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# THE INVARIANT SUBRINGS OF DEMEYER–KANZAKI GALOIS EXTENSIONS

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**Abstract:** Let $B$ be a ring with 1, $G$ a finite automorphism group of $B$, $C$ the center of $B$, $B^G$ the set of elements in $B$ fixed under each element in $G$. When $B$ is a DeMeyer–Kanzaki Galois extension of $B^G$ with Galois group $G$, it was shown that a separable subring $S$ of $B$ over $B^G$ is equal to $B^K$ for some subgroup $K$ of $G$ if and only if $CJ_g^{(S)}$ is a faithful $C$-module for each $g \notin K$ where $J_g^{(S)} = \{s - g(s) \mid s \in S\}$. Moreover, the invariant subrings of $C$ over $C^G$ (i.e., $S = C^K$ for some subgroup $K$ of $G$) and of $B * G$ over $(B * G)^{\bar{G}}$ are characterized in terms of the faithful $B$-module $BJ_g^{(S)}$ and the faithful $C^G$-module $C^G J_g^{(S)}$ respectively for $g \in G$.

## 1 – Introduction

Throughout this paper, $B$ will represent a ring with 1, $G$ a finite automorphism group of $B$, $C$ the center of $B$, $B^G$ the set of elements in $B$ fixed under each element in $G$, $B * G$ a skew group ring over $B$ in which the multiplication is given by $gb = g(b)g$ for $b \in B$ and $g \in G$, and $\bar{G}$ the inner automorphism group of $B * G$ induced by $G$, that is, $\bar{g}(f) = gfg^{-1}$ for each $f \in B * G$ and $g \in G$. We note that $\bar{G}$ restricted to $B$ is $G$.

Following the notations and facts in [5], $B$ is called a Galois extension of $B^G$ with Galois group $G$ if there exist elements $\{c_i, d_i$ in $B$, $i = 1, 2, ..., m\}$ for some integer $m$ such that $\sum_{i=1}^{m} c_i\, g(d_i) = \delta_{1,g}$ for each $g \in G$. Such a set $\{c_i, d_i\}$ is

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called a $G$-Galois system for $B$. $B$ is called a center Galois extension of $B^G$ if $C$ is a Galois algebra over $C^G$ with Galois group $G|_C \cong G$. Let $A$ be a subring of a ring $B$ with the same identity 1. $V_B(A)$ denotes the commutator subring of $A$ in $B$. We call $B$ a separable extension of $A$ if there exist $\{a_i, b_i$ in $B$, $i = 1, 2, ..., m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all $b$ in $B$ where $\otimes$ is over $A$, and an Azumaya algebra is a separable extension of its center. $B$ is called a DeMeyer–Kanzaki Galois extension with Galois group $G$ if $B$ is an Azumaya $C$-algebra and a center Galois extension with Galois group $G$. A ring $F$ is called a $H$-separable extension of $B$ if $F \otimes_B F$ is isomorphic to a direct summand of a finite direct sum of $F$ as a $F$-bimodule. $S$ is called a $D$-$S$-separable extension of $A$ in $B$ if $S$ is a separable extension of $A$ in $B$ and a direct summand of a finite direct sum of $B$ as a bimodule over $S$ ([3]). We denote $\{s - g(s) \mid s \in S\}$ by $J_g^{(S)}$ and the $A$-module generated by $J_g^{(S)}$ by $AJ_g^{(S)}$ for $g \in G$.

The fundamental theorem for Galois extensions of a field or a commutative ring with no idempotents but 0 and 1 states that there exists a one-to-one correspondence between the set of subgroups of the Galois group $G$ and the set of separable subrings of the Galois extension ([1], Chapter 3). In general, there exists no such a correspondence for Galois extensions of rings although there are some kind of correspondeces between certain sets of separable extensions of rings ([2]). For a Galois extension $B$ it is easy to see that the map from the set of subgroups of $G$ to the set of separable extensions of $B^G$ in $B$ given by $K \to B^K$ is one-to-one but not necessarily onto. So it is interesting to know what kind of separable subrings of $B$ is invariant under a subgroup $K$ of $G$. The purpose of the present paper is to characterize for a DeMeyer–Kanzaki Galois extension $B$ the invariant separable subrings $S$ of $B$ over $B^G$, of $C$ over $C^G$, and of $B * G$ over $(B * G)^{\bar{G}}$ respectively.

