PORTUGALIAE MATHEMATICA
Vol. 61 Fasc. 2 – 2004
Nova Série

STABILIZATION OF A WAVE-WAVE SYSTEM

N. Aissa and D. Hamroun

Abstract: In this paper, we use the multiplier method to study the energy decay of a coupling of a wave equation posed on a square Ω of $\mathbb{R}^2$ with a wave equation posed on one of its sides Γ_0. We prove that if the dissipation is located both on Ω and Γ_0, then the energy decays exponentially.

1 – Introduction

Let Ω be the open square of $\mathbb{R}^2$

$$\Omega \,=\,]0,1[\times]0,1[\;.$$

Micu and Zuazua in [8] and [9] studied a wave-wave coupling and they shown that when the dissipation is weak (say located on a part of the boundary of Ω), the system is not exponentially stable, while W. Littman and B. Liu in [7] had obtained only a strong stability for another model of a wave-wave coupling. As there was no dissipation in Ω in the previous papers, we have thought to add a dynamical controller of advection form which induces this dissipation in order to obtain the exponential stability.

Stabilization of couplings has been studied by several authors, we cite [1], [2], [6] and [3] and the references therein. Following [1] and [2], we use the multiplier method.

Received: September 12, 2002; *Revised*: May 12, 2003.
AMS Subject Classification: 35B40, 35C20, 35L20, 35P20, 49N35.
Keywords: wave equation; semigroup; stability; multipliers.

We consider the hybrid system of equations arising in the control of noise

$$
(1) \qquad \begin{cases} u_{tt} = \Delta u - d.\nabla u_t \quad \text{in } \mathbb{R}_+ \times \Omega\,, \\ u = 0 \ \text{ in } \mathbb{R}_+ \times \Gamma_1\,, \quad \dfrac{\partial u}{\partial n} = -(-\partial_{xx})^{1/2}\, v_t \ \text{ in } \mathbb{R}_+ \times \Gamma_0\,, \\ v_{tt} = v_{xx} + (-\partial_{xx})^{1/2}(u_{t|_{\Gamma_0}}) \quad \text{in } \mathbb{R}_+ \times \Gamma_0\,, \\ v(t,0) = v(t,1) = 0 \ \text{ in } \mathbb{R}_+\,, \quad u(0) = u_0,\ u_t(0) = u_1 \ \text{ in } \Omega\,, \\ v(0) = v_0,\ v_t(0) = v_1 \quad \text{in } \Gamma_0\,, \end{cases}
$$

where

$$
\Gamma_0 = \,]0,1[\, \times \{0\}\,, \qquad \Gamma_1 = \partial\Omega \backslash \Gamma_0\;.
$$

Let us first precise the definition of the operator $(-\partial_{xx})^{\frac{1}{2}}$. We consider an orthogonal basis $(v_n)_n$ of $H^1_0(0,1)$, orthonormal in $L^2(0,1)$ consisting of renormalized eigenfunctions of the operator $(-\partial_{xx})$ in $H^1_0(0,1)$. We recall that the eigenvalues $(\lambda_n)_n$ and the corresponding eigenvectors $(v_n)_n$ of this operator are given by

$$
\lambda_n = n^2\pi^2\,, \qquad v_n = \sqrt{2}\,\sin(n\pi x) \qquad n \in \mathbb{N}^\star\,.
$$

Since $(-\partial_{xx})$ is self-adjoint and positive in $H^1_0(0,1)$, $(-\partial_{xx})^{\frac{1}{2}}$ is defined in $H^1_0(0,1)$ by

$$
(-\partial_{xx})^{\frac{1}{2}} w \;=\; \sum_{n\geq 1} \lambda_n^{\frac{1}{2}} (w, v_n)_{L^2(0,1)}\, v_n\,, \qquad \forall\, w \in H^1_0(0,1)
$$

then we extend it to $L^2(0,1)$ by duality

$$
(2) \qquad \Big\langle (-\partial_{xx})^{\frac{1}{2}} w, v \Big\rangle_{H^{-1}(0,1), H^1_0(0,1)} = \Big(w, (-\partial_{xx})^{\frac{1}{2}} v \Big)_{L^2(0,1)}
$$

for all $w \in L^2(0,1)$ and $v \in H^1_0(0,1)$. Therefore, we have

$$
(3) \qquad \Big| (-\partial_{xx})^{\frac{1}{2}} w \Big|_{L^2(0,1)} = |\partial_x w|_{L^2(0,1)}\,, \qquad \forall\, w \in H^1_0(0,1)\;.
$$

