PORTUGALIAE MATHEMATICA
Vol. 61 Fasc. 4 – 2004
Nova Série

SEMILINEAR PROBLEMS WITH A NON-SYMMETRIC LINEAR PART HAVING AN INFINITE DIMENSIONAL KERNEL

J. Berkovits and C. Fabry

Recommended by L. Sanchez

Abstract: We consider semilinear equations, where the linear part L is non-symmetric and has a possibly infinite dimensional kernel. We shall show that, under certain monotonicity conditions for the nonlinearity, a generalized Leray–Schauder degree can be defined for these problems. In order to build the degree theory, we introduce, for the nonlinearity N, monotonicity properties with respect to a linear map T, e.g. T-pseudomonotonicity or maps of class $(S_+)_T$. As applications, we obtain new existence results for semilinear equations, in particular in resonance situations. In this latter case, we modify the standard inequalities of Landesman–Lazer type by replacing the identity map I by a linear homeomorphism $\mathcal{J}$, which will then appear in the monotonicity conditions.

1 – Introduction

We consider equations of the form

$$Lu = N(u) + h\,, \qquad u \in D(L)\ , \tag{1}$$

where $L\colon D(L) \subset H \to H$ is a densely defined unbounded closed linear operator on a real separable Hilbert space H, $N\colon H \to H$ is nonlinear and $h \in H$.

Received: October 14, 2003; *Revised*: December 30, 2003.
AMS Subject Classification: 47J25, 47H11.
Keywords: non-symmetric linear operators; topological degree; resonance; Landesman–Lazer conditions.

We assume that the range $\operatorname{Im} L$ is closed, implying that the partial inverse of L, denoted by $K\colon \operatorname{Im} L \to \operatorname{Im} L^* \cap D(L)$, is bounded. Our main objective is to consider problem (1) under the assumptions that K is compact, that

$$\operatorname{Ker} L \neq \operatorname{Ker} L^* \quad \text{and} \quad \dim \operatorname{Ker} L = \dim \operatorname{Ker} L^* \leq \infty \ .$$

It is well-known that the solvability of equation (1), as well as the methods available, depend crucially on the kernels of L and L^*. Indeed, if L is selfadjoint or normal, then $\operatorname{Ker} L = \operatorname{Ker} L^*$, and (1) can be equivalently written as

$$\tilde{P}\Big(u - K\tilde{P}N(u)\Big) + PN(u) = \hat{h}\,, \qquad u \in H \ ,$$

where $P\colon H \to \operatorname{Ker} L$ is the orthogonal projection, $\tilde{P} = I - P$ and $\hat{h} = K\tilde{P}h - Ph$. If $\dim \operatorname{Ker} L < \infty$ and K is compact, then the classical Leray–Schauder degree or the coincidence degree can be applied. A more challenging situation is encountered when $\dim \operatorname{Ker} L = \infty$. If K is compact and N is of class (S_+) or of more general class $(S_+)_P$, then the degree theory constructed in [2] and [1] can be used.

The application of the topological degree to equation (1) is based on the use of homotopies and on suitable a priori bounds. Essential for obtaining a priori bounds is an inequality of the form

$$\|Lu\|^2 \geq \rho \, \langle Lu, u\rangle \quad \text{for all} \ \ u \in D(L) \ , \tag{2}$$

where $\rho \in \mathbb{R}$. It is easy to see that inequality (2) is satisfied for some constant $\rho \neq 0$, when L is self-adjoint or normal. However, when $\operatorname{Ker} L \not\subset \operatorname{Ker} L^*$, the inequality (2) fails for any $\rho \neq 0$ (see Lemma 4.1). To circumvent this difficulty, we shall replace, as in [8], the identity map I by some linear homeomorphism $\mathcal{J}\colon H \to H$ and consider, instead of (2), the inequality

$$\|Lu\|^2 \geq \rho \, \langle Lu, \mathcal{J}u\rangle \quad \text{for all} \ \ u \in D(L) \ . \tag{3}$$

Inequalities of the type (3) and its implications are studied in [8] for the case $\dim \operatorname{Ker} L < \infty$, $\dim \operatorname{Ker} L^* < \infty$, and in a more general setting in [7]. Of course, if (3) is used, the hypotheses about the nonlinearity have to be modified accordingly. In relation with (3), it is useful to study, for $\mathcal{J}$ given, the set

$$\mathcal{A}_{\mathcal{J}} \,=\, \Big\{\rho \in \mathbb{R} \mid \ \|Lu\|^2 \geq \rho \, \langle Lu, \mathcal{J}u\rangle \ \ \text{for all} \ \ u \in D(L)\Big\} \ .$$

As recalled in Lemma 4.1, if $\mathcal{J}(\operatorname{Ker} L) \subset \operatorname{Ker} L^*$, the set $\mathcal{A}_{\mathcal{J}}$ is a closed interval containing the origin in its interior.

The paper is organized as follows. In Section 2, we give the definitions of some generalized classes of mappings of monotone type, e.g., the class $(S_+)_T$ and T-pseudomonotone maps. We present briefly the basic properties of the generalized Leray–Schauder degree needed in this paper. In Section 3, we show how the problem (1) can be reformulated using a linear homeomorphism $\mathcal{J} \colon H \to H$, for which it turns advantageous to have $\mathcal{J}(\operatorname{Ker} L) = \operatorname{Ker} L^*$. Section 4 is devoted to the study of the set $\mathcal{A}_{\mathcal{J}}$, whereas Section 5 proposes a particular construction of a homeomorphism $\mathcal{J}$ having the desired properties in the case $\operatorname{Ker} L \neq \operatorname{Ker} L^*$, $\dim \operatorname{Ker} L = \dim \operatorname{Ker} L^* < \infty$. In Section 6, we prove existence results which generalize those obtained earlier in the case $\dim \operatorname{Ker} L < \infty$, (see [6], [8], [9]), or in the case $\dim \operatorname{Ker} L = \infty$, $\mathcal{J} = I$, (see [4], [5], [6], for instance). In Section 7, we particularize the existence results to the case of two-component systems, with a diagonal linear part. We close this paper by results concerning semi-abstract equations, giving some indication on the kind of problem to which the abstract results of Sections 6 and 7 can be applied.

2 – Prerequisites

Throughout this paper, H will denote a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$. We recall some basic definitions. A mapping $F \colon H \to H$ is

- *bounded*, if it takes any bounded set into a bounded set;
- *demicontinuous*, if $u_j \to u$ (norm convergence) implies $F(u_j) \rightharpoonup F(u)$ (weak convergence);
- *compact*, if it is continuous and the image of any bounded set is relatively compact;
- of *Leray–Schauder type*, if it is of the form $I - C$, where C is compact.

Let $T \colon H \to H$ be a bounded linear operator. Then a mapping $F \colon H \to H$ is said

- *T-monotone*, if $\langle F(u) - F(v),\, T(u-v) \rangle \geq 0$ for all $u, v \in H$;
- *of class $(S_+)_T$*, if for any sequence (u_j), $u_j = v_j + z_j$, $v_j \in \operatorname{Ker} T$, $z_j \in (\operatorname{Ker} T)^\perp$ with $u_j \rightharpoonup u$ and $v_j \to v$ such that $\limsup \langle F(u_j), T(u_j - u) \rangle \leq 0$, it follows that $u_j \to u$;

– *T-pseudomonotone* ($F \in (PM)_T$), if for any sequence (u_j), $u_j = v_j + z_j$, $v_j \in \operatorname{Ker} T$, $z_j \in (\operatorname{Ker} T)^\perp$ with $u_j \rightharpoonup u$ and $v_j \to v$ such that $\limsup \langle F(u_j), T(u_j - u)\rangle \le 0$, it follows that $F(u_j) \rightharpoonup F(u)$ and $\langle F(u_j), T(u_j - u)\rangle \to 0$;
– *T-quasimonotone* ($F \in (QM)_T$), if for any sequence (u_j), $u_j = v_j + z_j$, $v_j \in \operatorname{Ker} T$, $z_j \in (\operatorname{Ker} T)^\perp$ with $u_j \rightharpoonup u$ and $v_j \to v$, we have $\limsup \langle F(u_j), T(u_j - u)\rangle \ge 0$.

