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EVERY FUNCTION IS THE REPRESENTATION FUNCTION OF AN ADDITIVE BASIS FOR THE INTEGERS

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Abstract: Let A be a set of integers. For every integer n, let $r_{A,h}(n)$ denote the number of representations of n in the form $n = a_1 + a_2 + \cdots + a_h$, where $a_1, a_2, ..., a_h \in A$ and $a_1 \leq a_2 \leq \cdots \leq a_h$. The function

$$r_{A,h} \colon \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$$

is the *representation function of order h for A.* The set A is called an *asymptotic basis of order h* if $r_{A,h}^{-1}(0)$ is finite, that is, if every integer with at most a finite number of exceptions can be represented as the sum of exactly h not necessarily distinct elements of A. It is proved that every function is a representation function, that is, if $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ is any function such that $f^{-1}(0)$ is finite, then there exists a set A of integers such that $f(n) = r_{A,h}(n)$ for all $n \in \mathbb{Z}$. Moreover, the set A can be arbitrarily sparse in the sense that, if $\varphi(x) \geq 0$ for $x \geq 0$ and $\varphi(x) \to \infty$, then there exists a set A with $f(n) = r_{A,h}(n)$ and $\operatorname{card}(\{a \in A : |a| \leq x\}) < \varphi(x)$ for all x.

It is an open problem to construct dense sets of integers with a prescribed representation function. Other open problems are also discussed.

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1 – Additive bases and the Erdős–Turán conjecture

Let $\mathbb{N}, \mathbb{N}_0$, and $\mathbb{Z}$ denote the positive integers, nonnegative integers, and integers, respectively. Let A be a set of integers. For every positive integer h, we define the *sumset*

$$hA = \left\{a_1 + \cdots + a_h \colon\ a_i \in A \text{ for all } i = 1, ..., h\right\} .$$

We denote by $r_{A,h}(n)$ the number of representations of n in the form $n = a_1 + a_2 + \cdots + a_h$, where $a_1, a_2, ..., a_h \in A$ and $a_1 \leq a_2 \leq \cdots \leq a_h$. The function $r_{A,h}$ is called the *representation function of order* h of the set A.

In this paper we consider additive bases for the set of all integers. The set A of integers is called a *basis of order* h *for* $\mathbb{Z}$ if every integer can be represented as the sum of h not necessarily distinct elements of A. The set A of integers is called an *asymptotic basis of order* h *for* $\mathbb{Z}$ if every integer with at most a finite number of exceptions can be represented as the sum of h not necessarily distinct elements of A. Equivalently, the set A is an asymptotic basis of order h if the representation function $r_{A,h} : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ satisfies the condition

$$\text{card}\left(r_{A,h}^{-1}(0)\right) < \infty .$$

For any set X, let $\mathcal{F}_0(X)$ denote the set of all functions

$$f\colon\ X \to \mathbb{N}_0 \cup \{\infty\}$$

such that

$$\text{card}\left(f^{-1}(0)\right) < \infty .$$

We ask: Which functions in $\mathcal{F}_0(\mathbb{Z})$ are representation functions of asymptotic bases for the integers? This question has a remarkably simple and surprising answer. In the case $h = 1$ we observe that $f \in \mathcal{F}_0(\mathbb{Z})$ is a representation function if and only if $f(n) = 1$ for all integers $n \notin f^{-1}(0)$. For $h \geq 2$ we shall prove that *every* function in $\mathcal{F}_0(\mathbb{Z})$ is a representation function. Indeed, if $f \in \mathcal{F}_0(\mathbb{Z})$ and $h \geq 2$, then there exist infinitely many sets A such that $f(n) = r_{A,h}(n)$ for every $n \in \mathbb{Z}$. Moreover, we shall prove that we can construct arbitrarily sparse asymptotic bases A with this property. Nathanson [7] previously proved this theorem for $h = 2$ and the function $f(n) = 1$ for all $n \in \mathbb{Z}$.

This result about asymptotic bases for the integers contrasts sharply with the case of the nonnegative integers. The set A of nonnegative integers is called an *asymptotic basis of order* h *for* $\mathbb{N}_0$ if every sufficiently large integer can be represented as the sum of h not necessarily distinct elements of A. Very little is known about the class of representation functions of asymptotic bases for $\mathbb{N}_0$. However, if $f \in \mathcal{F}_0(\mathbb{N}_0)$, then Nathanson [5] proved that there exists at most one set A such that $r_{A,h}(n) = f(n)$.

Many of the results that have been proved about asymptotic bases for $\mathbb{N}_0$ are negative. For example, Dirac [2] showed that the representation function of an asymptotic basis of order 2 cannot be eventually constant. Erdős and Fuchs [4] proved that the average value of a representation function of order 2 cannot even be approximately constant, in the sense that, for every infinite set A of nonnegative integers and every real number $c > 0$,

$$\sum_{n \le N} r_{A,2}(n) \ne cN + o\left(N^{1/4} \log^{-1/2} N\right) .$$

Erdős and Turán [3] conjectured that if A is an asymptotic basis of order h for the nonnegative integers, then the representation function $r_{A,h}(n)$ must be unbounded, that is,

$$\limsup_{n \to \infty} r_{A,h}(n) = \infty .$$

This famous unsolved problem in additive number theory is only a special case of the general problem of classifying the representation functions of asymptotic bases of finite order for the nonnegative integers.