In this paper, we make the following hypothesis

$$
(\mathrm{H}) \qquad \begin{cases} d = (d_1, d_2) \in C^1(\mathbb{R}^2, \mathbb{R}) \quad \text{and} \quad \exists\, C_0 > 0 \quad \text{such that} \\ -\operatorname{div} d \geq C_0\,, \ \ \forall\, (x,y) \in \overline{\Omega}\,, \qquad d_2(x,0) \leq -C_0, \ \ \forall\, x \in [0,1]\;. \end{cases}
$$

If (u,v) is a solution of (1), we define its energy

$$
E(t) \;=\; \frac{1}{2}\bigg(\int_\Omega |\nabla u|^2\, dx\, dy \;+\; \int_\Omega |u_t|^2\, dx\, dy \;+\; \int_0^1 |v_x|^2\, dx \;+\; \int_0^1 |v_t|^2\, dx \bigg)
$$

and a formal computation shows that for a smooth solution (u,v)

$$\begin{aligned} E'(t) &= -\int_\Omega (d.\nabla u_t) u_t \, dx\, dy \\ &= \frac{1}{2}\int_\Omega (\nabla.\, d) u_t^2 \, dxdy + \frac{1}{2}\int_0^1 u_t^2(x,0)\, d_2(x,0)\, dx \end{aligned}$$

so according to the hypothesis (H), the energy is decreasing and we see that this dissipation is given both by Ω and Γ_0.

The considerations above suggest that the natural wellposedeness space for (1) is

$$X = H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times H^1_0(\Gamma_0) \times L^2(\Gamma_0)$$

where

$$H^1_{\Gamma_1}(\Omega) = \left\{ u \in H^1(\Omega)\,;\; u|_{\Gamma_1} = 0 \right\} .$$

From now on, we denote by $(\cdot,\cdot)$ the inner product of $L^2(\Omega)$ or $L^2(0,1)$ and $|\cdot|$ the associated norm; the inner product $\langle\cdot,\cdot\rangle_X$ of X, is given by

$$\Big\langle (u,\,p,\,v,\,q),\,(U,\,P,\,V,\,Q) \Big\rangle_X = (\nabla u, \nabla U) + (p,P) + (v_x, V_x) + (q,Q)$$

and the corresponding norm will be denoted $\|\cdot\|_X$.

Let A the linear operator defined on the hilbertian space X by

$$\text{(4)}\quad D(A) = \left\{ (u,p,v,q) \in H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)\times H^1_0(\Gamma_0)\times H^1_0(\Gamma_0) \Big/ \Delta u \in L^2(\Omega)\,, \right.$$
$$\left. \frac{\partial u}{\partial n} = -(-\partial_{xx})^{1/2} q \;\text{ on } \Gamma_0, \;\; (-\partial_{xx})^{1/2}(p\,|_{\Gamma_0}) + v_{xx} \in L^2(\Gamma_0) \right\},$$

$$A = \begin{pmatrix} 0 & I & 0 & 0 \\ \Delta & -d.\nabla & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & (-\partial_{xx})^{1/2}\circ\gamma_0 & \partial_{xx} & 0 \end{pmatrix},$$

γ_0 being the trace operator on Γ_0. Then we can write the system (1) in the form

$$\text{(5)}\qquad \begin{cases} Y'(t) = A\,Y(t)\,, & t \in \mathbb{R}_+ \\ Y(0) = Y_0 \;. \end{cases}$$

We will prove in section 2 the existence and the uniqueness of a solution of (1); then in section 3 we will study the asymptotic behavior of the solution.

2 – Existence and uniqueness

Proposition 1. *Assume the hypothesis (H) hold, then for every $Y_o \in D(A)$, the system* (5) *admits a unique solution*

$$Y \in C\Big([0,+\infty[\,;\, D(A)\Big) \cap C^1\Big(]0,+\infty[\,;\, X\Big) .$$

Proof: We will use the theorem of Hille–Yoshida. Let $Y=(u,p,v,q)\in D(A)$. We have, thanks to Green's formula (4) and (H),

$$\begin{aligned}\langle AY,Y\rangle_X &= (\nabla p,\nabla u) + (\Delta u - d.\nabla p,\, p) + (q_x, v_x) + \Big((-\partial_{xx})^{\frac12}(p|_{\Gamma_0}) + v_{xx},\, q\Big)\\ &= -(d.\nabla p,\, p) + \Big\langle \frac{\partial u}{\partial n}|_{\Gamma_0},\, p\Big\rangle_{H^{-1/2}(\Gamma_0),H^{1/2}(\Gamma_0)} + (q_x, v_x)\\ &\quad - \Big\langle \frac{\partial u}{\partial n}|_{\Gamma_0},\, p\Big\rangle_{H^{-1}(\Gamma_0),H^1_0(\Gamma_0)} + \Big\langle v_{xx}, q\Big\rangle_{H^{-1}(\Gamma_0),H^1_0(\Gamma_0)} .\end{aligned}$$