With $T = I$, we get the standard definitions for monotonicity and the classes (S_+), (PM), (QM) widely used in the literature (we denote here $(S_+)_I = (S_+)$, etc...). Assuming that all mappings are bounded and demicontinuous, it is easy to prove that $(S_+)_T \subset (PM)_T \subset (QM)_T$ and $(S_+)_T + (QM)_T = (S_+)_T$, i.e., the class $(S_+)_T$ is stable under T-quasimonotone perturbations. Notice also that, when $\dim(\operatorname{Ker} T)^\perp < \infty$, all mappings are of class $(S_+)_T$.

In this note, we shall deal with the following cases:

(**a**) Let $T = \mathcal{J}$, a given linear homeomorphism. Then $\operatorname{Ker} \mathcal{J} = \{0\}$ and we obviously have $N \in (S_+)_{\mathcal{J}}$ if and only if $\mathcal{J}^* N \in (S_+)$. Similar observation holds for $\mathcal{J}$-pseudomonotone and $\mathcal{J}$-quasimonotone mappings.

(**b**) Let $T = P$, an orthogonal projection and denote $\tilde{P} = I - P$. A detailed study of the classes $(S_+)_P$, $(PM)_P$ and $(QM)_P$ can be found in [1], where also a topological degree theory is constructed for mappings of the type

$$F = \tilde{P}(I - C) + PN : \ \overline{G} \to H \ , \tag{4}$$

where G is an open bounded set in H, C is compact and N is a bounded demicontinuous map of class $(S_+)_P$. Any mapping of the type (4) is called *admissible for degree*. Since each Leray–Schauder type map is of class (S_+) and $(S_+) \subset (S_+)_P$, we can write any Leray–Schauder type map in the form (4), i.e., $I - C = \tilde{P}(I - C) + P(I - C)$. Hence, the degree theory constructed in [1] is an extension of the classical Leray–Schauder degree in Hilbert space. It is unique, single-valued and has the usual properties of degree, such as additivity of domains and invariance under homotopies. Let us denote the corresponding degree function by d_H.

(**c**) Let $T = \mathcal{J}P$, where $\mathcal{J}$ a linear homeomorphism and P an orthogonal projection. Then $\operatorname{Ker} T = \operatorname{Ker} P$ and it is easy to see that $N \in (S_+)_{\mathcal{J}P}$ if and only if $\mathcal{J}^* N \in (S_+)_P$. A similar observation holds for $\mathcal{J}P$-pseudomonotone and $\mathcal{J}P$-quasimonotone mappings. Note that a mapping F is admissible for degree if it is of the form

$$F = \tilde{P}(I - C) + P\mathcal{J}^* N : \ \overline{G} \to H \ ,$$

where G is an open bounded set in H, C is compact and N is a bounded demicontinuous map of class $(S_+)_{\mathcal{J}P}$. This observation will be used in Section 3, where a definition of the degree is presented for maps of the above type. The degree can then be used to obtain existence results for (1) when N is of class $(S_+)_{\mathcal{J}P}$. Moreover, using a standard perturbation procedure, the treatment of equation (1) can be extended to situations where N satisfies weaker conditions, namely $N \in (PM)_{\mathcal{J}P}$ or even $N \in (QM)_{\mathcal{J}P}$. These existence results appear in Section 6.

The following observation may be useful. With $\mathcal{J}$ and P as above, we notice that $(S_+)_{\mathcal{J}} \subset (S_+)_{\mathcal{J}P}$ and by the results given in [1], $(S_+)_{\mathcal{J}} = (S_+)_{\mathcal{J}P}$ if and only if $\dim \operatorname{Ker} P < \infty$.

3 – Reformulation of the equation

Let $L \colon D(L) \subset H \to H$ be a densely defined closed linear operator with closed range $\operatorname{Im} L$. Then the adjoint $L^* \colon D(L^*) \subset H \to H$ of L inherits these properties, i.e., also L^* is a densely defined closed linear operator having closed range. Since

$$\operatorname{Im} L^* = (\operatorname{Ker} L)^\perp \quad \text{and} \quad \operatorname{Im} L = (\operatorname{Ker} L^*)^\perp \ ,$$

the space H has the orthogonal direct sum decompositions

$$H = \operatorname{Ker} L \oplus \operatorname{Im} L^* = \operatorname{Ker} L^* \oplus \operatorname{Im} L \ .$$

Denote the corresponding orthogonal projections by $P \colon H \to \operatorname{Ker} L$, $\tilde{P} = I - P \colon H \to \operatorname{Im} L^*$, $Q \colon H \to \operatorname{Ker} L^*$ and $\tilde{Q} = I - Q \colon H \to \operatorname{Im} L$. Let L_0 stand for the restriction of L to $\operatorname{Im} L^* \cap D(L)$. Hence L_0 is injective and by the assumptions, its inverse $K = L_0^{-1} \colon \operatorname{Im} L \to \operatorname{Im} L^* \cap D(L)$ is bounded. Let $N \colon H \to H$ be a given mapping and $h \in H$. Let $\mathcal{J} \colon \mathcal{H} \to \mathcal{H}$ be a linear homeomorphism. We shall frequently apply the following lemma for $\mathcal{J}$ and $\mathcal{J}^{-1}$.

Lemma 3.1. *Let $T \colon H \to H$ be a linear homeomorphism and E, M closed linear subspaces of H. Then $E \subset T(M)$ if and only if $M^\perp \subset T^*(E^\perp)$.* ∎

Lemma 3.2.

(**a**) *With L, P, Q as above, let $\mathcal{J}: \mathcal{H} \to \mathcal{H}$ be a linear homeomorphism such that*

$$\mathcal{J}(\operatorname{Ker} L) \subset \operatorname{Ker} L^* \ .$$

Assume that $u \in D(L)$ satisfies

$$Lu - N(u) = h \ . \tag{5}$$

Then

$$\tilde{P}\Big(u - K\tilde{Q}N(u)\Big) + P\mathcal{J}^* N(u) \ = \ \hat{h} \ , \tag{6}$$

where $\hat{h} = K\tilde{Q}h - P\mathcal{J}^* h$.

(**b**) *Let $\mathcal{J}: \mathcal{H} \to \mathcal{H}$ be a linear homeomorphism such that $\operatorname{Ker} L^* \subset \mathcal{J}(\operatorname{Ker} L)$. If $u \in H$ is a solution of (6), then $u \in D(L)$ and u is a solution of (5).*

Hence, the solution sets of the equations (5) and (6) coincide whenever $\operatorname{Ker} L^* = \mathcal{J}(\operatorname{Ker} L)$.