2 – Two lemmas

We use the following notation. For sets A and B of integers and for any integer t, we define the *sumset*

$$A + B = \{a + b\colon\ a \in A,\ b \in B\} ,$$

the *translation*

$$A + t = \{a + t\colon\ a \in A\} ,$$

and the *difference set*

$$A - B = \{a - b\colon\ a \in A,\ b \in B\} .$$

For every nonnegative integer h we define the *h-fold sumset* hA by induction:

$$0A = \{0\} ,$$

$$hA = A + (h-1)A = \{a_1 + a_2 + \cdots + a_h\colon\ a_1, a_2, ..., a_h \in A\} .$$

We denote the cardinality of a set S by card(S). The *counting function* for the set A is

$$A(y,x) \;=\; \mathrm{card}\Big(\{a \in A:\; y \le a \le x\}\Big)\;.$$

In particular, $A(-x,x)$ counts the number of integers $a \in A$ with $|a| \le x$. If A is a finite set of integers, we denote the maximum element of A by max(A).

Let $[x]$ denote the integer part of the real number x.

Lemma 1. *Let $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ be a function such that $f^{-1}(0)$ is finite. Let Δ denote the cardinality of the set $f^{-1}(0)$. Then there exists a sequence $U = \{u_k\}_{k=1}^{\infty}$ of integers such that, for every $n \in \mathbb{Z}$ and $k \in \mathbb{N}$,*

$$f(n) \;=\; \mathrm{card}\Big(\{k \ge 1:\; u_k = n\}\Big)$$

and

$$|u_k| \;\le\; \left[\frac{k+\Delta}{2}\right]\;.$$

Proof: Every positive integer m can be written uniquely in the form

$$m = s^2 + s + 1 + r\;,$$

where s is a nonnegative integer and $|r| \le s$. We construct the sequence

$$\begin{aligned} V \;&=\; \{0,-1,0,1,-2,-1,0,1,2,-3,-2,-1,0,1,2,3,...\} \\ &=\; \{v_m\}_{m=1}^{\infty}\;, \end{aligned}$$

where

$$v_{s^2+s+1+r} = r \qquad \text{for} \;\; |r| \le s\;.$$

For every nonnegative integer k, the first occurrence of $-k$ in this sequence is $v_{k^2+1} = -k$, and the first occurrence of k in this sequence is $v_{(k+1)^2} = k$.

The sequence U will be the unique subsequence of V constructed as follows. Let $n \in \mathbb{Z}$. If $f(n) = \infty$, then U will contain the terms $v_{s^2+s+1+n}$ for every $s \ge |n|$. If $f(n) = \ell < \infty$, then U will contain the ℓ terms $v_{s^2+s+1+n}$ for $s = |n|, |n|+1, ..., |n|+\ell-1$ in the subsequence U, but not the terms $v_{s^2+s+1+n}$ for $s \ge |n|+\ell$. Let $m_1 < m_2 < m_3 < \cdots$ be the strictly increasing sequence of positive integers such that $\{v_{m_k}\}_{k=1}^{\infty}$ is the resulting subsequence of V. Let $U = \{u_k\}_{k=1}^{\infty}$, where $u_k = v_{m_k}$. Then

$$f(n) \;=\; \mathrm{card}\Big(\{k \ge 1:\; u_k = n\}\Big)\;.$$

Let $\operatorname{card}\left(f^{-1}(0)\right) = \Delta$. The sequence U also has the following property: If $|u_k| = n$, then for every integer $m \notin f^{-1}(0)$ with $|m| < n$ there is a positive integer $j < k$ with $u_j = m$. It follows that

$$\{0, 1, -1, 2, -2, ..., n-1, -(n-1)\} \backslash f^{-1}(0) \subseteq \{u_1, u_2, ..., u_{k-1}\} ,$$

and so

$$k - 1 \geq 2(n-1) + 1 - \Delta .$$

This implies that

$$|u_k| = n \leq \frac{k + \Delta}{2} .$$

Since u_k is an integer, we have

$$|u_k| \leq \left[\frac{k+\Delta}{2}\right] .$$

This completes the proof. ∎

Lemma 1 is best possible in the sense that for every nonnegative integer Δ there is a function $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ with $\operatorname{card}\left(f^{-1}(0)\right) = \Delta$ and a sequence $U = \{u_k\}_{k=1}^{\infty}$ of integers such that

$$|u_k| = \left[\frac{k+\Delta}{2}\right] \qquad \text{for all } \; k \geq 1 . \tag{1}$$

For example, if $\Delta = 2\delta + 1$ is odd, define the function f by

$$f(n) = \begin{cases} 0 & \text{if } |n| \leq \delta \\ 1 & \text{if } |n| \geq \delta + 1 \end{cases}$$

and the sequence U by

$$\begin{aligned} u_{2i-1} &= \delta + i , \\ u_{2i} &= -(\delta + i) \end{aligned}$$

for all $i \geq 1$.

