_\alpha \in U_\alpha$, whenever $|\lambda| \leqslant 1$) and *pseudoconvex* (i.e. $U_\alpha + U_\alpha \subset \mu U_\alpha$ for some $\mu \geqslant 2$) sets. Moreover, a locally pseudoconvex algebra A is *locally A-pseudoconvex* if for each $U_\alpha \in \mathcal{B}$ and $a \in A$ there is a number $\mu_a > 0$ such that aU_α, $U_\alpha a \subset \mu_a U_\alpha$ and is *locally m-pseudoconvex* if $U_\alpha^2 \subset U_\alpha$ for each $U_\alpha \in \mathcal{B}$.

2. Let now A be a real topological algebra,

$$Z(A) = \left\{z \in A : \ za = az \ \text{ for each } \ a \in A\right\}$$

the center of A and B a closed subalgebra of $Z(A)$ in the subset topology. An ideal $M \in m(B)$ is called *extendible to A* if

$$I(M) = \mathrm{cl}_A\left\{\sum_{k=1}^{n} a_k m_k : \ n \in \mathbb{N}, \ a_1, ..., a_n \in A; \ m_1, ..., m_n \in M\right\} \neq A,$$

where $\mathrm{cl}_A(M)$ denotes the closure of M in the topology of A. We denote by $m_e(B)$ the set of all ideals $M \in m(B)$, which are extendible to A.

3. Let A be a (real or complex) topological algebra, M a maximal regular left (right) ideal of A and $P_M = \{a \in A : aA \subset M\}$ ($P_M = \{a \in A : Aa \subset M\}$, respectively) the *primitive ideal* of A defined by M. If $\{\theta_A\}$ is a primitive ideal of A, then A is called a *primitive algebra* and if there is a closed maximal regular left (right) ideal M of A such that $P_M = \{\theta_A\}$, then A is called a *topologically primitive algebra* (see [2], p. 21).

4. Properties of real Banach algebras have been studied in several books and articles (see, for example, [7], [8], [9] and [10]); of real k-normed and real k-Banach algebras in [6]; of real Waelbroeck algebras in [6] and in [12]; of real locally m-convex algebras in [11] and of real locally pseudoconvex division algebras in [5]. Properties of several classes of real Gelfand–Mazur algebras are studied and conditions for a real topological algebra A that the trace $M \cap B$ of a closed maximal left (right) ideal M of A in a subalgebra B of the center $Z(A)$ to be a closed maximal ideal in B are given in the present paper.

2 – Properties of the center and of the quotient algebra

Let (A, τ) be a real topological algebra, I a closed two-sided ideal of A and π_I the canonical homomorphism of A onto A/I. By $\tau_{A/I}$ we denote the quotient topology on A/I, defined by τ and π_I, and by τ_I the subset topology on $Z(A/I)$ defined by $\tau_{A/I}$. Similarily as in the complex case (see [2], pp. 26–28) we have the following result.

Proposition 1. *Let A be a real topological algebra and I a closed two-sided ideal of A. If there exists a topology τ on A such that*

- **a)** *(A, τ) is locally pseudoconvex, then $(A/I, \tau_{A/I})$ and $(Z(A/I), \tau_I)$ are real locally pseudoconvex algebras;*
- **b)** *(A, τ) is locally A-pseudoconvex (in particular, locally m-pseudoconvex), then $(A/I, \tau_{A/I})$ and $(Z(A/I), \tau_I)$ are real locally A-pseudoconvex (respectively, locally m-pseudoconvex) algebras;*
- **c)** *(A, τ) is an exponentially galbed algebra with bounded elements, then $(A/I, \tau_{A/I})$ and $(Z(A/I), \tau_I)$ are real exponentially galbed algebras with bounded elements;*
- **d)** *(A, τ) is a locally pseudoconvex Fréchet algebra, then $(A/I, \tau_{A/I})$ and $(Z(A/I), \tau_I)$ are real locally pseudoconvex Fréchet algebras;*
- **e)** *(A, τ) is a real topological algebra with jointly continuous multiplication, then $(A/I, \tau_{A/I})$ and $(Z(A/I), \tau_I)$ are real topological algebras with jointly continuous multiplication.*