Proof: (**a**) Assume that $\mathcal{J}(\operatorname{Ker} L) \subset \operatorname{Ker} L^*$ and $u \in D(L)$ is a solution of (5). Then $Q(N(u) + h) = 0$ and $\tilde{P}u = K\tilde{Q}(N(u) + h)$. By Lemma 3.1, $\mathcal{J}^*(\operatorname{Im} L) \subset \operatorname{Im} L^*$ and, since $N(u) + h \in \operatorname{Im} L$, we obtain $\mathcal{J}^*(N(u) + h) \in \operatorname{Im} L^*$, i.e., $P\mathcal{J}^*(N(u) + h) = 0$. Consequently,

$$\tilde{P}\Big(u - K\tilde{Q}N(u)\Big) + P\mathcal{J}^* N(u) \ = \ K\tilde{Q}h - P\mathcal{J}^* h \ .$$

(**b**) By (6), $\tilde{P}(u - K\tilde{Q}(N(u) + h)) = 0$. Hence, $u - K\tilde{Q}(N(u) + h) \in \operatorname{Ker} L$, implying $u \in D(L)$ and

$$Lu - \tilde{Q}(N(u) + h) \ = \ 0 \ .$$

By (6), we also have $P\mathcal{J}^*(N(u) + h) = 0$, i.e., $\mathcal{J}^*(N(u) + h) \in \operatorname{Im} L^*$. By Lemma 3.1, $\operatorname{Im} L^* \subset \mathcal{J}^*(\operatorname{Im} L)$, and thus we get $N(u) + h \in \operatorname{Im} L$, i.e., $Q(N(u) + h) = 0$, completing the proof. ∎

Notice that, by Lemma 3.1, the condition

$$\operatorname{Ker} L^* = \mathcal{J}(\operatorname{Ker} L) \ .$$

is equivalent to

$$\operatorname{Im} L^* = \mathcal{J}^*(\operatorname{Im} L) \ .$$

Define
$$F(u) \,=\, \tilde{P}\Big(u - K\tilde{Q}N(u)\Big) + P\mathcal{J}^*N(u) \;.$$
Assume that K is compact and $N\colon H \to H$ is bounded, demicontinuous and of class $(S_+)_{\mathcal{J}P}$. Then $\mathcal{J}^*N \in (S_+)_P$ and F is admissible for degree. Assume $\operatorname{Ker} L^* \subset \mathcal{J}(\operatorname{Ker} L)$. In order to simplify our notations we define a further degree function 'deg' by setting
$$\deg(L-N,G,h) \,\equiv\, d_H\Big(\tilde{P}(I-K\tilde{Q}N) + P\mathcal{J}^*N,\, G,\, K\tilde{Q}h - P\mathcal{J}^*h\Big)$$
for any open bounded set $G \subset H$ such that $h \notin (L-N)(\partial G \cap D(L))$. Definition is relevant in view of Lemma 3.2 (b).

4 – About the set $\mathcal{A}_{\mathcal{J}}$

Let L be as in section 3, $\mathcal{J}$ being a linear homeomorphism. As explained in the introduction, our aim is to use a generalized form of condition (2) by replacing I by $\mathcal{J}$. Denote
$$\mathcal{A}_{\mathcal{J}} \,=\, \Big\{\rho \in \mathbb{R} \mid\; \|Lu\|^2 \geq \rho\,\langle Lu, \mathcal{J}u\rangle \quad \text{for all} \;\; u \in D(L)\Big\} \;.$$
It is easy to see that
$$\mathcal{A}_{\mathcal{J}} \,=\, \left\{\rho \in \mathbb{R} \;\mid\; \left\|Lu - \frac{\rho}{2}\,\mathcal{J}u\right\| \geq \left\|\frac{\rho}{2}\,\mathcal{J}u\right\| \quad \text{for all} \;\; u \in D(L)\right\}.$$
The following important result is proved in [8] in a slightly different setting, but the same proof applies here.

Lemma 4.1. *Let $\mathcal{J}\colon \mathcal{H} \to \mathcal{H}$ be a linear homeomorphism. The set $\mathcal{A}_{\mathcal{J}}$ is a closed interval containing* 0. *If $\mathcal{J}(\operatorname{Ker} L) \subset \operatorname{Ker} L^*$, then* 0 *is an interior point of $\mathcal{A}_{\mathcal{J}}$. Otherwise, $\mathcal{A}_{\mathcal{J}} = \{0\}$.* ∎

As will appear below, in order to obtain a priori bounds for the solutions of (1), it is advantageous to have $\mathcal{A}_{\mathcal{J}} \neq \{0\}$. But, by the above lemma, $\mathcal{A}_I = \{0\}$ when $\dim \operatorname{Ker} L = \dim \operatorname{Ker} L^* < \infty$ and $\operatorname{Ker} L \neq \operatorname{Ker} L^*$, which justifies the interest of replacing I by some other map $\mathcal{J}$. On the other hand, an elementary calculation shows that $[-\rho_1, \rho_1] \subset \mathcal{A}_{\mathcal{J}}$, where $\rho_1 = \|\mathcal{J}K\|^{-1}$, whenever $\mathcal{J}: H \to H$ is such that $\mathcal{J}(\operatorname{Ker} L) \subset \operatorname{Ker} L^*$. The following possible characterization of the set $\mathcal{A}_{\mathcal{J}}$ is given in [8].

Lemma 4.2. *Let $\mathcal{J}:\mathcal{H}\to\mathcal{H}$ be a linear homeomorphism such that $\mathcal{J}(D(L))\subset D(L^*)$ and $\mathcal{J}(\operatorname{Ker} L)\subset \operatorname{Ker} L^*$. Assume that the right inverse K of L is compact. Then, $\sup\mathcal{A}_{\mathcal{J}}$ (resp. $\inf\mathcal{A}_{\mathcal{J}}$) is the least positive (resp. greatest negative) eigenvalue of the problem*

$$2\,L^*Lu \;=\; \lambda(\mathcal{J}^*Lu + L^*\mathcal{J}u)\ ,$$

where $u\in D(L^*L)\cap \operatorname{Im} L$. ∎

In our next result, we make some observations about the boundedness of the interval $\mathcal{A}_{\mathcal{J}}$.

Lemma 4.3. *Let $\mathcal{J}:\mathcal{H}\to\mathcal{H}$ be a linear homeomorphism such that $\mathcal{J}(\operatorname{Ker} L)\subset \operatorname{Ker} L^*$. Then*

(**1**) *$\mathcal{A}_{\mathcal{J}}$ is unbounded below if and only if L is $\mathcal{J}$-monotone.*

(**2**) *$\mathcal{A}_{\mathcal{J}}$ is unbounded above if and only if $-L$ is $\mathcal{J}$-monotone.*

Proof: We shall prove the first assertion. Assume that L is $\mathcal{J}$-monotone. Then

$$\|Lu\|^2 \geq \rho\,\langle Lu, \mathcal{J}u\rangle \quad \text{for all } \ u\in D(L) \ \text{ and all } \ \rho\leq 0\ .$$

Thus $]-\infty,0]\subset\mathcal{A}_{\mathcal{J}}$. On the other hand, assume that $\mathcal{A}_{\mathcal{J}}$ is unbounded below. Then $-n\in\mathcal{A}_{\mathcal{J}}$ (recall that $\mathcal{A}_{\mathcal{J}}$ is an interval with 0 as an interior point). Consequently,

$$-\frac{1}{n}\,\|Lu\|^2 \leq \langle Lu, \mathcal{J}u\rangle \quad \text{for all } \ u\in D(L) \ \text{ and all } \ n\in\mathbb{Z}_+\ .$$

Hence necessarily $\langle Lu,\mathcal{J}u\rangle\geq 0$ for all $u\in D(L)$. ∎

Assume that $\mathcal{J}(\operatorname{Ker} L)\subset\operatorname{Ker} L^*$. By the previous lemma, we conclude that $\mathcal{A}_{\mathcal{J}}=\mathbb{R}$ if and only if $\langle Lu,\mathcal{J}u\rangle=0$ for all $u\in D(L)$. Moreover, $\mathcal{A}_{\mathcal{J}}$ is a bounded closed interval with 0 as an interior point if and only if neither L nor $-L$ is $\mathcal{J}$-monotone.

By Lemma 4.1, since we want (3) to be satisfied for some $\rho\neq 0$, it is natural to require that $\mathcal{J}(\operatorname{Ker} L)\subset\operatorname{Ker} L^*$. Actually, we shall frequently assume

$$\mathcal{J}(\operatorname{Ker} L)=\operatorname{Ker} L^*\ , \tag{7}$$

in order to have the equivalence between (5) and (6). Note that the condition (7) does not imply $\mathcal{J}(\operatorname{Im} L^*)=\operatorname{Im} L$, except when $\mathcal{J}^*=\mathcal{J}^{-1}$.