Consequently,

$$\tag{6} \begin{aligned}\langle AY,Y\rangle_X &= -(d.\nabla p,\, p)\\ &= \frac12\int_\Omega (\nabla.d)p^2\,dx\,dy \;+\; \frac12\int_0^1 p^2(x,0)\,d_2(x,0)\,dx \;\le\; 0 .\end{aligned}$$

Now let us prove that A is maximal; take $Z=(U,P,V,Q)\in X$, we will look for $Y=(u,p,v,q)\in D(A)$ such that $(I-A)Y=Z$. This system is equivalent to

$$\tag{7} \begin{cases} u - \Delta u + d.\nabla u \;=\; P + U + d.\nabla U ,\\ v - (-\partial_{xx})^{\frac12}(u|_{\Gamma_0}) - v_{xx} \;=\; Q + V - (-\partial_{xx})^{\frac12}(U|_{\Gamma_0}) ,\\ p = u - U , \qquad q = v - V . \end{cases}$$

Let $(\varphi,\psi)\in H = H^1_{\Gamma_1}(\Omega)\times H^1_0(\Gamma_0)$, then from (7), we get

$$\begin{aligned}&(u-\Delta u + d.\nabla u,\, \varphi) \;+\; \Big\langle v - (-\partial_{xx})^{\frac12}(u|_{\Gamma_0}) - v_{xx},\, \psi\Big\rangle_{H^{-1}(\Gamma_0),H^1_0(\Gamma_0)} =\\ &\quad = (P+U+d.\nabla U,\, \varphi) \;+\; \Big\langle Q+V-(-\partial_{xx})^{\frac12}(U|_{\Gamma_0}),\, \psi\Big\rangle_{H^{-1}(\Gamma_0),H^1_0(\Gamma_0)}\end{aligned}$$

so, thanks to (4) and (2), we get

$$\begin{aligned}&(u+d.\nabla u,\varphi) + (\nabla u,\nabla U) + (v,\psi) + (v_x,\psi_x) \;-\\ &\qquad\qquad -\; (u|_{\Gamma_0}, (-\partial_{xx})^{\frac12}\psi) + ((-\partial_{xx})^{\frac12} v, \varphi|_{\Gamma_0}) \;=\\ &\quad = (P+U+d.\nabla U,\varphi) + (Q+V,\psi) + ((-\partial_{xx})^{\frac12} V, \varphi|_{\Gamma_0}) - (U|_{\Gamma_0}, (-\partial_{xx})^{\frac12}\psi) .\end{aligned}$$

This last equality is of the form

$$B\Big((u,v),(\varphi,\psi)\Big) \;=\; L(\varphi,\psi)\;. \tag{8}$$

it is clear that B is a continuous bilinear form on $H\times H$ and that L is a continuous linear form on H endowed with the norm

$$\|(u,v)\|_H^2 \;=\; |\nabla u|^2 + |v_x|^2\;.$$

Moreover for every $(u,v)\in H$,

$$\begin{aligned} B\Big((u,v),(u,v)\Big) &= |u|^2+|\nabla u|^2-\frac{1}{2}\int_\Omega(\nabla.d)u^2\,dx\,dv\;-\;\frac{1}{2}\int_0^1 u^2(x,0)\,d_2(x,0)\,dx\\ &\quad+\;|v|^2+|v_x|^2\\ &\geq\;\|(u,v)\|_H^2 \end{aligned}$$

thanks to (H), thus B is coercitive. Therefore, using Lax–Milgram theorem, the problem (8) has a unique solution (u,v) in H. Consequently, the system (7) has a solution

$$(u,p,v,q)\;\in\;H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)\times H^1_0(\Gamma_0)\times H^1_0(\Gamma_0)\;.$$

Furthermore

$$\Delta u \;=\; u+d.\nabla u - P - U - d.\nabla U\;\in\;L^2(\Omega)$$

and

$$(-\partial_{xx})^{\frac12}(p|_{\Gamma_0})+v_{xx} \;=\; (-\partial_{xx})^{\frac12}(u|_{\Gamma_0})-(-\partial_{xx})^{\frac12}(U|_{\Gamma_0})+v_{xx}\;=\;v-Q-V$$

thus

$$(-\partial_{xx})^{\frac12}(p|_{\Gamma_0})+v_{xx}\;\in\;L^2(\Gamma_0)\;.$$

Now using (8), the Green formula and (7), we get

$$\frac{\partial u}{\partial n} = -(-\partial_{xx})^{\frac12}\,q \quad \text{ on } \;\Gamma_0$$

so $Y\in D(A)$. Therefore, we conclude that the operator $(A,D(A))$ is m-accretif so $(-A)$ generates a semi-group $(S_A(t))_{t\geq 0}$ of contractions in X. ∎