Moreover, if I is a regular ideal, u a right modular unit for I and for each $a \in A$ there is a $\lambda \in \mathbb{R}$ such that $a - \lambda u \in I$, then $\mathrm{sp}_{A/I}(x)$ is not empty for each $x \in A/I$ and $\mathrm{sp}_{Z(A/I)}(y) = \mathrm{sp}_{A/I}(y)$ for each $y \in Z(A/I)$. ∎

3 – Complexification of real algebras

1. Let A be a (not necessarily topological) real algebra and let $\tilde{A} = A + iA$ be the comlexification of A. Then every element $\tilde{a}$ of $\tilde{A}$ is representable in the form $\tilde{a} = a + ib$, where $a, b \in A$ and $i^2 = -1$. If we define the addition in $\tilde{A}$, the multiplication over $\mathbb{C}$ and the multiplication in $\tilde{A}$ by

$$(a+ib)+(c+id) = (a+c)+i(b+d),$$

$$(\alpha+i\beta)(a+ib) = (\alpha a - \beta b) + i(\alpha b + \beta a),$$

and

$$(a+ib)(c+id) = (ac-bd)+i(ad+bc)$$

for all $a, b, c, d \in A$ and $\alpha, \beta \in \mathbb{R}$, then $\tilde{A}$ is a complex algebra with zero element $\theta_{\tilde{A}} = \theta_A + i\theta_A$ (here and later on θ_A denotes the zero element of A). If A is an algebra with unit element e_A, then $e_{\tilde{A}} = e_A + i\theta_A$ is the unit element of $\tilde{A}$. Herewith, $\tilde{A}$ is an associative (commutative) algebra if A is an associative (respectively, commutative) algebra. We can consider A as a real subalgebra of $\tilde{A}$ under the imbedding ν from A into $\tilde{A}$ defined by $\nu(a) = a + i\theta_A$ for each $a \in A$.

2. Let A be an algebra over $\mathbb{K}$ with unit e_A and

$$\mathrm{sp}_A(a) = \{\lambda \in \mathbb{K} : a - \lambda e_A \notin \mathrm{Inv} A\}$$

for each $a \in A$. Then $\mathrm{sp}_A(a)$ is the *spectrum* of a. Herewith, elements of $\mathrm{sp}_A(a)$ are complex numbers if A is a complex algebra and real numbers if A is a real algebra.

A real (not necessarily topological) algebra A is *formally real* if from $a, b \in A$ and $a^2 + b^2 = \theta_A$ follows that $a = b = \theta_A$ and is *strictly real* if $\mathrm{sp}_{\tilde{A}}(a + i\theta_A) \subset \mathbb{R}$. It is known (see, for example, [6], Proposition 1.9.14) that every formally real division algebra is strictly real and every commutative strictly real division algebra is formally real. Moreover, the complexification $\tilde{A}$ of a commutative real division algebra A is division algebra if and only if A is formally normal (see [6], Proposition 1.6.20).

Lemma 1. *Let A be a real algebra and I a two-sided ideal of A. Then the quotient algebra A/I is formally real if and only if I satisfies the condition*

(α) *from $a, b \in A$ and $a^2 + b^2 \in I$ follows that $a, b \in I$.*

Proof: Let A be a real algebra, I a two-sided ideal in A, π_I the quotient map of A onto A/I and let $a,\ b \in A$ be such that $a^2 + b^2 \in I$. Then

$$\pi_I(a)^2 + \pi_I(b)^2 = \pi_I(a^2 + b^2) = \theta_{A/I}.$$

If A/I is formally real, then $\pi_I(a) = \pi_I(b) = \theta_{A/I}$ or $a \in I$ and $b \in I$.