5 – A special choice for $\mathcal{J}$

In this section, we indicate how a linear homeomorphism $\mathcal{J}$ having the desired properties can easily be built in the case

$$\dim \operatorname{Ker} L \;=\; \dim \operatorname{Ker} L^* < \infty \; . \tag{8}$$

This construction can be helpful in the treatment of problems of the form

$$Lu = g(x,u) + h \,, \qquad (u \in D(L)) \;,$$

where L acts, for instance, on the space $H = L^2(\Omega;\mathbb{R}^m)$, where Ω is a bounded domain in some space $\mathbb{R}^p$. For such problems, it is advantageous to have a linear homeomorphism $\mathcal{J}$ is induced by a function $J \in L^\infty(\Omega;\mathbb{R}^m\times\mathbb{R}^m)$, so that $\mathcal{J}$ acts pointwise on $u \in L^2(\Omega;\mathbb{R}^m)$. With such a $\mathcal{J}$, the hypotheses required on the nonlinearity N for the existence results of Section 6 can be deduced from pointwise conditions on g.

Under (8), let $\{\phi^{(j)}\}$, $\{\psi^{(j)}\}$ $(j=1,...,n)$ denote bases of $\operatorname{Ker} L \subset L^2(\Omega;\mathbb{R}^m)$ and $\operatorname{Ker} L^*$ respectively. We introduce the matrices

$$\Phi(x) = \left(\phi_i^{(j)}\right)_{i,j=1}^{m,n} , \qquad \Psi(x) = \left(\psi_i^{(j)}\right)_{i,j=1}^{m,n} .$$

We are looking for a matrix $J(x)$ such that $J(x)\,\Phi(x) = \Psi(x)$. Let $(\Phi(x))^\dagger$ be the generalized inverse of $\Phi(x)$. If rank $\Phi(x) = n$, defining the $m{\times}m$ matrix $J(x)$ by $J(x) = \Psi(x)\,(\Phi(x))^\dagger$, it is immediate by definition of the generalized inverse that

$$J(x)\,\phi^{(j)}(x) = \psi^{(j)}(x) \qquad (j=1,...,n) \; .$$

Therefore, assuming that $J \in L^\infty(\Omega;\mathbb{R}^m{\times}\mathbb{R}^m)$, and that $J(x)$ is regular for a.e. $x \in \Omega$, the operator $\mathcal{J}\colon L^2(\Omega;\mathbb{R}^m) \to L^2(\Omega;\mathbb{R}^m)$ defined by $(\mathcal{J}u)(x) = J(x)\,u(x)$, is such that $\mathcal{J}(\operatorname{Ker} L) = \operatorname{Ker} L^*$.

If $m=1$ and $\dim\operatorname{Ker} L = \dim\operatorname{Ker} L^* = 1$, the above construction leads simply to $J(x) = \psi_1^{(1)}(x)/\phi_1^{(1)}(x)$, a choice that has been used in [8]. It turns out in Section 7 that the above construction can also be used in certain cases, where $\dim\operatorname{Ker} L = \dim\operatorname{Ker} L^* = \infty$.

6 – Abstract resonance results

Let H be a real separable Hilbert space and $L : D(L) \subset H \to H$ be a densely defined closed linear operator with closed range $\operatorname{Im} L$ and with compact partial inverse $K\colon \operatorname{Im} L \to \operatorname{Im} L^* \cap D(L)$. As above, let P, Q denote the orthogonal projections onto $\operatorname{Ker} L$ and $\operatorname{Ker} L^*$ respectively; the kernels may be infinite dimensional. Let $\mathcal{J}\colon H \to H$ be a linear homeomorphism such that

$$\mathcal{J}(\operatorname{Ker} L) = \operatorname{Ker} L^* \; . \tag{9}$$

We can now generalize the existence results obtained in [8], where $\dim \operatorname{Ker} L = \dim \operatorname{Ker} L^* < \infty$. Our first result is actually a nonresonance theorem giving the surjectivity of $L-N$, if $N \in (S_+)_{\mathcal{J}P}$ or $N \in (PM)_{\mathcal{J}P}$. Recall that $(S_+)_{\mathcal{J}P} \subset (PM)_{\mathcal{J}P} \subset (QM)_{\mathcal{J}P}$ and that the class $(S_+)_{\mathcal{J}P}$ is stable under $\mathcal{J}P$-quasi-monotone perturbations. We shall use the fact (see [1]) that for any linear injection $L-S$ admissible for degree

$$\deg\Big(L-S,\, B_R(0),\, 0\Big) \neq 0$$

for all $R > 0$.

Theorem 6.1. *Let L, $\mathcal{J}$ be as indicated above. Assume that (9) holds and $N\colon H \to H$ is a bounded demicontinuous map. Suppose that there exist $\rho \in \,]0, \sup \mathcal{A}_{\mathcal{J}}]$, $\mu \in [0, \rho/2[$ and $\alpha \in [0,1[$ such that*

$$\Big\|N(u) - \frac{\rho}{2}\,\mathcal{J}u\Big\| \leq \mu\,\|\mathcal{J}u\| + O(\|u\|^\alpha) \tag{10}$$

for $u \in H$, $\|u\| \to \infty$. If N is $\mathcal{J}P$-pseudomonotone, then the equation

$$Lu - N(u) = h\,, \qquad u \in D(L) \tag{11}$$

admits a solution for any $h \in H$. In case N is only $\mathcal{J}P$-quasimonotone, the range of $L-N$ is dense in H.

Proof: We consider the homotopy equation

$$Lu \;=\; (1-t)\,\frac{\rho}{2}\,\mathcal{J}u + t\Big(N(u)+h\Big)\,, \qquad 0 \leq t \leq 1\;. \tag{12}$$

We notice first that for $t=0$ the operator $L - \frac{\rho}{2}\mathcal{J}$ is injective since $\rho \in \mathcal{A}_{\mathcal{J}}$ and thus (12) with $t=0$ has only the trivial solution. Moreover,

$$\deg\left(L - \frac{\rho}{2}\,\mathcal{J},\, B_R(0),\, 0\right) \neq 0$$

for all $R > 0$. Let us show that the solution set of (12) remains bounded. Indeed, by the definition of $\mathcal{A}_{\mathcal{J}}$, for any solution $u \in D(L)$, we have the estimate

$$\begin{aligned}\frac{\rho}{2}\,\|\mathcal{J}u\| \;&\le\; \left\|Lu - \frac{\rho}{2}\,\mathcal{J}u\right\| \;=\; t\,\left\|N(u) - \frac{\rho}{2}\,\mathcal{J}u + h\right\| \\ &\le\; \left\|N(u) - \frac{\rho}{2}\mathcal{J}u\right\| + \|h\| \;\le\; \mu\,\|\mathcal{J}u\| + O(\|u\|^\alpha) \quad \text{for } \;\|u\| \to \infty\;.\end{aligned}$$

Consequently, there exists $R > 0$ such that

$$Lu \;\ne\; (1-t)\,\frac{\rho}{2}\mathcal{J}u + t\Big(N(u) + h\Big), \qquad \text{for all } \;0 \le t \le 1, \quad u \in D(L), \quad \|u\| \ge R\;.$$

Assume first that $N \in (S_+)_{\mathcal{J}P}$. Then

$$\deg\Big(L - N,\; B_R(0),\; h\Big) \;=\; \deg\left(L - \frac{\rho}{2}\,\mathcal{J},\; B_R(0),\; 0\right) \;\ne\; 0$$

and the conclusion follows.