If $\Delta = 2\delta$ is even, define f by

$$f(n) = \begin{cases} 0 & \text{if } -\delta \leq n \leq \delta - 1 \\ 1 & \text{if } n \geq \delta \text{ or } n \leq -\delta - 1 \end{cases}$$

and the sequence U by $u_1 = \delta$ and

$$\begin{aligned} u_{2i} &= \delta + i , \\ u_{2i+1} &= -(\delta + i) \end{aligned}$$

for all $i \geq 1$. In both cases the sequence U satisfies (1).

The set A is called a *Sidon set of order* h if $r_{A,h}(n) = 0$ or 1 for every integer n. If A is a Sidon set of order h, then A is a Sidon set of order j for all $j = 1, 2, ..., h$.

Lemma 2. *Let A be a finite Sidon set of order h and $d = \max(\{|a| : a \in A\})$. If $|c| > (2h-1)d$, then $A \cup \{c\}$ is also a Sidon set of order h.*

Proof: Let $n \in h\,(A \cup \{c\})$. Suppose that

$$n \;=\; a_1 + \cdots + a_j + (h-j)\,c \;=\; a'_1 + \cdots + a'_\ell + (h-\ell)\,c \ ,$$

where

$$0 \le j \le \ell \le h \ ,$$

$$a_1, ..., a_j, a'_1, ..., a'_\ell \;\in\; A \ ,$$

and

$$a_1 \le \cdots \le a_j \qquad \text{and} \qquad a'_1 \le \cdots \le a'_\ell \ .$$

If $j < \ell$, then

$$\begin{aligned} |c| \;&\le\; |(\ell-j)c| \\ &=\; \left|a'_1 + \cdots + a'_\ell - (a_1 + \cdots + a_j)\right| \\ &\le\; (\ell+j)d \\ &\le\; (2h-1)d \\ &<\; |c| \ , \end{aligned}$$

which is absurd. Therefore, $j = \ell$ and $a_1 + \cdots + a_j = a'_1 + \cdots + a'_j$. Since A is a Sidon set of order j, it follows that $a_i = a'_i$ for all $i = 1, ..., j$. Consequently, $A \cup \{c\}$ is a Sidon set of order h. ∎

3 – Construction of asymptotic bases

We can now construct asymptotic bases of order h for the integers with arbitrary representation functions.

Theorem 1. *Let $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ be a function such that the set $f^{-1}(0)$ is finite. Let $\varphi : \mathbb{N}_0 \to \mathbb{R}$ be a nonnegative function such that $\lim_{x\to\infty} \varphi(x) = \infty$. For every $h \ge 2$ there exist infinitely many asymptotic bases A of order h for the integers such that*

$$r_{A,h}(n) = f(n) \quad \textit{for all} \;\; n \in \mathbb{Z} \ ,$$

and

$$A(-x,x) \;\le\; \varphi(x)$$

for all $x \ge 0$.

Proof: By Lemma 1, there is a sequence $U = \{u_k\}_{k=1}^{\infty}$ of integers such that

$$f(n) \;=\; \mathrm{card}\Big(\{k \ge 1\colon\; u_k = n\}\Big)$$

for every integer n.

Let $h \ge 2$. We shall construct an ascending sequence of finite sets $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ such that, for all positive integers k and for all integers n,

(i)

$$r_{A_k,h}(n) \le f(n) \;,$$

(ii)

$$r_{A_k,h}(n) \;\ge\; \mathrm{card}\Big(\{i\colon\; 1 \le i \le k \;\text{ and }\; u_i = n\}\Big) \;,$$

(iii)

$$\mathrm{card}(A_k) \le 2k \;,$$

(iv)

$$A_k \text{ is a Sidon set of order } h-1 \;.$$

Conditions (i) and (ii) imply that the infinite set

$$A \;=\; \bigcup_{k=1}^{\infty} A_k$$

is an asymptotic basis of order h for the integers such that $r_{A,h}(n) = f(n)$ for all $n \in \mathbb{Z}$.

We construct the sets A_k by induction. Since the set $f^{-1}(0)$ is finite, there exists a nonnegative integer d_0 such that $f(n) \ge 1$ for all integers n with $|n| \ge d_0$. If $u_1 \ge 0$, choose a positive integer $c_1 > 2hd_0$. If $u_1 < 0$, choose a negative integer $c_1 < -2hd_0$. Then

$$|c_1| > 2hd_0 \;.$$

Let

$$A_1 = \{-c_1,\, (h{-}1)c_1 + u_1\} \;.$$

The sumset hA_1 is the finite arithmetic progression

$$\begin{aligned} hA_1 &= \{-hc_1 + (hc_1+u_1)i : \ i = 0,1,...,h\} \\ &= \{-hc_1,\, u_1,\, hc_1+2u_1,\, 2hc_1+3u_1,\, ...,\, (h-1)hc_1+hu_1\}\ . \end{aligned}$$

Then $|n| \geq h|c_1| > d_0$ for all $n \in hA_1 \setminus \{u_1\}$. Since $f(u_1) \geq 1$, we have $r_{A_1,h}(n) = 1 \leq f(n)$ for all $n \in hA_1$. Similarly, since $r_{A_1,h}(n) = 0$ for all $n \notin hA_1$, it follows that

$$r_{A_1,h}(n) \leq f(n)$$

for all $n \in \mathbb{Z}$. The set A_1 is a Sidon set of order h, hence also a Sidon set of order $h-1$. Thus, the set A_1 satisfies conditions (i)–(iv).