## 2 – Main results

In this section, we first characterize for a DeMeyer–Kanzaki Galois extension $B$ the invariant separable subrings $S$ of $B$ over $B^G$, and then characterize for a center Galois extension $B$ the invariant separable subrings $S$ of $C$ over $C^G$ and of $B * G$ over $(B * G)^{\bar{G}}$ respectively. Consequently, results are derived for a DeMeyer–Kanzaki Galois extension $B$ of $B^G$. We first give three lemmas.

**Lemma 1.** *Let $B$ be a ring. Then, $BJ_g^{(C)}$ is a faithful $B$-module for each $g \neq 1$ if and only if $V_{B*G}(C) = B$ where $J_g^{(C)} = \{c - g(c) \mid s \in C\}$ and $V_{B*G}(C)$ is the commutator subring of $C$ in $B * G$.*

**Proof:** ($\Rightarrow$) Clearly, $B \subset V_{B*G}(C)$. Let $\sum_{g\in G} b_g g$ in $V_{B*G}(C)$ for some $b_g \in B$. Then $c(\sum_{g\in G} b_g g) = (\sum_{g\in G} b_g g)c$ for each c in $C$, so $cb_g = b_g g(c)$, that is, $b_g(c - g(c)) = 0$ for each $g \in G$ and $c \in C$. Since $BJ_g^{(C)}$ is a faithful $B$-module for each $g \neq 1$, $b_g = 0$ for each $g \neq 1$. But then $\sum_{g\in G} b_g g = b_1 \in B$. Hence $V_{B*G}(C) \subseteq B$, and so $V_{B*G}(C) = B$.

($\Leftarrow$) By the above argument, we have that $V_{B*G}(C) = \{\sum_{g\in G} b_g g \mid b_g J_g^{(C)} = \{0\}$ for each $g \in G\}$. Thus, $V_{B*G}(C) = B$ implies that $BJ_g^{(C)}$ is a faithful $B$-module for each $g \neq 1$. ∎

**Lemma 2.** *Let $B$ be a ring such that $B = B^G C$, $S$ a subring of $B$ over $B^G$, and $K = \{g \in G \mid g(s) = s$ for all $s \in S\}$. Then $CJ_g^{(S)}$ is a faithful $C$-module for each $g \notin K$ if and only if $V_{B*G}(S) = C * K$.*

**Proof:** By hypothesis, $B = B^G C$. So $V_B(B^G) = V_B(B^G C) = V_B(B) = C$. Hence $V_{B*G}(B^G) = V_B(B^G) * G = C * G$. But $B^G \subset S$, so $V_{B*G}(S) \subset V_{B*G}(B^G) = C * G$. Thus, $V_{B*G}(S) = V_{C*G}(S)$. By a direct computation, $V_{C*G}(S) = C * K \oplus \sum_{g\notin K} I_g g$ where $I_g = \{c \in C \mid c(s - g(s)) = 0$ for each $s \in S\} = \mathrm{Ann}_C(J_g^{(S)})$, the annihilator of the $C$-module $CJ_g^{(S)}$. Therefore, $CJ_g^{(S)}$ is a faithful $C$-module for each $g \notin K$ if and only if $V_{B*G}(S) = C * K$. ∎

**Lemma 3.** *Assume that $B$ is a ring such that $B = B^G C$ and $BJ_g^{(C)}$ is a faithful $B$-module for each $g \neq 1$. Let $S$ be a subring of $B$ over $B^G$ and $K = \{g \in G \mid g(s) = s$ for all $s \in S\}$. Then, $S = B^K$ and $C * K$ satisfies the double centralizer property in $B * G$ if and only if $CJ_g^{(S)}$ is a faithful $C$-module for each $g \notin K$ and $S$ satisfies the double centralizer property in $B * G$.*