3 – Exponential stability

Theorem 2. *Under the hypothesis (H), the system* (5) *is exponentially stable.*

We recall that a system of the form (5) is said to be exponentially stable if the semigroup of contractions $(S_A(t))_{t\geq 0}$ satisfies the property

$$\|S_A(t)\| \leq M \exp(-\omega t)\,, \qquad \forall\, t \geq 0 \tag{9}$$

for some constants $\omega > 0$ and $M \geq 1$.

For the proof of the theorem, we need the following result

Proposition 3. *Assume that* $g \in \mathcal{C}([0,+\infty[; H_0^1(0,1)) \cap \mathcal{C}^1(]0,+\infty[; L^2(0,1))$, *then there exists a unique solution* $f \in \mathcal{C}([0,+\infty[; H^1(\Omega)) \cap \mathcal{C}^1(]0,+\infty[; L^2(\Omega))$ *solving the boundary value problem*

$$\begin{cases} -\Delta f = 0 & \textit{in } \ \mathbb{R}^+_\star \times \Omega\,, \\ f = (-\partial_{xx})^{-\frac{1}{2}}\, g & \textit{in } \ \mathbb{R}^+_\star \times \Gamma_0\,, \\ f = 0 & \textit{in } \ \mathbb{R}^+_\star \times \Gamma_1\,. \end{cases} \tag{10}$$

Moreover, f_t *is continuous on* $\mathbb{R}^+ \times \overline{\Omega}$ *and there exists* $C > 0$, $C' > 0$ *such that*

$$\|f(t,\cdot)\|_{H^1(\Omega)} \;\leq\; C|g(t,\cdot)| \qquad \forall\, t > 0\,, \tag{11}$$

$$|f_t|_{\Gamma_0}(t,\cdot)| \;\leq\; C'|g_t(t,\cdot)| \qquad \forall\, t > 0\,. \tag{12}$$

Proof: The uniqueness is immediate since a harmonic function null on the boundary is identically null. For the existence, first we take

$$g(t,x) = \alpha_k(t) \sin(k\pi x)$$

for some integer $k \in \mathbb{N}^\star$ and a function $\alpha_k \in \mathcal{C}^{\,1}(\mathbb{R}^+)$. Then a direct computation shows that

$$f(t,x,y) \;=\; \alpha_k(t)\, \frac{\sin(k\pi x)}{k\pi}\, \frac{\sinh(k\pi(1-y))}{\sinh(k\pi)}\,.$$

Since $(v_k = \sqrt{2}\sin(k\pi x))_{k\geq 1}$ is a complete orthonormal system of $H_0^1(0,1)$, for general g in $\mathcal{C}([0,+\infty[; H_0^1(0,1)) \cap \mathcal{C}^1(]0,+\infty[; L^2(0,1))$, we write

$$g(t,x) \;=\; \sum_{k\geq 1} (g(t,\cdot), v_k)\, v_k$$

so putting

$$f_k(t,x,y) \;=\; (g(t,\cdot),v_k)\,\frac{\sin(k\pi x)}{k\pi}\,\frac{\sinh(k\pi(1-y))}{\sinh(k\pi)}$$

we get that $f=\sum_{k\geq 1} f_k$ is a solution of (10). Indeed the series $\sum_{k\geq 1} f_k$ converges uniformly in $[0,T]\times\overline{\Omega}$, $\forall\, T>0$ since

$$\begin{aligned}\Big|\sum_{k=m}^{n} f_k(t,x,v)\Big| \;&\leq\; \Big(\sum_{k=m}^{n}\Big|(g(t,\cdot),v_k)\Big|^2\Big)^{\frac12}\Big(\sum_{k=m}^{n}\frac{1}{k^2\pi^2}\Big)^{\frac12}\\ &\leq\; |g(t,\cdot)|\Big(\sum_{k=m}^{n}\frac{1}{k^2\pi^2}\Big)^{\frac12}\end{aligned}$$

for all $(t,x,y)\in[0,T]\times\overline{\Omega}$, then

$$\sup_{[0,T]\times\overline{\Omega}}\Big|\sum_{k=m}^{n} f_k(t,x,v)\Big| \;\leq\; \sup_{t\in[0,T]}|g(t,\cdot)|\Big(\sum_{k=m}^{n}\frac{1}{k^2\pi^2}\Big)^{\frac12} \;<\; \infty\;.$$