We assume that for some integer $k \geq 2$ we have constructed a set A_{k-1} satisfying conditions (i)–(iv). If

$$r_{A_{k-1,h}}(n) \ \geq \ \mathrm{card}\Big(\{i : \ 1 \leq i \leq k \ \text{ and } \ u_i = n\}\Big)$$

for all $n \in \mathbb{Z}$, then the set $A_k = A_{k-1}$ satisfies conditions (i)–(iv). Otherwise,

$$r_{A_{k-1},h}(u_k) \ = \ \mathrm{card}\Big(\{i : \ 1 \leq i \leq k \ \text{ and } \ u_i = u_k\}\Big) - 1 \ < \ f(u_k)\ .$$

We shall construct a Sidon set A_k of order $h-1$ such that

$$\mathrm{card}(A_k) \ = \ \mathrm{card}(A_{k-1}) + 2$$

and

$$r_{A_k,h}(n) \ = \ \begin{cases} r_{A_{k-1},h}(n) + 1 & \text{if } \ n = u_k \\ r_{A_{k-1},h}(n) & \text{if } \ n \in hA_{k-1} \setminus \{u_k\} \\ 1 & \text{if } \ n \in hA_k \setminus (hA_{k-1} \cup \{u_k\})\ . \end{cases} \tag{2}$$

Define the nonnegative integer

$$d_{k-1} = \max\Big(\{|a| : \ a \in A_{k-1} \cup \{u_k\}\}\Big)\ . \tag{3}$$

Then

$$A_{k-1} \subseteq [-d_{k-1}, d_{k-1}]\ .$$

If $u_k \geq 0$, choose a positive integer c_k such that $c_k > 2hd_{k-1}$. If $u_k < 0$, choose a negative integer c_k such that $c_k < -2hd_{k-1}$. Then

$$|c_k| > 2hd_{k-1}\ . \tag{4}$$

Let

$$A_k \ = \ A_{k-1} \cup \{-c_k, (h-1)c_k + u_k\}\ .$$

Then

$$\mathrm{card}(A_k) \;=\; \mathrm{card}(A_{k-1}) + 2 \;\le\; 2k \;.$$

We shall assume that $u_k \ge 0$, hence $c_k > 0$. (The argument in the case $u_k < 0$ is similar.) We decompose the sumset hA_k as follows:

$$hA_k \;=\; \bigcup_{\substack{r+i+j=h\\ r,i,j\ge 0}} \Bigl(r(h-1)c_k + ru_k - ic_k + jA_{k-1}\Bigr) \;=\; \bigcup_{r=0}^{h} B_r \;,$$

where

$$B_r \;=\; r(h-1)c_k + ru_k + \bigcup_{i=0}^{h-r} \Bigl(-ic_k + (h-r-i)A_{k-1}\Bigr) \;.$$

If $n \in B_r$, then there exist integers $i \in \{0,1,...,h-r\}$ and $y \in (h-r-i)A_{k-1}$ such that

$$n \;=\; r(h-1)c_k + ru_k - ic_k + y \;.$$

Since

$$|y| \le (h-r-i)d_{k-1} \;,$$

it follows that

$$n \;\ge\; r(h-1)c_k + ru_k - ic_k - (h-r-i)d_{k-1} \tag{5}$$

and

$$n \;\le\; r(h-1)c_k + ru_k - ic_k + (h-r-i)d_{k-1} \;.$$

Let $m \in B_{r-1}$ and $n \in B_r$ for some $r \in \{1,...,h\}$. There exist nonnegative integers $i \le h-r$ and $j \le h-r+1$ such that

$$\begin{aligned} n-m \;&\ge\; \Bigl(r(h-1)c_k + ru_k - ic_k - (h-r-i)d_{k-1}\Bigr)\\ &\quad - \Bigl((r-1)(h-1)c_k + (r-1)u_k - jc_k + (h-r+1-j)d_{k-1}\Bigr)\\ &=\; (h-1+j-i)c_k + u_k - (2h-2r-i-j+1)d_{k-1}\\ &\ge\; (h-1-i)c_k - 2hd_{k-1} \;. \end{aligned}$$

If $r \ge 2$, then $i \le h-2$ and inequality (4) implies that

$$n - m \ge c_k - 2hd_{k-1} > 0 \;.$$

Therefore, if $m \in B_{r-1}$ and $n \in B_r$ for some $r \in \{2,...,h\}$, then $m < n$.

In the case $r = 1$ we have $m \in B_0$ and $n \in B_1$. If $i \le h-2$, then (4) implies that

$$n - m \;\ge\; (h-1-i)c_k - 2hd_{k-1} \;\ge\; c_k - 2hd_{k-1} \;>\; 0$$

and (5) implies that

$$n \geq (h-1-i)c_k + u_k - (h-1-i)d_{k-1} > c_k - hd_{k-1} > d_0 .$$

If $r = 1$ and $i = h-1$, then $n = u_k$. Therefore, if $m \in B_0$ and $n \in B_1$, then $m < n$ unless $m = n = u_k$. It follows that the sets $B_0, B_1 \backslash \{u_k\}, B_2, ..., B_h$ are pairwise disjoint.