**Proof:** ($\Leftarrow$) Since $CJ_g^{(S)}$ is a faithful $C$-module for each $g \notin K$, $V_{B*G}(S) = C*K$ by Lemma 2. Hence $V_{B*G}(V_{B*G}(S)) = V_{B*G}(C * K) = (V_{B*G}(C))^{\bar{K}} = B^K$ by Lemma 1 (for $BJ_g^{(C)}$ is a faithful $B$-module for each $g \neq 1$). But $V_{B*G}(V_{B*G}(S)) = S$ by hypothesis, so $S = B^K$, and $V_{B*G}(V_{B*G}(C * K)) = V_{B*G}(S) = C * K$.

($\Rightarrow$) By hypothesis, $BJ_g^{(C)}$ is a faithful $B$-module for each $g \neq 1$, so $V_{B*G}(C) = B$ by Lemma 1. Hence $V_{B*G}(C * K) = (V_{B*G}(C))^{\bar{K}} = B^K = S$ by hypothesis. Thus $V_{B*G}(S) = V_{B*G}(V_{B*G}(C * K)) = C * K$. Therefore, $CJ_g^{(S)}$ is a faithful $C$-module for each $g \notin K$ by Lemma 2. Moreover, $V_{B*G}(V_{B*G}(S)) = V_{B*G}(C * K) = S$. This completes the proof. ∎

We now show a characterization for an invariant separable subring $S$ of $B$ over $B^G$ for a DeMeyer–Kanzaki Galois extension.

**Theorem 4.** *If $B$ is a DeMeyer–Kanzaki Galois extension of $B^G$ with Galois group $G$, $S$ a separable subring of $B$ over $B^G$, and $K = \{g \in G \mid g(s) = s$ for all $s \in S\}$. Then, $S = B^K$ if and only if $CJ_g^{(S)}$ is a faithful $C$-module for each $g \notin K$.*

**Proof:** Since $B$ is a DeMeyer–Kanzaki Galois extension of $B^G$ with Galois group $G$, $B$ is an Azumaya $C$-algebra and $B$ is a center Galois extension of $B^G$. Hence, by Theorem 3.2 and Lemma 3.1 in [5], $V_{B*G}(B) = C$, so $V_{B*G}(B * G) = (V_{B*G}(B))^{\bar{G}} = C^G$, that is, $C^G$ is the center of $B * G$. Since $B$ is a center Galois extension of $B^G$ again, $B * G$ is $H$-separable over $B$ and $C$ is separable over $C^G$. Hence, $B * G$ is separable over $C^G$ by the transitivity of separable extensions. Thus, $B * G$ is an Azumaya $C^G$-algebra. Since $S$ is a separable extension over $B^G$ which is separable over $C^G$, $S$ is a separable $C^G$-subalgebra of the Azumaya algebra $B * G$ by the transitivity of separabe extensions. Hence $S$ satisfies the double centralizer property in $B * G$ ([1], Theorem 4.3, page 57). On the other hand, by definition of the DeMeyer–Kanzaki Galois extension, $C$ is a commutative Galois extension of $C^G$ with Galois group $G$, so for any subgroup $K$ of $G$, $C$ is a Galois extension of $C^K$ with Galois group $K$ with the same Galois system. Hence $C * K$ is an Azumaya $C^K$-algebra and $C^K$ is separable over $C^G$, and so $C * K$ is separable over $C^G$ by the transitivity of separabe extensions. Thus, $C * K$ also satisfies the double centralizer property in $B * G$ for $B * G$ is an Azumaya $C^G$-algebra. Moreover, since $B$ is a center Galois extension of $B^G$ with Galois group $G$, by Theorem 3.2 in [5], $B = B^G C$ and $BJ_g^{(C)} = B$, which is a faithful $B$-module, for each $g \neq 1$ in $G$. Therefore, Theorem 4 holds by Lemma 3. ∎

To characterize for a center Galois extension $B$ the invariant separable subrings $S$ of $C$ over $C^G$ and of $B * G$ over $(B * G)^{\bar{G}}$ respectively, Theorem 1 in [3] plays an important role. For convenient, we state it here as a proposition.