Similarly, using again the Cauchy–Shwartz inequality, we can easily show the uniform convergence of the series $\sum_{k\geq 1}\frac{\partial f_k}{\partial x}$ and $\sum_{k\geq 1}\frac{\partial f_k}{\partial y}$ in $[0,T]\times\overline{\Omega}$, $\forall\, T>0$. Furthermore

$$\begin{aligned}\int_\Omega\Big|\frac{\partial f}{\partial x}\Big|^2 dx\,dy \;&=\; \sum_{k\geq 1}\int_\Omega (g(t,\cdot),v_k)^2\,\cos^2(k\pi x)\,\frac{\sinh^2(k\pi(1-y))}{\sinh^2(k\pi)}\,dx\,dy\\ &\leq\; C|g(t,\cdot)|^2\;.\end{aligned}$$

The same can be done with $\int_\Omega\Big|\frac{\partial f}{\partial y}\Big|^2 dx\,dy$ and we obtain (11). For the proof of (12), we consider the series $\sum_{k\geq 1}(g_t(t,\cdot),v_k)\,\frac{\sin(k\pi x)}{k\pi}\,\frac{\sinh(k\pi(1-y))}{\sinh(k\pi)}$ and we show as above that it is uniformly convergent in $[0,T]\times\overline{\Omega}$, for all $T>0$, so we conclude that g_t is continuous on $\mathbb{R}^+\times\overline{\Omega}$ and

$$g_t(t,x,y) \;=\; \sum_{k\geq 1}(g_t(t,\cdot),v_k)\,\frac{\sin(k\pi x)}{k\pi}\,\frac{\sinh(k\pi(1-y))}{\sinh(k\pi)}\;.$$

Therefore

$$g_t(t,x,0) \;=\; \sum_{k\geq 1} (g_t(t,\cdot),v_k)\,\frac{\sin(k\pi x)}{k\pi}$$

and

$$|g_t|_{\Gamma_0}|^2 \;=\; \frac{1}{2}\sum_{k\geq 1}\Big|(g_t(t,\cdot),v_k)\Big|^2\frac{1}{k^2\pi^2}$$

so we obtain (12) thanks to Cauchy–Shwartz inequality. ∎

Proof of the Theorem: Let $Y_0=(u_0,u_1,v_0,v_1)\in D(A)$ and $Y(t)=(u,u_t,v,v_t)$ the solution of (4) provided by Proposition 1. Using the Proposition 3, we introduce the function q solution of the problem

$$\left\{\begin{array}{ll} -\Delta q=0 & \text{in } \mathbb{R}^+_\star\times\Omega\,, \\ q=(-\partial_{xx})^{-\frac{1}{2}}v_t & \text{in } \mathbb{R}^+_\star\times\Gamma_0\,, \\ q=0 & \text{in } \mathbb{R}^+_\star\times\Gamma_1\;. \end{array}\right. \tag{13}$$

Let ε, a and b be positive constants. We consider the function

$$\varrho_{\varepsilon,a,b}(t) \;=\; \frac{1}{2}\,\|Y(t)\|_X^2+\varepsilon\Big((u_t,u)+a(v_t,v)+b(u_t,q)-(u|_{\Gamma_0},(-\partial_{xx})^{\frac{1}{2}}v)\Big)\;. \tag{14}$$

Then the proof of the theorem will result from the following proposition.

Proposition 4. *There exists ε_0, a, $b>0$ such that*

$$M(\varepsilon)\,\|Y(t)\|_X^2\leq \varrho_{\varepsilon,a,b}(t)\leq N(\varepsilon)\,\|Y(t)\|_X^2\,,\qquad \forall\,t\geq 0,\quad \forall\,\varepsilon>0\;, \tag{15}$$

$$\varrho_{\varepsilon,a,b}(t)\leq \varrho_{\varepsilon,a,b}(0)\exp(-C(\varepsilon)\,t)\,,\qquad \forall\,t\geq 0,\quad \forall\,\varepsilon<\varepsilon_0\;, \tag{16}$$

where $M(\varepsilon)$, $N(\varepsilon)$ and $C(\varepsilon)$ are positive constants independent of Y.