Assume secondly that N is $\mathcal{J}P$-quasimonotone. Let $\bar{t} \in \,]0,1[$ be arbitrary but fixed. Then $(1-\bar{t})\,\frac{\rho}{2}\mathcal{J} + \bar{t}(N+h)$ is of class $(S_+)_{\mathcal{J}P}$ and by the above reasoning

$$\deg\left(L - (1-\bar{t})\,\frac{\rho}{2}\mathcal{J} - \bar{t}(N+h),\; B_R(0),\; h\right) \;=\; \deg\left(L - \frac{\rho}{2}\,\mathcal{J},\; B_R(0),\; 0\right) \ne\; 0\;.$$

Hence there exists $\bar{u} \in D(L)$ such that

$$L\bar{u} \;=\; (1-\bar{t})\,\frac{\rho}{2}\,\mathcal{J}\bar{u} \;+\; \bar{t}\Big(N(\bar{u}) + h\Big)\;.$$

Consequently, for any sequence $(t_n) \subset [0,1[$, $t_n \to 1_-$, we conclude by letting $\bar{t} = t_n$ with corresponding solution $\bar{u} = u_n \in D(L)$, that there exists a sequence $(u_n) \subset D(L) \cap B(0,R)$ such that

$$Lu_n - (1-t_n)\,\frac{\rho}{2}\,\mathcal{J}u_n - \;t_n\Big(N(u_n) + h\Big) \;=\; 0\;.$$

Clearly $Lu_n - N(u_n) \to h$, that is, $h \in \overline{R(L-N)}$. If N is $\mathcal{J}P$-pseudomonotone we can continue the reasoning. Taking a subsequence if necessary we can assume that $u_n \rightharpoonup u$. On the other hand, $QN(u_n) + Qh \to 0$ from which follows that

$$\lim\Big\langle QN(u_n) + Qh,\; \mathcal{J}P(u_n - u)\Big\rangle \;=\; 0$$

implying, since $\mathcal{J}(\operatorname{Ker} L) = \operatorname{Ker} L^*$,

$$\lim\Big\langle N(u_n),\; \mathcal{J}P(u_n - u)\Big\rangle = 0\;.$$

From the $\mathcal{J}P$-pseudomonotonicity of N, we deduce that $N(u_n) \rightharpoonup N(u)$. Since L is closed, we get $u \in D(L)$ and $Lu - N(u) = h$, completing the proof. ∎

If we allow $\mu = \rho/2$ in condition (10) we need a further h-dependent resonance type condition and the restriction $\rho < \sup \mathcal{A}_{\mathcal{J}}$.

Theorem 6.2. *Assume that (9) holds and $N: H \to H$ is a bounded demicontinuous map. Suppose that there exist $\rho \in\,]0, \sup \mathcal{A}_{\mathcal{J}}[$ and $\alpha \in [0,1[$ such that*

$$\left\| N(u) - \frac{\rho}{2}\,\mathcal{J}u \right\| \le \frac{\rho}{2}\,\|\mathcal{J}u\| + O(\|u\|^\alpha) \tag{13}$$

for $u \in H$, $\|u\| \to \infty$. Let $h \in H$ be given and assume that for any sequence $(u_n) \subset D(L)$ such that $\|u_n\| \to \infty$ and $\|Lu_n\| = o(\|u_n\|)$ for $n \to \infty$, there exists n_0 such that

$$\Big\langle N(u_n) + h,\ \mathcal{J}Pu_n \Big\rangle \ge 0 \quad \text{for all } n \ge n_0\ . \tag{14}$$

If N is $\mathcal{J}P$-pseudomonotone, then the equation (11) admits a solution. If N is only $\mathcal{J}P$-quasimonotone, then $h \in \overline{R(L{-}N)}$.

Proof: As in the previous theorem, we consider the homotopy equation (12). We prove that the set of solutions of (12) remains bounded. Assume, by contradiction, that there exist sequences $(u_n) \subset D(L)$ and $(t_n) \subset\,]0,1[$ such that $\|u_n\| \to \infty$ and

$$Lu_n = (1-t_n)\,\frac{\rho}{2}\,\mathcal{J}u_n + t_n\Big(N(u_n) + h\Big)\ . \tag{15}$$

Take any $\bar\rho > \rho$, $\bar\rho \in \mathcal{A}_{\mathcal{J}}$. We then have the useful estimate

$$\left\| Lu - \frac{\rho}{2}\,\mathcal{J}u \right\|^2 \ge \left(1 - \frac{\rho}{\bar\rho}\right) \|Lu\|^2 + \left(\frac{\rho}{2}\,\|\mathcal{J}u\|\right)^2 \quad \text{for all } u \in D(L)\ .$$

Hence we obtain

$$\begin{aligned}
\left(1 - \frac{\rho}{\bar\rho}\right) \|Lu_n\|^2 + \left(\frac{\rho}{2}\,\|\mathcal{J}u_n\|\right)^2 &\le \left\| Lu_n - \frac{\rho}{2}\,\mathcal{J}u_n \right\|^2 \\
&\le t^2 \left(\left\| N(u_n) - \frac{\rho}{2}\,\mathcal{J}u_n \right\| + \|h\| \right)^2 \\
&\le \left(\frac{\rho}{2}\,\|\mathcal{J}u_n\| + \|h\| + O(\|u_n\|^\alpha) \right)^2
\end{aligned}$$

implying

$$\|Lu_n\| = o(\|u_n\|) = o(\|\mathcal{J}u_n\|)\ .$$

Denote $z_n = \frac{u_n}{\|\mathcal{J} u_n\|}$ and $w_n = L z_n$. Then $w_n \to 0$ and hence $\tilde{P} z_n = K w_n \to 0$. By (15),

$$\Big\langle L u_n - \frac{\rho}{2}\, \mathcal{J} u_n,\, \mathcal{J} P u_n \Big\rangle = t_n \Big\langle N(u_n) + h - \frac{\rho}{2}\, \mathcal{J} u_n,\, \mathcal{J} P u_n \Big\rangle$$

and, since $\mathcal{J} P u_n \in \operatorname{Ker} L^* = (\operatorname{Im} L)^\perp$, we get

$$\Big\langle N(u_n) + h,\, \mathcal{J} P u_n \Big\rangle \;=\; -(1-t_n)\, t_n^{-1} \frac{\rho}{2} \Big\langle \mathcal{J} u_n, \mathcal{J} P u_n \Big\rangle\ .$$

Writing $\mathcal{J} P u_n = \mathcal{J} u_n - \mathcal{J} \tilde{P} u_n$ leads to the equality

$$\Big\langle N(u_n) + h,\, \mathcal{J} P u_n \Big\rangle \;=\; -(1-t_n)\, t_n^{-1}\, \frac{\rho}{2}\, \|\mathcal{J} u_n\|^2 \Big[1 - \langle \mathcal{J} z_n, \mathcal{J} \tilde{P} z_n \rangle\Big]\ .$$

Clearly the righthandside will be negative for sufficiently large n, thus contradicting the assumption (14). We can proceed exactly like in the proof of Theorem 6.1 to obtain the conclusions. ∎

Corollary 6.1. *Assume that (9) holds and $N\colon H \to H$ is a bounded demicontinuous map. Suppose that there exist $\rho \in \,]0, \sup \mathcal{A}_{\mathcal{J}}[$ and $\alpha \in [0,1[$ such that (13) holds. Assume that $h \in H$ and*

$$\langle -h, \mathcal{J} v\rangle < \limsup \Big\langle N(s_n v_n),\, \mathcal{J} P v_n \Big\rangle \tag{16}$$

for any sequences $(v_n) \subset D(L)$, $(s_n) \subset \mathbb{R}$ with $v_n \rightharpoonup v \in \operatorname{Ker} L$ and $s_n \to \infty$. If N is $\mathcal{J}P$-pseudomonotone, then the equation (11) admits a solution. If N is $\mathcal{J}P$-quasimonotone, then $h \in \overline{R(L-N)}$.