Let now a two-sided ideal I satisfy the condition (α) and $x,\ y \in A/I$ be such that $x^2 + y^2 = \theta_{A/I}$. Then there are $a,\ b \in A$ such that $x = \pi(a)$, $y = \pi(b)$ and

$$\pi_I(a^2 + b^2) = x^2 + y^2 = \theta_{A/I}.$$

Hence, from $a^2 + b^2 \in I$ follows that $x = y = \theta_{A/I}$ by the condition (α). ∎

3. Let now (A, τ) be a real topological algebra and $\{U_\alpha : \alpha \in \mathcal{A}\}$ a base of neigbourhoods of zero of (A, τ). As usual (see [6] or [12]), we endow $\tilde{A}$ with the topology $\tilde{\tau}$ in which $\{U_\alpha + i\,U_\alpha : \alpha \in \mathcal{A}\}$ is a base of neighbourhoods of zero. It is known that $(\tilde{A}, \tilde{\tau})$ is a complex topological algebra and the multiplication in $(\tilde{A}, \tilde{\tau})$ is jointly continuous if the multiplication in (A, τ) is jointly continuous (see [6], Proposition 2.2.10). Moreover, the underlying topological space of $(\tilde{A}, \tilde{\tau})$ is a Hausdorff space if (A, τ) is a Hausdorff algebra.

Let M be a maximal regular left (right or two-sided) ideal of A. Then (see [6], Proposition 1.6.12, p. 46) $\widetilde{M} = M + iM$ is a maximal regular left (right or two-sided) ideal in $\widetilde{A}$.

Proposition 2. *Let A be a real topological algebra, M a closed maximal regular left (right) ideal of A and P_M the primitive ideal of A defined by M. Then*

- **a**) *the primitive ideal $\widetilde{P}_{\widetilde{M}}$ of $\widetilde{A}$ defined by $\widetilde{M}$ is representable in the form $\widetilde{P}_{\widetilde{M}} = P_M + iP_M$;*
- **b**) $\widetilde{A}/\widetilde{P}_{\widetilde{M}} = A/P_M + iA/P_M$;
- **c**) $Z(\widetilde{A}) = Z(A) + iZ(A)$.

Proof: **a**) Let A be a real topological algebra, $a,\ b \in P_M$ and $v + iw \in \widetilde{A}$. Since

$$(a + ib)(v + iw) = av - bw + i(aw + bv) \in \widetilde{M},$$

then $P_M + iP_M \subset \widetilde{P}_{\widetilde{M}}$. Let now $a + ib \in \widetilde{P}_{\widetilde{M}}$ and $v + i\theta_A \in \widetilde{A}$. Then

$$(a + ib)(v + i\theta_A) = av + ibv \in \widetilde{M}$$

if and only if $av,\ bv \in M$ or $a,\ b \in P_M$. Thus $\widetilde{P}_{\widetilde{M}} \subset P_M + iP_M$.

b) Let $a,\ b \in A$. Then

$$a + P_M + i(b + P_M) = (a + ib) + (P_M + iP_M) = (a + ib) + \widetilde{P}_{\widetilde{M}} \in \widetilde{A}/\widetilde{P}_{\widetilde{M}}.$$

Hence, $A/P_M + iA/P_M \subset \widetilde{A}/\widetilde{P}_{\widetilde{M}}$ and similarily $\widetilde{A}/\widetilde{P}_{\widetilde{M}} \subset A/P_M + iA/P_M$.

c) It is clear that $Z(A) + iZ(A) \subset Z(\widetilde{A})$. Let now $a_0 + ib_0 \in Z(\widetilde{A})$. Since

$$aa_0 + iab_0 = (a + i\theta_A)(a_0 + ib_0) = (a_0 + ib_0)(a + i\theta_A) = a_0 a + ib_0 a$$

for each $a \in A$, then $a_0, b_0 \in Z(A)$. ∎

Corollary 1. *If A is a real topologically primitive topological algebra, then the complexification of A is a complex topologically primitive topological algebra.*