Proof: One can easily obtain (15) by using (3) and Proposition 3. Now for (16), first since

$$\frac{1}{2}\frac{d}{dt}\|Y(t)\|_X^2=\Big\langle AY(t),Y(t)\Big\rangle_X=\frac{1}{2}\int_\Omega(\operatorname{div}d)u_t^2\,dx\,dy+\frac{1}{2}\int_0^1 u_t^2(t,x,0)\,d_2(x,0)\,dx$$

so according to (H), we obtain

$$\frac{1}{2}\frac{d}{dt}\|Y(t)\|_X^2\leq -C_0\Big(|u_t|^2+|u_t|_{\Gamma_0}|^2\Big)\;. \tag{17}$$

In the other hand, we have

$$\begin{aligned}\frac{d}{dt}(u_t,u) &= |u_t|^2 + (\Delta u - d.\nabla u_t, u)\\ &= |u_t|^2 - |\nabla u|^2 - ((-\partial_{xx})^{\frac12} v_t, u|_{\Gamma_0}) + (d.\nabla u, u_t)\\ &\quad + ((\operatorname{div} d)u_t, u) + \int_0^1 u_t\, u(t,x,0)\, d_2(x,0)\, dx\ ,\end{aligned}\tag{18}$$

$$\begin{aligned}\frac{d}{dt}(v_t,v) &= |v_t|^2 + \Big(v_{xx} + (-\partial_{xx})^{\frac12}(u_t|_{\Gamma_0}),\ v\Big)\\ &= |v_t|^2 - |v_x|^2 + (u_t|_{\Gamma_0}, (-\partial_{xx})^{\frac12} v)\ ,\end{aligned}\tag{19}$$

$$\begin{aligned}\frac{d}{dt}(u_t,q) &= (\Delta u - d.\nabla u_t, q) + (u_t, q_t)\\ &= -(\nabla u, \nabla q) - \Big((-\partial_{xx})^{\frac12} v_t, (-\partial_{xx})^{-\frac12} v_t\Big) + (d.\nabla q, u_t)\\ &\quad + (u_t \operatorname{div} d, q) + \int_0^1 u_t\, q(t,x,0)\, d_2(x,0)\, dx\ +\ (u_t, q_t)\ .\end{aligned}\tag{20}$$

Now, we consider the function h solving the problem

$$\begin{cases} -\Delta h = u_t & \text{in } \mathbb{R}_*^+ \times \Omega\,,\\ h = 0 & \text{in } \mathbb{R}_*^+ \times \Gamma\ .\end{cases}$$

Since $(-\Delta)$ is an isomorphism from $H^2(\Omega)\cap H_0^1(\Omega)$ into $L^2(\Omega)$, $h \in C([0,+\infty[; H^2(\Omega))$ and

$$\left|\frac{\partial h}{\partial n}|_{\Gamma_0}\right| \le C_1\, \|h\|_{H^2(\Omega)} \le C_1' |\Delta h|$$

so

$$\left|\frac{\partial h}{\partial n}|_{\Gamma_0}\right| \le C_1' |u_t| \tag{21}$$

with some positive constants C_1 and C_1'. Then since q satisfies (13), we get thanks to Proposition 3

$$(u_t,q_t) = -(\Delta h, q_t) = \left(\frac{\partial q_t}{\partial n}|_{\Gamma_0},\, h\right) - \left(\frac{\partial h}{\partial n}|_{\Gamma_0},\, q_t|_{\Gamma_0}\right) = \left(\frac{\partial h}{\partial n}|_{\Gamma_0},\, (-\partial_{xx})^{-\frac12} v_{tt}\right)$$

thus

$$(u_t,q_t) = -\left(\frac{\partial h}{\partial n}|_{\Gamma_0},\, (-\partial_{xx})^{\frac12} v + u_t|_{\Gamma_0}\right)\ . \tag{22}$$

Consequently (20) becomes

$$\begin{aligned}\frac{d}{dt}(u_t,q) &= -(\nabla u,\nabla q) - |v_t|^2 + (d.\nabla q, u_t) + ((\operatorname{div} d)u_t, q)\\ &\quad + \int_0^1 u_t\, q(t,x,0)\, d_2(x,0)\, dx - \left(\frac{\partial h}{\partial n}|_{\Gamma_0},\, (-\partial_{xx})^{\frac12} v\right) - \left(\frac{\partial h}{\partial n}|_{\Gamma_0},\, u_t|_{\Gamma_0}\right)\end{aligned}\tag{23}$$

whereas the last term of (14) gives

$$\frac{d}{dt}\Big(u|_{\Gamma_0}, (-\partial_{xx})^{\frac{1}{2}}v\Big) \;=\; \Big(u_t|_{\Gamma_0}, (-\partial_{xx})^{\frac{1}{2}}v\Big) + \Big(u|_{\Gamma_0}, (-\partial_{xx})^{\frac{1}{2}}v_t\Big)\;. \tag{24}$$

Note that the last term of (24) simplifies with the third term of the right hand side of (18).