**Proposition 5.** ([3], Theorem 1) *Let $A$ be a $H$-separable extension of $E$. Then if $A$ is left or right $E$-finitely generated projective, there exists a one-to-one correspondence $V\colon S \to V_A(S)$ such that $V^2$ is an identity between the set of $D$-$S$-separable extensions of $E$ in $A$ and the set of $Z(A)$-separable subalgebras of $V_A(E)$ where $Z(A)$ is the center of $A$.* ∎

**Theorem 6.** *Let $B$ be a center Galois extension of $B^G$, $S$ a separable extension of $C^G$ in $C$ and $K = \{g \in G \mid g(s) = s$ for all $s \in S\}$. Then, $S = C^K$ if and only if $BJ_g^{(S)}$ is a faithful $B$-module for each $g \notin K$.*

**Proof:** (⇐) By a direct computation, we have $V_{B*G}(S) = B*K \oplus \sum_{g\notin K} I_g\, g$ where $I_g = \mathrm{Ann}_B(BJ_g^{(S)})$. But, $BJ_g^{(S)}$ is a faithful $B$-module for each $g \notin K$, so $I_g = \{0\}$ for each $g \notin K$; and so $V_{B*G}(S) = B*K$. Hence, $V_{B*G}(V_{B*G}(S)) = V_{B*G}(B*K) = (V_{B*G}(B))^{\bar{K}} = C^K$. Next, we prove that $S$ satisfies the double centralizer property in $B*G$; and so $S = V_{B*G}(V_{B*G}(S)) = C^K$. In fact, since $B$ is a center Galois extension of $B^G$, $B = BJ_g^{(C)}$ for each $g \neq 1$ in $G$ ([5], Theorem 3.2). Hence, $B*G$ is $H$-separable over $B$ and $B$-finitely generated projective ([5], Lemma 3.1-(3)). Moreover, by Lemma 3.1-(4) in [5], $V_{B*G}(B) = C$. Therefore, $S$ is a separable $C^G$-subalgebra of $V_{B*G}(B)(= C)$. Thus, $V_{B*G}(V_{B*G}(S)) = S$ by Proposition 5.