Proof: Let A be a real topologically primitive topological algebra. Then there exists a closed maximal regular left (right) ideal M in A such that $P_M = \{\theta_A\}$. Since

$$\widetilde{P}_{\widetilde{M}} = P_M + iP_M = \{\theta_A + i\theta_A\} = \{\theta_{\tilde{A}}\}$$

and $\widetilde{M}$ is a closed maximal regular left (right) ideal of $\widetilde{A}$, then $\widetilde{A}$ is a complex topologically primitive topological algebra. ∎

4 – Commutative real Gelfand–Mazur algebras

To describe real Gelfand–Mazur algebras, we need the following result proved in [5], Corollary 5.5:

Proposition 3. *Let A be a commutative strictly real division algebra. If A has a topology*(1) *τ such that (A, τ) is*

- **a**) *a locally pseudoconvex Hausdorff algebra with continuous inversion;*
- **b**) *a locally A-pseudoconvex (in particular, locally m-pseudoconvex) Hausdorff algebra;*
- **c**) *a locally pseudoconvex Fréchet algebra;*
- **d**) *an exponentially galbed Hausdorff algebra with jointly continuous multiplication and bounded elements;*

(1) Which can be different from the preliminary topology of A.

e) *a topological Hausdorff algebra for which the spectrum* $\mathrm{sp}_A(a)$ *is not empty for each* $a \in A$,

then A *is a commutative real Gelfand–Mazur division algebra.* ∎

Now we prove

Theorem 1. *Let* A *be a commutative real topological algebra. If* A *has a topology*(2) τ *such that* (A, τ) *satisfies the condition* (α) *for each* $I \in m((A, \tau))$ *and belongs in one of the following classes of topological algebras:*

a) *locally pseudoconvex Waelbroeck algebras;*

b) *locally* A*-pseudoconvex (in particular, locally* m*-pseudoconvex) algebras;*

c) *locally pseudoconvex Fréchet algebras;*

d) *exponentially galbed algebras with jointly continuous multiplication and bounded elements;*

e) *topological algebras in which for any element* $a \in A$ *and* $M \in m((A, \tau))$ *there is a* $\lambda \in \mathbb{R}$ *such that* $a - \lambda u \in M$ *(here* u *is a modular unit for* M*),*

then A *is a commutative real Gelfand–Mazur algebra.*

Proof: Let (A, τ) be a commutative real topological algebra which satisfies the condition (α) for each $I \in m((A, \tau))$ and M a fixed element of $m((A, \tau))$. Then $(A/M, \tau_{A/M})$ is a commutative strictly real topological division Hausdorff algebra by Lemma 1. If now (A, τ) satisfies

1) the condition a), then $(A/M, \tau_{A/M})$ is a commutative strictly real locally pseudoconvex Waelbroeck division algebra by the statement a) of Proposition 1 and Corollary 3.6.27 from [6];

2) the condition b), then $(A/M, \tau_{A/M})$ is a commutative strictly real locally A-pseudoconvex (in particular, m-pseudoconvex) Hausdorff division algebra by the statement b) of Proposition 1;

3) the condition c), then $(A/M, \tau_{A/M})$ is a commutative strictly real locally pseudoconvex Fréchet division algebra by the statement d) of Proposition 1;

4) the condition d), then $(A/M, \tau_{A/M})$ is a commutative strictly real exponentially galbed Hausdorff division algebra with jointly continuous multiplication and bounded elements by the statements c) and f) of Proposition 1;

---

(2) See the footnote 1.

5) the condition e), then $(A/M, \tau_{A/M})$ is a commutative strictly real topological Hausdorff algebra for which the spectrum $\mathrm{sp}_{A/M}(x)$ is not empty for each $x \in A/M$ by Proposition 1.