Let $n \in B_r$ for some $r \geq 1$. Suppose that $0 \leq i \leq j \leq h-r$, and that

$$n = r(h-1)c_k + ru_k - ic_k + y \quad \text{for some } \ y \in (h-r-i)A_{k-1}$$

and

$$n = r(h-1)c_k + ru_k - jc_k + z \quad \text{for some } \ z \in (h-r-j)A_{k-1} .$$

Subtracting these equations, we obtain

$$z - y = (j-i)c_k .$$

Recall that $|a| \leq d_{k-1}$ for all $a \in A_{k-1}$. If $i < j$, then

$$\begin{aligned} c_k &\leq (j-i)c_k = z - y \\ &\leq |y| + |z| \leq (2h - 2r - i - j)d_{k-1} \\ &< 2hd_{k-1} < c_k , \end{aligned}$$

which is impossible. Therefore, $i = j$ and $y = z$. Since $0 \leq h - r - i \leq h-1$ and A_{k-1} is a Sidon set of order $h-1$, it follows that

$$r_{A_{k-1},h-r-i}(y) = 1$$

and so

$$r_{A_k,h}(n) = 1 \leq f(n) \quad \text{for all } \ n \in (B_1 \backslash \{u_k\}) \cup \bigcup_{r=2}^{h} B_r .$$

Next we consider the set

$$B_0 = hA_{k-1} \cup \bigcup_{i=1}^{h} \Big(-ic_k + (h-i)A_{k-1} \Big) .$$

For $i = 1, ..., h$, we have

$$c_k > 2hd_{k-1} \geq (2h - 2i + 1)d_{k-1}$$

and so

$$\begin{aligned}\max\Big(-ic_k+(h-i)A_{k-1}\Big) &\le -ic_k+(h-i)d_{k-1}\\ &< -(i-1)c_k-(h-i+1)d_{k-1}\\ &\le \min\Big(-(i-1)c_k+(h-i+1)A_{k-1}\Big)\ .\end{aligned}$$

Therefore, the sets $-ic_k+(h-i)A_{k-1}$ are pairwise disjoint for $i=0,1,...,h$. In particular, if $n\in B_0\backslash hA_{k-1}$, then

$$n\le\max\Big(-c_k+(h-1)A_{k-1}\Big)\ \le\ -c_k+(h-1)d_{k-1}\ <\ -d_{k-1}\ \le\ -d_0$$

and $f(n)\ge 1$. Since A_{k-1} is a Sidon set of order $h-1$, it follows that

$$r_{A_k,h}(n)=1\le f(n)$$

for all

$$n\in\bigcup_{i=1}^{h}\Big(-ic_k+(h-i)A_{k-1}\Big)\ =\ B_0\backslash hA_{k-1}\ .$$

It follows from (3) that for any $n\in B_0\backslash hA_{k-1}$ we have

$$n<-d_{k-1}\le u_k\ ,$$

and so $u_k\notin B_0\backslash hA_{k-1}$. Therefore,

$$r_{A_k,h}(u_k)=r_{A_{k-1},h}(u_k)+1\ ,$$

and the representation function $r_{A_k,h}$ satisfies the three requirements of (2).

We shall prove that

$$A_k\ =\ A_{k-1}\cup\{-c_k,(h-1)c_k+u_k\}\ .$$

is a Sidon set of order $h-1$. Since A_{k-1} is a Sidon set of order $h-1$ with $d_{k-1}\ge\max\{|a|:a\in A_{k-1}\}$, and since

$$c_k>2hd_{k-1}>\Big(2(h-1)-1\Big)d_{k-1}\ ,$$

Lemma 2 implies that $A_{k-1}\cup\{-c_k\}$ is a Sidon set of order $h-1$.

Let $n\in(h-1)A_k$. Suppose that

$$\begin{aligned}n\ &=\ r(h-1)c_k+ru_k-ic_k+x\\ &=\ s(h-1)c_k+su_k-jc_k+y\ ,\end{aligned}$$

where

$$0 \leq r \leq s \leq h-1 \ ,$$

$$0 \leq i \leq h-1-r \ ,$$

$$0 \leq j \leq h-1-s \ ,$$

$$x \ \in \ (h-1-r-i)A_{k-1} \ ,$$

and

$$y \ \in \ (h-1-s-j)A_{k-1} \ .$$

Then

$$|x| \ \leq \ (h-1-r-i)d_{k-1}$$

and

$$|y| \ \leq \ (h-1-s-j)d_{k-1} \ .$$

If $r < s$, then $j \leq h-2$ and

$$\begin{aligned} (h-1)c_k \ &\leq \ (s-r)(h-1)c_k + (s-r)u_k \\ &= \ (j-i)c_k + x - y \\ &\leq \ (j-i)c_k + (2h-2-r-s-i-j)d_{k-1} \\ &\leq \ (h-2)c_k + 2hd_{k-1} \\ &< \ (h-1)c_k \ , \end{aligned}$$

which is absurd. Therefore, $r = s$ and

$$-ic_k + x = -jc_k + y \ \in \ (h-1-r)\left(A_k \cup \{-c_k\}\right) \ .$$

Since $A_k \cup \{-c_k\}$ is a Sidon set of order $h-1$, it follows that $i = j$ and that x has a unique representation as the sum of $h-1-r-i$ elements of A_k. Thus, A_k is a Sidon set of order $h-1$.