(⇒) By the above argument, $V_{B*G}(B) = C$ and $V_{B*G}(S) = B*K \oplus \sum_{g\notin K} I_g\, g$. Hence, to show that $BJ_g^{(S)}$ is a faithful $B$-module for each $g \notin K$, that is, $I_g = \{0\}$ for each $g \notin K$, it suffices to show that $V_{B*G}(S) = B*K$. Since $S = C^K$, $V_{B*G}(S) = V_{B*G}(C^K) = V_{B*G}((V_{B*G}(B))^{\bar{K}}) = V_{B*G}(V_{B*G}(B*K))$. Therefore, we only need to show that $B*K$ satifies the double centralizer property in $B*G$. Since $B*G$ is $H$-separable over $B$ and $B$-finitely generated projective again, $V_{B*G}(S)$ is a $D$-$S$-separable extension of $B$ in $B*G$ by Proposition 5 (for $S$ is a separable $C^G$-subalgebra of $C(= V_{B*G}(B))$). Next we claim that $B*K$ is a $D$-$S$-separable extension of $B$ in $B*G$, and so $V_{B*G}(V_{B*G}(B*K)) = B*K$ by Proposition 5. In fact, since $C$ is a Galois extension of $C^G$, $C$ is a Galois extension of $C^K$ with the same Galois system. Hence $B*K$ is separable over $B$ by Lemma 3.1-(3) in [5]. Moreover, Since $V_{B*G}(S)$ is a direct summand of a finite direct sum of $B*G$ as a bimodule over $V_{B*G}(S)$ and $V_{B*G}(S) = B*K \oplus \sum_{g\notin K} I_g\, g$, $B*K$ will be a direct summand of a finite direct sum of $B*G$ as a bimodule over $B*K$ if we can show that $\sum_{g\notin K} I_g g$ is a $B*K$-bimodule. In fact, for any $b \in B$ and $k \in K$ and for any $b_g \in I_g$ with $g \notin K$, $(bk)(b_g g) = bk(b_g)(kg)$. Since $k \in K$ and $g \notin K$, $kg \notin K$. Moreover, for any $s \in S$, $(bk(b_g))\,(s - (kg)(s)) = bk(b_g)\,(k(s) - (kg)(s)) = bk(b_g(s - g(s)) = 0$ since $b_g \in I_g$. Hence $bk(b_g) \in I_{kg}$, and so $bk(b_g)(kg) \in \sum_{h\notin K} I_h\, h$. Thus $\sum_{h\notin K} I_h\, h$ is a left $B*K$-module. Similarly, $(b_g g)(bk) = (b_g g(b))(gk)$ with $gk \notin K$ and for any $s \in S$, $(b_g)\, g(b)\,(s - (gk)(s)) = b_g\, g(b)\,(s - g(s)) = (b_g(s - g(s))\, g(b) = 0$ since $b_g \in I_g$. Hence $(b_g\, g(b)) \in I_{gk}$, and so $(b_g\, g)(bk) \in \sum_{h\notin K} I_h\, h$. Thus $\sum_{h\notin K} I_h\, h$ is a right $B*K$-module. Therefore, $\sum_{g\notin K} I_g\, g$ is a $B*K$-bimodule. This completes the proof. ∎

**Corollary 7.** *Let $B$ be a DeMeyer–Kanzaki Galois extension of $B^G$, $S$ a separable extension of $C^G$ in $C$ and $K = \{g \in G \mid g(s) = s$ for all $s \in S\}$. Then, $S = C^K$ if and only if $BJ_g^{(S)}$ is a faithful $B$-module for each $g \notin K$.*

**Proof:** Since a DeMeyer–Kanzaki Galois extension is a center Galois extension, the Corollary is an immediate consequence of Theorem 6. ∎

**Corollary 8.** *Let $C$ be a commutative Galois extension of $C^G$, $S$ a separable extension of $C^G$ in $C$ and $K = \{g \in G \mid g(s) = s$ for all $s \in S\}$. Then, $S = C^K$ if and only if $CJ_g^{(S)}$ is a faithful $C$-module for each $g \notin K$.*

**Proof:** Let $B = C$ in Theorem 6 or Theorem 4. ∎

Next, we give a characterization for an invariant separable subring of $B * G$ over $(B * G)^{\bar{G}}$ for some subgroup $K$ of $G$.

**Theorem 9.** *Let $B$ be a center Galois extension of $B^G$ with Galois group $G$ of order $n$ invertible in $B$, $S$ a $D$-$S$-separable extension of $(B*G)^{\bar{G}}$ in $B*G$, and $K = \{g \in G \mid \bar{g}(s) = s$ for all $s \in S\}$. Then, $S = (B*G)^{\bar{K}}$ if and only if $C^G J_g^{(S)}$ is a faithful $C^G$-module for each $g \notin K$.*