Proof: It suffices to show that condition (14) is valid. Indeed, take $(u_n) \subset D(L)$ such that $\|u_n\| \to \infty$ and $L u_n = o(\|u_n\|)$. Assuming that (14) is not valid, and taking a subsequence if necessary, we can assume that

$$\Big\langle N(u_n) + h,\, \mathcal{J} P u_n \Big\rangle < 0 \quad \text{ for all } \; n\ .$$

Denote $s_n = \|u_n\|$ and $v_n = \|u_n\|^{-1} u_n$. Since $\tilde{P} v_n \to 0$ we can write $v_n \rightharpoonup v \in \operatorname{Ker} L$ at least for a subsequence. Thus by (16)

$$\langle -h, \mathcal{J} v\rangle \;<\; \limsup \Big\langle N(s_n v_n),\, \mathcal{J} P v_n \Big\rangle \;\le\; \langle -h, \mathcal{J} v\rangle\ ,$$

a contradiction completing the proof. ∎

We point out the close connection of (16) with the recession function introduced by Brezis and Nirenberg in [6]. Condition (16) gives, as a special case, the classical Landesman–Lazer condition (cf. [3], [6], [9], [10]).

7 – Two-component systems

Let H_1, H_2 be real separable Hilbert spaces and denote $H = H_1 \times H_2$. We use the same symbols for the scalar products in H_1, H_2, H; the same remark applies to the norms. For $k = 1, 2$, let $L_k\colon D(L_k) \subset H_k \to H_k$ be a linear, densely defined, closed operator with closed range $\operatorname{Im} L_k = (\operatorname{Ker} L_k^*)^\perp$. The inverse $K_k\colon \operatorname{Im} L_k \to \operatorname{Im} L_k^*$ of the restriction of each L_k to $\operatorname{Im} L_k^* \cap D(L_k)$ is assumed to be a compact linear operator. We define the diagonal operator $L\colon D(L) \subset H \to H$ by setting

$$Lu = (L_1 u_1, L_2 u_2)\,, \qquad u = (u_1, u_2) \in D(L)\ ,$$

where $D(L) = D(L_1) \times D(L_2)$. The inverse $K = L^{-1}\colon \operatorname{Im} L \to \operatorname{Im} L^*$ is compact, with $Ku = (K_1 u_1, K_2 u_2)$ for $u = (u_1, u_2) \in \operatorname{Im} L$. We denote by P_k and Q_k the orthogonal projections onto $\operatorname{Ker} L_k$ and $\operatorname{Ker} L_k^*$ respectively ($k = 1, 2$), and by P, Q the orthogonal projections onto $\operatorname{Ker} L$ and $\operatorname{Ker} L^*$; obviously

$$Pu = (P_1 u_1, P_2 u_2) \quad \text{and} \quad Qu = (Q_1 u_1, Q_2 u_2)$$

for any $u = (u_1, u_2) \in H_1 \times H_2$. As before, we denote $\tilde{P}_k = I - P_k$, $\tilde{Q}_k = I - Q_k$ ($k = 1, 2$) and $\tilde{P} = I - P$, $\tilde{Q} = I - Q$. Let $N\colon H \to H$ be a (possibly nonlinear) bounded demicontinuous map; we will write $N(u)$ as

$$N(u) = \Big(N_1(u_1, u_2),\, N_2(u_1, u_2)\Big)\ ,$$

where, for $k = 1, 2$, $u_k \in H_k$, $N_k(u_1, u_2) \in H_k$. We will consider the equation

$$Lu - N(u) = 0\,, \qquad u \in D(L)\ . \tag{17}$$

For $k = 1, 2$, let $\mathcal{J}_k\colon H_k \to H_k$ be linear homeomorphisms, $\mathcal{J}$ being naturally defined, for $u = (u_1, u_2)$, by $\mathcal{J} u = (\mathcal{J}_1 u_1, \mathcal{J}_2 u_2)$. Assuming that

$$\operatorname{Ker} L_1^* = \mathcal{J}_1(\operatorname{Ker} L_1) \quad \text{and} \quad \operatorname{Ker} L_2^* = \mathcal{J}_2(\operatorname{Ker} L_2)\ , \tag{18}$$

we have $\operatorname{Ker} L^* = \mathcal{J}(\operatorname{Ker} L)$. Hence by Lemma 3.2 equation (17) is equivalent to $F(u) = 0$, where F is defined by

$$F(u) = \tilde{P}\Big(u - K\tilde{Q}N(u)\Big) + P\mathcal{J}^* N(u)\,, \qquad u \in H\ .$$

Moreover, F is admissible for degree provided $N \in (S_+)_{\mathcal{J}P}$.

We shall consider the following special case, where $\dim \operatorname{Ker} L_1 = \dim \operatorname{Ker} L_1^* = \infty$, $\dim \operatorname{Ker} L_2 = \dim \operatorname{Ker} L_2^* < \infty$. Note that if $\operatorname{Ker} L_1 = \operatorname{Ker} L_1^*$, then it is possible to take $J_1 = I$ and in certain cases use the procedure given in Section 5 to find J_2.

The following lemma provides conditions under which N is of class $(S_+)_{\mathcal{J}P}$.

Lemma 7.1. *Let L_1, L_2 and $\mathcal{J}$ be as indicated above, assume (18) holds and $\dim \operatorname{Ker} L_1 = \infty$, $\dim \operatorname{Ker} L_2 < \infty$. Let $N\colon H \to H$ be bounded and demicontinuous. Assume that*

- **(i)** *For each $u_2 \in H_2$, the mapping $N_1(\cdot, u_2)\colon H_1 \to H_1$ is of class $(S_+)_{\mathcal{J}_1 P_1}$.*
- **(ii)** *$N_1(u_1, \cdot)\colon H_2 \to H_2$ is continuous, uniformly for u_1 in any bounded set $B \subset H_1$.*

Then $N \in (S_+)_{\mathcal{J}P}$.

Proof: Let $(u^{(j)}) \subset H$ be a sequence such that, with $u^{(j)} = v^{(j)} + z^{(j)}$, $v^{(j)} \in \operatorname{Ker} \mathcal{J}P = \operatorname{Im} L^*$, $z^{(j)} \in (\operatorname{Ker} \mathcal{J}P)^\perp = \operatorname{Ker} L$,

$$u^{(j)} \rightharpoonup u, \quad v^{(j)} \to v \quad \text{and} \quad \limsup_{j\to\infty} \Big\langle N(u^{(j)}),\, \mathcal{J}P(u^{(j)} - u)\Big\rangle \le 0 \ .$$

We have to show that $u^{(j)} \to u$. For $k = 1, 2$, denote respectively by $u_k^{(j)}, v_k^{(j)}, z_k^{(j)}$, u_k, z_k the components of $u^{(j)}, v^{(j)}$, $z^{(j)}, u, z$ in H_k. Since $\dim \operatorname{Ker} L_2 < \infty$ and N is bounded,

$$\lim_{j\to\infty} \Big\langle N_2(u_1^{(j)}, u_2^{(j)}),\, \mathcal{J}_2 P_2(u_2^{(j)} - u_2)\Big\rangle = 0 \ .$$

Consequently, taking into account the diagonal structure of $\mathcal{J}P$, we have

$$\limsup_{j\to\infty} \Big\langle N_1(u_1^{(j)}, u_2^{(j)}),\, \mathcal{J}_1 P_1(u_1^{(j)} - u_1)\Big\rangle \le 0 \ .$$

But, by hypothesis (ii),

$$\lim_{j\to\infty} \Big[\Big\langle N_1(u_1^{(j)}, u_2^{(j)}),\, \mathcal{J}_1 P_1(u_1^{(j)} - u_1)\Big\rangle - \Big\langle N_1(u_1^{(j)}, u_2),\, \mathcal{J}_1 P_1(u_1^{(j)} - u_1)\Big\rangle\Big] = 0 \ .$$

Subtracting this from the previous inequality gives

$$\limsup_{j\to\infty} \Big\langle N_1(u_1^{(j)}, u_2),\, \mathcal{J}_1 P_1(u_1^{(j)} - u_1)\Big\rangle \le 0 \ .$$

As $N_1(\cdot, u_2)\colon H_1 \to H_1$ is assumed to be of class $(S_+)_{\mathcal{J}_1 P_1}$, we conclude that $(u_1^{(j)})$ converges to u_1. Since $\operatorname{Ker} L_2$ is finite dimensional, $(u_2^{(j)})$ also converges to u_2; this proves that $N \in (S_+)_{\mathcal{J}P}$. ∎

The main point in the previous lemma is that there is no monotonicity-type hypothesis on the component N_2. Note that to there is no obvious pseudomonotone variant of Lemma 7.1 due to the requirement $N_2(u^{(j)}) \rightharpoonup N_2(u)$ as $u^{(j)} \rightharpoonup u$ needed for $N \in (PM)_{\mathcal{J}P}$.