Hence, in all these cases A/M (in the quotient topology defined by the preliminary topology of A) is topologically isomorphic to $\mathbb{R}$ for each $M \in m(A)$ by Proposition 3. Therefore A is a commutative real Gelfand–Mazur algebra. ∎

5 – Maximality of traces of ideals

Let A be a unital real topological algebra, B a subalgebra of $Z(A)$ and M a closed maximal left (right) ideal of A. It is easy to see that the trace $M \cap B$ of M is a closed ideal in B. To find the conditions for A that the trace $M \cap B$ of M to be maximal in B, we need

Proposition 4. *Let A be a real locally A-pseudoconvex algebra* (*or a real locally pseudoconvex Fréchet algebra*) *with a unit element e_A, M a closed maximal left* (*right*) *ideal of A and P_M a primitive ideal of A defined by M. If P_M satisfies the condition*

(β) *from $a, b \in A$ and $a^2 + b^2 \in P_M$ follows that $a, b \in P_M$,*

then $Z(A/P_M)$ is topologically isomorphic to $\mathbb{R}$.

Proof: Let (A, τ) be a unital real locally A-pseudoconvex (locally pseudoconvex Fréchet) algebra, M a closed maximal regular left (right) ideal of A, P_M a primitive ideal in A defined by M, π_M the canonical homomorphism of A onto A/P_M and τ_M the quotient topology on A/P_M defined by τ and π_M. Then $(A/P_M, \tau_M)$ is a unital real locally A-pseudoconvex Hausdorff (respectively, locally pseudoconvex Fréchet) algebra by Proposition 1. Since the complexification of A/P_M is $\widetilde{A}/\widetilde{P}_{\widetilde{M}}$, where $\widetilde{P}_{\widetilde{M}}$ is a closed primitive ideal in $\widetilde{A}$ by Proposition 2, then $(\widetilde{A}/\widetilde{P}_{\widetilde{M}}, \widetilde{\tau}_M)$ is a unital complex locally A-pseudoconvex Hausdorff (respectively, locally pseudoconvex Fréchet) algebra by Theorem 3.3 and Corollary 3.2 from [5]. Hence, $Z(\widetilde{A}/\widetilde{P}_{\widetilde{M}})$ is topologically isomorphic to $\mathbb{C}$ by Theorem 1 from [1] or by Theorem 2.17 from [2]. Therefore, $Z(\widetilde{A}/\widetilde{P}_{\widetilde{M}})$ is a complex division algebra. As $Z(\widetilde{A}/\widetilde{P}_{\widetilde{M}}) = Z(A/P_M) + iZ(A/P_M)$ by Proposition 2, then $Z(A/P_M)$ is formally real by Proposition 1.6.20 from [6] (by condition (β) the quotient algebra A/P_M is formally real by Lemma 1, hence $Z(A/P_M)$ is formally real too).

Now, every element $x \in Z(A/P_M)$ is representable in the form $x = \lambda_x e_A$ for some $\lambda_x \in \mathbb{R}$. Therefore, $Z(A/P_M)$ is isomorphic to $\mathbb{R}$. In the same way as in the complex case (see, e.g. [2], p. 47) it is easy to show that this isomorphism is a topological isomorphism because $Z(A/P_M)$ is a Hausdorff space in the subset topology. ∎

Corollary 2. *Let A be a real locally m-pseudoconvex topological algebra with unit, P_M a primitive ideal of A defined by a closed maximal regular left (right) ideal M of A. If P_M satisfies the condition (β), then $Z(A/P_M)$ is topologically isomorphic to $\mathbb{R}$.*

Proof: Since every locally m-pseudoconvex algebra is locally A-pseudoconvex, then $Z(A/P_M)$ is topologically isomorphic to $\mathbb{R}$ by Proposition 4. ∎

Corollary 3. *Let A be a unital strictly real topologically primitive locally A-pseudoconvex Hausdorff algebra or a unital real topologically primitive locally pseudoconvex Fréchet algebra. Then $Z(A)$ is topologically isomorphic to $\mathbb{R}$.* ∎