In the following, C will denote different positive constants which can be taken as large as we want, α, β, γ, δ and θ will denote positive constants which will be chosen later. We have

$$|(d.\nabla u, u_t)| \;\le\; C\Big[\alpha|\nabla u|^2 + \tfrac{1}{\alpha}|u_t|^2\Big]\;, \tag{25}$$

$$|((\operatorname{div} d)u_t, u)| \;\le\; C\Big[\alpha|\nabla u|^2 + \tfrac{1}{\alpha}|u_t|^2\Big]\;, \tag{26}$$

$$\left|\int_0^1 u_t\, u(t,x,0)\, d_2(x,0)\, dx\right| \;\le\; C\Big[\alpha|\nabla u|^2 + \tfrac{1}{\alpha}|u_t|_{\Gamma_0}|^2\Big]\;, \tag{27}$$

$$\Big|(u_t|_{\Gamma_0}, (-\partial_{xx})^{\frac{1}{2}}v)\Big| \;\le\; C\Big[\theta|u_t|_{\Gamma_0}|^2 + \tfrac{1}{\theta}|v_x|^2\Big]\;, \tag{28}$$

$$|(\nabla u, \nabla q)| \;\le\; C\Big[\beta|\nabla u|^2 + \tfrac{1}{\beta}|v_t|^2\Big]\;, \tag{29}$$

$$|(d.\nabla q, u_t)| \;\le\; C\Big[\gamma|u_t|^2 + \tfrac{1}{\gamma}|v_t|^2\Big]\;, \tag{30}$$

$$|(u_t \operatorname{div} d, q)| \;\le\; C\Big[\gamma|u_t|^2 + \tfrac{1}{\gamma}|v_t|^2\Big]\;, \tag{31}$$

$$\left|\int_0^1 u_t\, q(t,x,0)\, d_2(x,0)\, dx\right| \;\le\; C\Big[\gamma|u_t|_{\Gamma_0}|^2 + \tfrac{1}{\gamma}|v_t|^2\Big]\;, \tag{32}$$

$$\left|\Big(\frac{\partial h}{\partial n}|_{\Gamma_0}, (-\partial_{xx})^{\frac{1}{2}}v\Big)\right| \;\le\; C\Big[\delta|v_x|^2 + \tfrac{1}{\delta}|u_t|^2\Big]\;, \tag{33}$$

$$\left|\Big(\frac{\partial h}{\partial n}|_{\Gamma_0},\, u_t|_{\Gamma_0}\Big)\right| \;\le\; C\Big[|u_t|_{\Gamma_0}|^2 + |u_t|^2\Big]\;. \tag{34}$$

Note that the estimates (25), (26) and (30) derive from the hypothesis (H), (28) from (3), while (29), (30), (31) and (32) result from (13) and (11). Finally (33) and (34) follow from (21) and (3). All these inequalities as well as (17) give

$$\begin{aligned}
\frac{d}{dt}\varrho_{\varepsilon,a,b}(t) \;\le\;& \varepsilon\Big[-1 + C(3\alpha + b\beta)\Big]\,|\nabla u|^2 + \varepsilon\Big[a - b + C\,b\Big(\frac{1}{\beta} + \frac{3}{\gamma}\Big)\Big]\,|v_t|^2\\
&+ \varepsilon\Big[-a + C\Big(\frac{a+1}{\theta} + b\,\delta\Big)\Big]\,|v_x|^2\\
&+ \Big(-C_0 + \varepsilon\Big[1 + C\Big(\frac{2}{\alpha} + 2\,b\,\gamma + b\Big(\frac{1}{\delta} + 1\Big)\Big)\Big]\Big)\,|u_t|^2\\
&+ \Big(-C_0 + \varepsilon\Big[1 + \varepsilon\, C\Big(\theta(1+a) + \frac{1}{\alpha} + b(1+\gamma)\Big)\Big]\Big)\,|u_t|_{\Gamma_0}|^2\;.
\end{aligned} \tag{35}$$

Therefore according to (15), to get (16), it is enough to prove that for some constant $D(\varepsilon) > 0$

$$\frac{d}{dt}\varrho_{\varepsilon,a,b}(t) \;\leq\; -D(\varepsilon)\,\|Y(t)\|_X^2 \; . \tag{36}$$

But this last inequality hold true if we have

$$-1 + C(3\alpha + b\beta) < 0\,, \qquad a - b + C\,b\Big(\frac{1}{\beta} + \frac{3}{\gamma}\Big) < 0 \;, \tag{37}$$

$$-a + C\Big(\frac{a+1}{\theta} + b\,\delta\Big) < 0 \;, \tag{38}$$

$$-C_0 + \varepsilon\Big[1 + C\Big(\frac{2}{\alpha} + 2\,b\beta + b\Big(\frac{1}{\delta} + 1\Big)\Big)\Big] < \;0 \;, \tag{39}$$

$$-C_0 + \varepsilon\Big[1 + \varepsilon\,C\Big(\theta(1+a) + \frac{1}{\alpha} + b(1+\gamma)\Big)\Big] \leq \;0 \;. \tag{40}$$