The set A_k satisfies conditions (i)–(iv). It follows by induction that there exists an infinite increasing sequence $A_1 \subseteq A_2 \subseteq \cdots$ of finite sets with these properties, and that $A = \cup_{k=1}^{\infty} A_k$ is an asymptotic basis of order h with representation function $r_{A,h}(n) = f(n)$ for all $n \in \mathbb{Z}$.

Finally, we shall prove that, for every nonnegative function $\varphi(x)$ with $\lim_{x\to\infty} \varphi(x) = \infty$, there exist infinitely many asymptotic bases A of order h such that $r_{A,h}(n) = f(n)$ for all $n \in \mathbb{Z}$ and $A(-x,x) \leq \varphi(x)$ for all $x \in N_0$. Let $A_0 = \emptyset$, and let K' be the set of all positive integers k such that $A_k \neq A_{k-1}$. Then $1 \in K'$ and

$$A \ = \ \bigcup_{k\in K'} A_k \ = \ \bigcup_{k\in K'} \{-c_k, (h{-}1)c_k\} \ .$$

For each $k \in K'$, the only constraints on the choice of the number c_k in the construction of the set A_k were the sign of c_k and the growth condition (4)

$$|c_k| > 2hd_{k-1} \ .$$

Since $\varphi(x) \to \infty$ as $x \to \infty$, for every integer $k \geq 0$ there exists an integer w_k such that

$$\varphi(x) \geq 2k \quad \text{for all} \ \ x \geq w_k \ .$$

We now impose the following additional constraint: Choose c_k such that

$$|c_k| \geq w_k \quad \text{for all integers} \ \ k \in K' \ .$$

Then

$$A_1(-x, x) = 0 \leq \varphi(x) \quad \text{for} \ \ 0 \leq x < |c_1|$$

and

$$A_1(-x, x) \leq 2 \leq \varphi(x) \quad \text{for} \ \ x \geq |c_1| \geq w_1 \ .$$

Suppose that $k \geq 2$ and the set A_{k-1} satisfies $A_{k-1}(-x, x) \leq \varphi(x)$ for all $x \geq 0$. If $k \notin K'$, then $A_k = A_{k-1}$ and $A_k(-x, x) \leq \varphi(x)$ for all $x \geq 0$. If $k \in K$, then

$$A_k \cap (-|c_k|, |c_k|) \ = \ A_{k-1} \cap (-|c_k|, |c_k|) \ = \ A_{k-1} \ ,$$

and so

$$A_k(-x, x) = A_{k-1}(-x, x) \leq \varphi(x) \quad \text{for} \ \ 0 \leq x < |c_k|$$

and

$$A_k(-x, x) \leq 2k \leq \varphi(x) \quad \text{for} \ \ x \geq |c_k| \geq w_k \ .$$

It follows by induction that the finite sets A_k satisfy $A_k(-x, x) \leq \varphi(x)$ for all k and x. The infinite set $A = \cup_{k \in K'} A_k$ is an asymptotic basis with $r_{A,h}(n) = f(n)$ for all $n \in \mathbb{Z}$. Since $\lim_{k \to \infty} |c_k| = \infty$, for every nonnegative integer x we can choose $k \in K'$ such that $|c_k| > x$. It follows that

$$A(-x, x) = A_k(-x, x) \leq \varphi(x) \ .$$

For every integer $k \in K'$ we had infinitely many choices for the integer c_k to use in the construction of the set A_k, and so there are infinitely many asymptotic bases A with the property that $r_A(n) = f(n)$ for all $n \in \mathbb{Z}$ and $A(-x, x) \leq \varphi(x)$ for all $x \in \mathbb{N}_0$. This completes the proof. ∎

4 – Sums of pairwise distinct integers

Let A be a set of integers and h a positive integer. We define the sumset $h \wedge A$ as the set consisting of all sums of h pairwise distinct elements of A, and the *restricted representation function*

$$\hat{r}_{A,h} \colon \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$$

by

$$\hat{r}_{A,h}(n) = \text{card}\Big(\Big\{\{a_1, ..., a_h\} \subseteq A \colon \ a_1 + \cdots + a_h = n \ \text{ and } \ a_1 < \cdots < a_h\Big\}\Big) \ .$$

The set A of integers is called a *restricted asymptotic basis of order* h if $h \wedge A$ contains all but finitely many integers, or, equivalently, if $\hat{r}_{A,h}^{-1}(0)$ is a finite subset of Z.

We can obtain the following result by the same method used to prove Theorem 1.