For concrete situations it may be useful to introduce the following concepts:

If $\mathcal{J}\colon H \to H$ is a linear homeomorphism, and $P\colon H \to H$ an orthogonal projection, then a mapping $N\colon H \to H$ is *$\mathcal{J}$-strongly monotone*, if there is a constant $c_0 > 0$ such that

$$\Big\langle N(u) - N(v),\, \mathcal{J}(u-v) \Big\rangle \geq c_0 \, \|u-v\|^2 \,, \qquad \text{for all } \ u, v \in H$$

and correspondingly *$\mathcal{J}P$-strongly monotone*, if

$$\Big\langle N(u) - N(v),\, \mathcal{J}P(u-v) \Big\rangle \geq c_0 \, \|P(u-v)\|^2 \,, \qquad \text{for all } \ u, v \in H \ .$$

It is clear that any N which is strongly $\mathcal{J}P$-monotone belong to class $(S_+)_{\mathcal{J}P}$ and any N which is strongly $\mathcal{J}$-monotone belong to class $(S_+)_{\mathcal{J}} \subset (S_+)_{\mathcal{J}P}$.

Hence in the previous lemma $N_1(\cdot, u_2)\colon H_1 \to H_1$ is of class $(S_+)_{\mathcal{J}_1 P_1}$ if it is $\mathcal{J}_1 P_1$-strongly monotone, which is the case if it is $\mathcal{J}_1$-strongly monotone. Therefore, Lemma 7.1 can be useful for instance in the study of systems like

$$\begin{cases} L_1 u_1 \ = \ N_{1,1}(u_1) + N_{1,2}(u_2) + h_1 \ , \\ L_2 u_2 \ = \ N_2(u_1, u_2) + h_2 \ , \end{cases}$$

where one would assume $N_{1,1}$ to be $\mathcal{J}_1$-strongly monotone and $N_{1,2}$ to be continuous, whereas no monotonicity hypothesis would be made on $N_{1,2}$ and N_2.

Combining the above lemma with Theorem 6.1 provides existence results for two-component systems.

Corollary 7.1. *Let L_1, L_2 and $\mathcal{J}$ be as indicated above, assume (18) holds and* $\dim \operatorname{Ker} L_1 = \infty$, $\dim \operatorname{Ker} L_2 < \infty$. *Let $N\colon H \to H$ be bounded and demicontinuous and let $N_1\colon H \to H_1$, $N_2\colon H \to H_2$ satisfy the conditions of Lemma 7.1. Assume that there exist $\rho \in \,]0, \sup \mathcal{A}_{\mathcal{J}}]$, $\mu \in [0, \rho/2[$ and $\alpha \in [0,1[$ such that*

$$\left\| N_1(u_1, u_2) - \frac{\rho}{2} \mathcal{J}_1 u_1 \right\| \ \leq \ \mu \, \|\mathcal{J}_1 u_1\| + O(\|u\|^\alpha) \ , \tag{19}$$

$$\left\| N_2(u_1, u_2) - \frac{\rho}{2} \mathcal{J}_2 u_2 \right\| \ \leq \ \mu \, \|\mathcal{J}_2 u_2\| + O(\|u\|^\alpha) \ , \tag{20}$$

for $u \in H$, $\|u\| \to \infty$. *Then, for any* $h_1 \in H_1$, $h_2 \in H_2$, *the system*

$$\begin{cases} L_1 u_1 = N_1(u_1, u_2) + h_1 \\ L_2 u_2 = N_2(u_1, u_2) + h_2 \end{cases}$$

has a solution. ∎

Again, the interest of the above corollary lies in the fact that no monotonicity hypothesis is required on the component N_2. Notice that (19), (20) imply that the growth of N_1 with respect to u_2 and the growth of N_2 with respect to u_1 is sublinear, restricting the possible couplings between the two components of the system.

8 – Semi-abstract applications

We shall close this note by some examples in a semi-abstract setting. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and take $H = L^2(\Omega; \mathbb{R}^m)$. Let $L\colon D(L) \subset H \to H$ be a linear densely defined closed operator with closed range $\operatorname{Im} L$. Assume that the partial inverse K of L is compact. We denote by $(\cdot \mid \cdot)$ and $|\cdot|$ the usual inner product and norm in $\mathbb{R}^m$.

Throughout this section, we will assume that the linear homeomorphism $\mathcal{J}$ is induced by a function $J \in L^\infty(\Omega; \mathbb{R}^m \times \mathbb{R}^m)$, $J(x)$ being a regular matrix for a.e. $x \in \Omega$, and $\mathcal{J}$ being assumed to be such that $\mathcal{J}(\operatorname{Ker} L) = \operatorname{Ker} L^*$. With this assumption, $\mathcal{A}_{\mathcal{J}}$ is a closed interval containing the origin.

We will consider equations of the type

$$Lu = g(x, u) + h \qquad (u \in D(L)) ,$$

where g satisfies at least the usual Carathéodory conditions and a linear growth condition. We denote by N the Nemytski operator associated to g; with the above hypotheses, it is well defined as operator from $L^2(\Omega; \mathbb{R}^m)$ to $L^2(\Omega; \mathbb{R}^m)$, and is continuous and bounded. The following lemmas provide conditions under which N is $\mathcal{J}$-monotone and thus $\mathcal{J}$-pseudomonotone. The proof of the first one is trivial and hence omitted.

Lemma 8.1. *Let* $L, \mathcal{J}$ *be as indicated above,* $\mathcal{J}$ *being induced by a function* $J \in L^\infty(\Omega; \mathbb{R}^m \times \mathbb{R}^m)$. *Let* $g\colon \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ *satisfy* L^2*-Carathéodory conditions and a linear growth condition. Assume that*

$$\Big(g(x,u) - g(x,v) \mid J(x)(u-v)\Big) \geq 0 \quad \textit{for a.e. } x \in \Omega, \quad \textit{for all } u, v \in \mathbb{R}^m . \tag{21}$$

Then, the the Nemytski operator N, defined by $(N(u))(x) = g(x, u(x))$, is $\mathcal{J}$-monotone. ∎

Lemma 8.2. *Let $L, \mathcal{J}$ be as indicated above, $\mathcal{J}$ being induced by a function $J \in L^\infty(\Omega; \mathbb{R}^m \times \mathbb{R}^m)$. Assume that the matrix $J(x)$ is orthogonal for a.e. $x \in \Omega$. Let $f\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded function such that $f(x, \cdot)$ is continuous for a.e. $x \in \Omega$ and $f(\cdot, t)$ is measurable for all $t \geq 0$. Assume that the map $t \mapsto f(x,t)\, t$, $t \geq 0$, is nondecreasing. Define the mapping $N\colon H \to H$ by setting*

$$N(u)(x) = f(x, |u(x)|)\, J(x)\, u(x) \qquad \textit{for all} \;\; u \in H \quad \text{a.e. } x \in \Omega \;.$$

Then N is bounded, continuous and N is $\mathcal{J}$-monotone.