Since the inequalities (37) are equivalent to

$$\frac{1}{C} - b\beta > 3\alpha\,, \qquad \frac{b-a}{C} - \frac{b}{\beta} > \frac{3b}{\gamma} \tag{41}$$

we will choose the positive constants $a, b, \alpha, \beta, \gamma$ satisfying

$$\frac{1}{C} - b\beta > 0 \quad \text{and} \quad \frac{b-a}{C} - \frac{b}{\beta} > 0$$

thus we must have

$$\frac{bC}{b-a} < \beta < \frac{1}{bC} \quad \text{with} \;\; b - a > 0 \; .$$

Consequently, it is necessary to have

$$\frac{bC}{b-a} < \frac{1}{bC}$$

that is

$$C^2 b^2 - b + a < 0 \; .$$

We see that this last inequality is fulfilled when we take a and b such that

$$0 < a < \frac{1}{4C^2}\,, \qquad \frac{1-\sqrt{1-4aC^2}}{2C^2} < b < \frac{1+\sqrt{1-4aC^2}}{2C^2} \tag{42}$$

then according to what precede, we take β satisfying

$$\frac{bC}{b-a} < \beta < \frac{1}{bC} \; . \tag{43}$$

Therefore the inequalities (41) are satisfied if

$$\alpha < \frac{\frac{1}{C} - b\beta}{3}\,, \qquad \gamma > \frac{3b}{\frac{b-a}{C} - \frac{b}{\beta}} \tag{44}$$

and in order to obtain (38), we choose δ and θ such that

$$0 < \delta < \frac{a}{bC}\,, \qquad \theta > \frac{a+1}{\frac{a}{C} - b\delta}\,. \tag{45}$$

Finally, choosing ε small enough, we get (39) and (40), which ends the proof of (36).

Hence combining (36) and (15), we deduce that

$$\frac{d}{dt}\,\varrho_{\varepsilon,a,b}(t) \;\leq\; -\frac{D(\varepsilon)}{N(\varepsilon)}\,\varrho_{\varepsilon,a,b}(t)$$

which leads to (16) with $C(\varepsilon) = \dfrac{D(\varepsilon)}{N(\varepsilon)}$. Using again (15), we obtain

$$\|Y(t)\|_X^2 \leq \frac{1}{M(\varepsilon)}\,\varrho_{\varepsilon,a,b}(0)\,\exp\!\left(-\frac{D(\varepsilon)}{N(\varepsilon)}\,t\right), \qquad \forall\, t \geq 0$$

and (9), which ends the proof of the theorem. ∎

ACKNOWLEDGEMENTS – We make a point of thanking Professors Assia Benabdallah and Farid Ammar-Khodja for their precious help.

REFERENCES

[1] Ammar Khodja, F.; Benabdallah, A. and Téniou, D. – Sur la stabilisation d'un couplage ondes-advection, *Portugaliae Mathematica,* 54(3) (1997), 335–344.

[2] Ammar Khodja, F.; Benabdallah, A. and Téniou, D. – Coupled systems, *Abstract and Applied Analysis,* 1(3) (1996), 327–340.

[3] Ammar Khodja, F. and Bader, A. – Stabilizability of systems of one-dimensionnal wave equations by one internal or boundary control force, *SIAM Journal on Control and Optimization,* 39(6) (2001), 1833–1851.

[4] Cazenave, T. and Dickstein, F. – On the initial value problem for a linear model of well reservoir coupling, *Nonlinear World,* 3 (1996), 567–587.

[5] Grisvard, P. – *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston, 1985.

[6] Komornik, V. and Rao, B. – Stabilisation frontiére d'un système d'équations des ondes, *C.R. Acad. Sci. Paris, série I,* 320 (1995), 833–838.

[7] Littman, W. and Liu, B. – On the spectral properties and stabilization of acoustic flow, *Society for Industrial and Applied Mathematics,* (1998), 17–34.

[8] Micu, S. and Zuazua, E. – Stabilization and periodic solutions of a hybrid system arising in the control of noise, *C.R. Acad. Sci. Paris,* (1994).

[9] Micu, S. and Zuazua, E. – Propriétés qualitatives d'un modèle bi-dimensionnel intervenant dans le contrôle du bruit, *C.R. Acad. Sci. Paris, série I,* 319 (1994), 1263–1268.

N. Aissa and D. Hamroun,
USTHB, Faculté de Mathématiques,
BP 32 El Alia, Bab Ezzouar – ALGER
E-mail: `djamroun@yahoo.fr`