Theorem 2. *Let* $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ *be a function such that* $f^{-1}(0)$ *is a finite set of integers. Let* $\varphi : \mathbb{N}_0 \to \mathbb{R}$ *be a nonnegative function such that* $\lim_{x\to\infty} \varphi(x) = \infty$*. For every* $h \geq 2$ *there exist infinitely many sets* A *of integers such that*

$$\hat{r}_{A,h}(n) = f(n) \quad \textit{for all} \ \ n \in \mathbb{Z}$$

and

$$A(-x, x) \leq \varphi(x)$$

for all $x \geq 0$. ∎

5 – Open problems

Let X be an abelian semigroup, written additively, and let A be a subset of X. We define the h-fold sumset hA as the set consisting of all sums of h not necessarily distinct elements of A. The set A is called an *asymptotic basis of order* h *for* X if the sumset hA consists of all but at most finitely many elements of X. We also define the h-fold *restricted sumset* $h \wedge A$ as the set consisting of all sums of h pairwise distinct elements of A. The set A is called a *restricted asymptotic basis of order* h *for* X if the restricted sumset $h \wedge A$ consists of all but at most finitely many elements of X. The classical problems of additive number theory concern the semigroups $\mathbb{N}_0$ and $\mathbb{Z}$.

There are four different representation functions that we can associate to every subset A of X and every positive integer h. Let $(a_1, ..., a_h)$ and $(a'_1, ..., a'_h)$ be h-tuples of elements of X. We call these h-tuples *equivalent* if there is a permutation σ of the set $\{1, ..., h\}$ such that $a'_{\sigma(i)} = a_i$ for all $i = 1, ..., h$. For every $x \in X$, let $r_{A,h}(x)$ denote the number of equivalence classes of h-tuples $(a_1, ..., a_h)$ of elements of A such that $a_1 + \cdots + a_h = x$. The function $r_{A,h}$ is called the *unordered representation function* of A. This is the function that we studied in this paper. The set A is an asymptotic basis of order h if $r_{A,h}^{-1}(0)$ is a finite subset of X.

Let $R_{A,h}(x)$ denote the number of h-tuples $(a_1, ..., a_h)$ of elements of A such that $a_1 + \cdots + a_h = x$. The function $R_{A,h}$ is called the *ordered representation function* of A.

Let $\hat{r}_{A,h}(x)$ denote the number of equivalence classes of h-tuples $(a_1, ..., a_h)$ of pairwise distinct elements of A such that $a_1 + \cdots + a_h = x$, and let $\hat{R}_{A,h}(x)$ denote the number of h-tuples $(a_1, ..., a_h)$ of pairwise distinct elements of A such that $a_1 + \cdots + a_h = x$. These functions are called the *unordered restricted representation function* of A and the *ordered restricted representation function* of A, respectively. The two restricted representation functions are essentially identical, since $\hat{R}_{A,h}(x) = h!\,\hat{r}_{A,h}(x)$ for all $x \in X$.

In the discussion below, we use only the unordered representation function $r_{A,h}$, but each of the problems can be reformulated in terms of the other representation functions.

For every countable abelian semigroup X, let $\mathcal{F}(X)$ denote the set of all functions $f : X \to \mathbb{N}_0 \cup \{\infty\}$, and let $\mathcal{F}_0(X)$ denote the set of all functions $f : X \to \mathbb{N}_0 \cup \{\infty\}$ such that $f^{-1}(0)$ is a finite subset of X. Let $\mathcal{F}_c(X)$ denote the set of all functions $f : X \to \mathbb{N}_0 \cup \{\infty\}$ such that $f^{-1}(0)$ is a cofinite subset of X, that is, $f(x) \neq 0$ for only finitely many $x \in X$, or, equivalently,

$$\operatorname{card}\left(f^{-1}(\mathbb{N} \cup \{\infty\})\right) < \infty \ .$$

Let $\mathcal{R}(X, h)$ denote the set of all h-fold representation functions of subsets A of X. If $r_{A,h}$ is the representation function of an asymptotic basis A of order h for X, then $r_{A,h}^{-1}(0)$ is a finite subset of X, and so $r_{A,h} \in \mathcal{F}_0(X)$. Let $\mathcal{R}_0(X, h)$ denote the set of all h-fold representation functions of asymptotic bases A of order h for X. Let $\mathcal{R}_c(X, h)$ denote the set of all h-fold representation functions of finite subsets of X. We have

$$\mathcal{R}(X, h) \subseteq \mathcal{F}(X) \ ,$$

$$\mathcal{R}_0(X,h) \subseteq \mathcal{F}_0(X) \ ,$$

and

$$\mathcal{R}_c(X,h) \subseteq \mathcal{F}_c(X) \ .$$

In the case $h = 1$, we have, for every set $A \subseteq X$,

$$r_{A,1}(x) = \begin{cases} 1 & \text{if } x \in A \,, \\ 0 & \text{if } x \notin A \ , \end{cases}$$

and so

$$\mathcal{R}(X,1) = \Big\{ f\colon\ X \to \{0,1\} \Big\} \ ,$$

$$\mathcal{R}_0(X,1) \ = \ \Big\{ f\colon\ X \to \{0,1\}\colon\ \mathrm{card}(f^{-1}(0)) < \infty \Big\} \ ,$$

and

$$\mathcal{R}_c(X,1) \ = \ \Big\{ f\colon\ X \to \{0,1\}\colon\ \mathrm{card}\,\Big(f^{-1}(\mathbb{N} \cup \{\infty\}) \Big) < \infty \Big\} \ .$$