Theorem 2. *Let*(3) *A be a real locally A-pseudoconvex (in particular, a locally m-pseudoconvex) algebra with unit e_A or a real locally pseudoconvex Fréchet algebra with unit e_A, M a closed maximal left (right or two-sided) ideal of A, P_M the primitive ideal in A defined by M and B a closed subalgebra of $Z(A)$, containing e_A. If P_M satisfies the condition (β), then*

1) *every $b \in B$ defines a number $\lambda \in \mathbb{R}$ such that $b - \lambda e_A \in M$;*

2) *$M \cap B \in m_e(B)$.*

Proof: Similarily as in [1], the proof of Corollary 1, or in [2], the proof of Proposition 3.1, it is easy to show that Theorem 2 holds by Proposition 4 and Corollary 2. ∎

Corollary 4. *Let A be a real locally A-pseudoconvex (in particular, a locally m-pseudoconvex) algebra with unit e_A or a real locally pseudoconvex Fréchet algebra with unit e_A, M a closed maximal left (right or two-sided) ideal of A and P_M the primitive ideal in A defined by M. If P_M satisfies the condition (β), then*

1) *every $z \in Z(A)$ defines a number $\lambda \in \mathbb{R}$ such that $z - \lambda e_A \in M$;*

2) *$M \cap Z(A) \in m_e(Z(A))$.* ∎

(3) For complex locally A-pseudoconvex (in particular, locally m-pseudoconvex) algebras with unit and for complex locally pseudoconvex Fréchet algebras with unit similar result has been published in [1], Corollary 1, and in [2], Proposition 3.1.

REFERENCES

[**1**] Abel, Mart – Description of closed maximal ideals in Gelfand–Mazur algebras, General Topological Algebras (Tartu, 1999), 7–13, *Math. Stud.* (Tartu), 1, *Est. Math. Soc.*, Tartu, 2001.

[**2**] Abel, Mart – Structure of Gelfand–Mazur algebras, *Dissertationes Mathematicae Universitatis Tartuensis,* 31, Tartu Univ. Press, Tartu 2003.

[**3**] Abel, Mati – *Gelfand–Mazur algebras*, in "Topological Vector Spaces, Algebras and Related Areas" (Hamilton, ON, 1994), 116–129, Pitman Res. Notes Math. Ser., 316, Longman Sci. Tech., Harlow, 1994.

[**4**] Abel, Mati – *Survey of results on Gelfand–Mazur algebras*, in "Non-Normed Topological Algebras" (Rabat, 2000), 14–25, Ecole Normale Supérieure Takaddoum, Rabat, 2004.

[**5**] Abel, Mati and Panova, O. – Real Gelfand–Mazur division algebras, *Int. J. Math. Math. Sci.,* (2003), 1–12.

[**6**] Balachandran, V.K. – *Topological Algebras*, North-Holland Math. Studies, 185, Elsevier, Amsterdam, 2000.

[**7**] Bonsall, F.F. and Duncan, J. – *Complete Normed Algebras*, Springer-Verlag, Berlin, 1973.

[**8**] Ingelstam, L. – Real Banach algebras, *Ark. Math.,* 5 (1964), 239–270.

[**9**] Li, B. and Tam, P. – Real Banach $*$-algebras, *Acta Math. Sin. (Engl. Ser.),* 16(3) (2000), 469–486.

[**10**] Rickart, C.E. – *General Theory of Banach Algebras*, D. van Nostrand Co, Inc., Princeton, 1960.

[**11**] Żelazko, W. – On generalized topological divisors of zero in real m-convex algebras, *Studia Math.,* 28 (1966/1967), 241–244.

[**12**] Żelazko, W. – On m-convexity of commutative real Waelbroeck algebras, *Comment. Math. Prace Mat.,* 40 (2000), 235–240.

Olga Panova,
Institute of Pure Mathematics, University of Tartu,
2 Liivi Str., room 613, 50409 Tartu – ESTONIA
E-mail: `olpanova@math.ut.ee`