Proof: By the assumptions N is bounded and continuous and

$$\begin{aligned}
&\Big(f(x,|u|)\, J(x)\, u - f(x,|v|)\, J(x)\, v \;\;|\;\; J(x)u - J(x)v\Big) \;\geq \\
&\qquad\geq\; f(x,|u|)\, |J(x)u|^2 - f(x,|v|)\, |J(x)v|\, |J(x)u| \\
&\qquad\quad\; + f(x,|v|)\, |J(x)v|^2 - f(x,|u|)\, |J(x)u|\, |J(x)v| \\
&\qquad=\; \Big(f(x,|u|)\, |J(x)u| - f(x,|v|)\, |J(x)v|\Big) \Big(|J(x)u| - |J(x)v|\Big) \\
&\qquad=\; \Big(f(x,|u|)\, |u| - f(x,|v|)\, |v|\Big) \Big(|u| - |v|\Big) \;\geq\; 0
\end{aligned}$$

for all $u, v \in \mathbb{R}^m$, a.e. $x \in \Omega$. Integration over the set Ω gives

$$\Big\langle N(u) - N(v),\, \mathcal{J}(u-v) \Big\rangle \geq 0 \qquad \text{for all} \;\; u, v \in H \;.$$

Hence N is $\mathcal{J}$-monotone. ∎

Combining the above lemmas with Theorem 6.1, we can deduce existence results.

Theorem 8.1. *Let $L, \mathcal{J}$ be as in Lemma 8.1. Let $g\colon \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ satisfy L^2-Carathéodory conditions, as well as condition (21). Assume moreover that there exists $\rho \in \,]0, \sup \mathcal{A}_{\mathcal{J}}]$, $\mu \in [0, \rho/2[$, $\alpha \in [0,1[$ and functions $k_1, k_2 \in L^\infty(\Omega; \mathbb{R})$, such that*

$$\left| g(x,u) - \frac{\rho}{2}\, J(x) u \right| \;\leq\; \mu |J(x)u| + k_1(x)\, |u|^\alpha + k_2(x) \;. \tag{22}$$

for a.e. $x \in \Omega$, $u \in \mathbb{R}^m$. Then, the equation

$$Lu - N(u) = h \,, \qquad u \in D(L)$$

admits a solution for any $h \in H$.

Proof: With $(N(u))(x) = g(x, u(x))$, it is easily shown, integrating over Ω, that (22) implies that, for some $\mu' \in \,]\mu, \rho/2[$,

$$\left\| N(u) - \frac{\rho}{2}\,\mathcal{J}u \right\| \leq \mu' \|\mathcal{J}u\| + O\left(\|u\|^\alpha\right) \qquad \text{for} \quad \|u\| \to \infty \ .$$

On the other hand, by Lemma 8.1, N is $\mathcal{J}$-monotone and, consequently $\mathcal{J}$-pseudomonotone and, hence, $\mathcal{J}P$-pseudomonotone. Therefore, Theorem 6.1 applies. ∎

In the case of linear equations, the hypotheses can be simplified.

Corollary 8.1. *Let L, $\mathcal{J}$ be as in Lemma 8.1. Let $M \in L^\infty(\Omega; \mathbb{R}^m \times \mathbb{R}^m)$ be such that there exist $\rho \in \,]0, \sup \mathcal{A}_\mathcal{J}]$ and $\mu \in [0, \rho/2[$ such that*

$$\left| M(x)u - \frac{\rho}{2}\, J(x)u \right| \leq \mu\, |J(x)u| \qquad \textit{for all} \;\; u \in \mathbb{R}^m, \quad \text{a.e. } x \in \Omega \ . \tag{23}$$

Then the equation

$$Lu = M(x)u + h(x)\,, \qquad u \in D(L)$$

admits a solution for any $h \in H$.

Proof: With $g(x, u) = M(x)u$, condition (22) is clearly satisfied with $\alpha = 0$, $k_1 = k_2 \equiv 0$. On the other hand, rewriting (23) as

$$\Big(M(x)u \mid J(x)u\Big) \;\geq\; \frac{1}{\rho}\, |M(x)u|^2 + \frac{1}{\rho}\left[\left(\frac{\rho}{2}\right)^2 - \mu^2\right] |J(x)u|^2$$

shows that (21) holds. Hence, Theorem 8.1 applies. ∎

Theorem 8.2. *Let L, $\mathcal{J}$, N be as in Lemma 8.2. Assume that there exist constants a, b such that*

$$0 < a \leq f(x,t) \leq b < \sup \mathcal{A}_\mathcal{J} \qquad \textit{for all} \;\; t \geq 0, \quad \text{a.e. } x \in \Omega \ . \tag{24}$$

Then the equation

$$Lu = N(u) + h\,, \qquad u \in D(L)$$

admits a solution for any $h \in H$.

Proof: We shall again apply Theorem 8.1. In view of Lemma 8.2, it suffices to prove that condition (22) holds. Take any ρ such that $b < \rho \leq \sup \mathcal{A}_\mathcal{J}$ and

$$\mu \;=\; \max\left(\left|a - \frac{\rho}{2}\right|,\; \left|b - \frac{\rho}{2}\right|\right) \ .$$

Then $0 < \mu < \frac{\rho}{2}$ and

$$\left| f(x,|u|)\, J(x)u - \frac{\rho}{2}\, J(x)u \right| \le\, \mu |J(x)u| \quad \text{for all} \;\; u \in \mathbb{R}^m, \quad \text{a.e.}\;\; x \in \Omega\;,$$

giving the desired inequality with $k_1 = k_2 \equiv 0$. The proof is thus completed. ∎

REFERENCES

[1] Berkovits, J. and Fabry, C. – *An extension of the topological degree theory in Hilbert space*, 18p, to appear.

[2] Berkovits, J. and Mustonen, V. – An extension of Leray–Schauder degree and applications to nonlinear wave equations, *Differential Integral Equations,* 3 (1990), 945–963.

[3] Berkovits, J. and Mustonen, V. – On the resonance for semilinear equations with normal linear part, *J. Math. Maroc,* 2 (1994), 115–123.

[4] Berkovits, J. and Mustonen, V. – *On semilinear wave equations at resonance*, in “Proceedings of the First World Congress of Nonlinear Analysts”, Tampa 1992 (V. Lakshmikantham, Ed.) (Walter de Gruyter Publisher, Berlin), (1996), 2399–2407.

[5] Berkovits, J. and Tienari, M. – Topological degree for some classes of multis with applications to hyperbolic and elliptic problems involving dicontinuous nonlinearities, *Dynamic Systems and Applications,* 5 (1996), 1–18.

[6] Brezis, H. and Nirenberg, L. – Characterizations of the ranges of some nonlinear operators and applications to boundary value problems, *Ann. Scuola Norm. Pisa,* 5 (1978), 225–326.

[7] Fabry, C. – Inequalities verified by asymmetric nonlinear operators, *Nonlinear Analysis, TMA,* 33 (1998), 121–137.

[8] Fabry, C.; Fonda, A. and Munyamarere, F. – Semilinear equations at resonance with non-symmetric linear part, *J. Math. Anal. Appl.,* 180 (1993), 189–206.

[9] Landesman, E. and Lazer, A. – Nonlinear perturbation of linear elliptic boundary value problems at resonance, *J. Math. Mech.,* 19 (1970), 609–623.

[10] Mawhin, J. – Landesman–Lazer conditions for boundary-value problems: a nonlinear version of resonance, *Boletin de La Sociedad Espanola de Matematica Aplicada,* 16 (2000), 45–65.

[11] Taylor, A.E. and Lay, D.C. – *Introduction to Functional Analysis*, Wiley&Sons, New York, 1958.

[12] Zeidler, E. – *Nonlinear Functional Analysis and its Applications II/B, Nonlinear Monotone Operators*, Springer-Verlag, New York, 1985.

Juha Berkovits,
Department of Mathematical Sciences, University of Oulu,
P.O.Box 3000, FIN-90014 Oulu – FINLAND
E-mail: jberkovi@sun3.oulu.fi

and

Christian Fabry,
Institut de Mathématique Pures et Appliquée, Université Catholique de Louvain,
Chemin du Cyclotron 2, B-1348 Louvain-la-Neuve – BELGIUM
E-mail: fabry@math.ucl.ac.be