In this paper we proved that

$$\mathcal{R}_0(\mathbb{Z},h) = \mathcal{F}_0(\mathbb{Z}) \qquad \text{for all} \quad h \geq 2 \ .$$

Nathanson [8] has extended this result to certain countably infinite groups and semigroups. Let G be any countably infinite abelian group such that $\{2g : g \in G\}$ is infinite. For the unordered restricted representation function $\hat{r}_{A,2}$, we have

$$\mathcal{R}_0(G,2) = \mathcal{F}_0(G) \ .$$

More generally, let S is any countable abelian semigroup such that for every $s \in S$ there exist $s', s'' \in S$ with $s = s' + s''$. In the abelian semigroup $X = S \oplus G$, we have

$$\mathcal{R}_0(X,2) = \mathcal{F}_0(X) \ .$$

If $\{12g : g \in G\}$ is infinite, then $\mathcal{R}_0(X,2) = \mathcal{F}_0(X)$ for the unordered representation function $r_{A,2}$.

The following problems are open for all $h \geq 2$:

1. Determine $\mathcal{R}_0(\mathbb{N}_0,h)$. Equivalently, describe the representation functions of additive bases for the nonnegative integers. This is a major unsolved problem in additive number theory, of which the Erdős–Turán conjecture is only a special case.

2. Determine $\mathcal{R}(\mathbb{Z}, h)$. In this paper we computed $\mathcal{R}_0(\mathbb{Z}, h)$, the set of representation functions of additive bases for the integers, but it is not known under what conditions a function $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ with $f^{-1}(0)$ infinite is the representation function of a subset A of X. It can be proved that if $f^{-1}(0)$ is infinite but sufficiently sparse, then $f \in \mathcal{R}(\mathbb{Z}, h)$.

3. Determine $\mathcal{R}(\mathbb{N}_0, h)$. Is there a simple list of necessary and sufficient conditions for a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ to be the representation function of some set of nonnegative integers?

4. Determine $\mathcal{R}_c(\mathbb{Z}, h)$. Equivalently, describe the representation functions of finite sets of integers, and identify the functions $f \in \mathcal{F}_c(\mathbb{Z})$ such that $f(n) = r_{A,h}(n)$ for some finite set A of integers. If A is a set of integers and t is an integer, then for the translated set $t + A$ we have

$$r_{t+A,h}(n) = r_{A,h}(n - ht)$$

for all integers n. This implies that if $f(n) \in \mathcal{R}_c(\mathbb{Z}, h)$, then $f(n - ht) \in \mathcal{R}_c(\mathbb{Z}, h)$ for every integer t, so it suffices to consider only finite sets A of nonnegative integers with $0 \in A$. Similarly, if $\gcd(A) = d$, then $r_{A,h}(n) > 0$ only if d divides n. Setting $B = \{a/d : d \in A\}$, we have $r_{h,A}(n) = r_{B,h}(n/d)$. It follows that we need to consider only finite sets A of relatively prime nonnegative integers with $0 \in A$.

5. Determine $\mathcal{R}_0(G, 2)$, $\mathcal{R}(G, 2)$, and $\mathcal{R}_c(G, 2)$ for the infinite abelian group $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$. Note that $\{2g : g \in G\} = \{0\}$ for this group.

6. Determine $\mathcal{R}_0(G, h)$ and $\mathcal{R}(G, h)$, where G is an arbitrary countably infinite abelian group and $h \geq 2$.

7. There is a class of problems of the following type. Do there exist integers h and k with $2 \leq h < k$ such that

$$\mathcal{R}(\mathbb{Z}, h) \neq \mathcal{R}(\mathbb{Z}, k) \ ?$$

We can easily find sets of integers to show that that $\mathcal{R}_0(\mathbb{N}_0, h) \neq \mathcal{R}_0(\mathbb{N}_0, k)$. For example, let $A = \mathbb{N}$ be the set of all positive integers, and let $h \geq 1$. Then $r_{\mathbb{N},h}(0) = 0$ and $r_{\mathbb{N},h}(h) = 1$. If B is any set of nonnegative integers and $k > h$, then $r_{B,k}(h) = 0$, and so $r_{\mathbb{N},h} \notin \mathcal{R}_0(\mathbb{N}_0, k)$. Is it true that

$$\mathcal{R}_0(\mathbb{N}_0, h) \cap \mathcal{R}_0(\mathbb{N}_0, k) = \emptyset$$

for all $h \neq k$?

8. By Theorem 1, for every $h \geq 2$ and every function $f \in \mathcal{F}_0(\mathbb{Z})$, there exist arbitrarily sparse sets A of integers such that $r_{A,h}(n) = f(n)$ for all n. It is an open problem to determine how dense the sets A can be. For example, in the special case $h = 2$ and $f(n) = 1$, Nathanson [7] proved that there exists a set A such that $r_{A,2}(n) = 1$ for all n, and $\log x \ll A(-x,x) \ll \log x$. For an arbitrary representation function $f \in \mathcal{F}_0(\mathbb{Z})$, Nathanson [6] constructed an asymptotic basis of order h with $A(-x,x) \gg x^{1/(2h-1)}$. In the case $h = 2$, Cilleruelo and Nathanson [1] improved this to $A(-x,x) \gg x^{\sqrt{2}-1+o(1)}$.

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