**Proof:** $(\Rightarrow)$ Since $B$ is a Galois extension of $B^G$, $B*G$ is a Galois extension of $(B*G)^{\bar{G}}$ with the same Galois system for $B$. Hence $B*G$ is right $(B*G)^{\bar{G}}$-finitely generated projective. Moreover, since elements in $\bar{G}$ are inner, $B*G$ is $H$-separable over $(B*G)^{\bar{G}}$ by Corollary 3 in [4]. Noting that $K$ is a subgroup of $G$, we have that $C^G K \subset C^G G \subset V_{B*G}(V_{B*G}(C^G G)) = V_{B*G}((B*G)^{\bar{G}})$. Hence, $C^G K$ is a separable $C^G$-subalgebra of $V_{B*G}((B*G)^{\bar{G}})$. Thus $C^G K$ satisfies the double centralizer property in $B*G$ by Proposition 5. Now, since $S = (B*G)^{\bar{K}}$, $V_{B*G}(S) = V_{B*G}((B*G)^{\bar{K}}) = V_{B*G}(V_{B*G}(C^G K)) = C^G K$. For any $c \in C^G$ such that $cJ_g^{(S)} = \{0\}$, we have $cs = cg(s)$, and so $s(cg) = (sc)g = csg = cg(s)g = (cg)s$ for all $s \in S$. Hence $cg \in V_{B*G}(S)$. But $V_{B*G}(S) = C^G K$, so $c = 0$ for each $g \notin K$. This implies that $C^G J_g^{(S)}$ is a faithful $C^G$-module for each $g \notin K$.

$(\Leftarrow)$ By the above argument, $B*G$ is $H$-separable over $(B*G)^{\bar{G}}$ and right $(B*G)^{\bar{G}}$-finitely generated projective. Since $S$ is a $D$-$S$-separable extension of $(B*G)^{\bar{G}}$ in $B*G$, $S$ satisfies the double centralizer property in $B*G$ by Proposition 5. Since $B$ is a center Galois extension of $B^G$, $V_{B*G}(B) = C$, so $V_{B*G}(B*G) = (V_{B*G}(B))^{\bar{G}} = C^G$, that is, $C^G$ is the center of $B*G$. But, $n$ is invertible in $C^G$, so $C^G G$ is $C^G$-separable subalgebra of $V_{B*G}((B*G)^{\bar{G}})$. Hence $V_{B*G}(V_{B*G}(C^G G)) = C^G G$ by Proposition 5. Now, by hypothesis, $(B*G)^{\bar{G}} \subset S$. Hence $V_{B*G}(S) \subset V_{B*G}((B*G)^{\bar{G}}) = V_{B*G}(V_{B*G}(C^G G)) = C^G G$. Therefore, $V_{B*G}(S) = V_{C^G G}(S) = C^G K \oplus \sum_{g \notin K} I_g \, g$ where $I_g = \mathrm{Ann}_{C^G}(J_g^{(S)})$. Since $J_g^{(S)}$ is a faithful $C^G$-module for each $g \notin K$, $V_{B*G}(S) = C^G K$. Therefore, $S = V_{B*G}(V_{B*G}(S)) = V_{B*G}(C^G K) = (B*G)^{\bar{K}}$. This completes the proof. ∎

## 3 – Examples

In this section, we give two examples to demonstrate our results and show that Theorem 4 does not hold for a center Galois extension $B$ in general.

**Example 1.** Let $Q$ be the rational field, $C = Q \oplus Q \oplus Q \oplus Q \oplus Q \oplus Q$, $B = C[i,j,k]$ the quaternion algebra over $C$, and $G = \langle g \rangle$, the cyclic group generated by $g$, where $g(a_1,a_2,a_3,a_4,a_5,a_6) = (a_2,a_3,a_4,a_5,a_6,a_1)$ for $a_i \in Q$ and $g(c_1+c_2i+c_3j+c_4k) = g(c_1)+g(c_2)i+g(c_3)j+g(c_4)k$ for $c_1+c_2i+c_3j+c_4k \in B$. Then

(**1**) The center of $B$ is $C$.

(**2**) $C^G = \{(a,a,a,a,a,a) |\ a \in Q\} \cong Q$.

(**3**) $B^G = C^G[i,j,k] \cong Q[i,j,k]$.

(**4**) $B$ is an Azumaya $C$-algebra.

(**5**) $C$ is a Galois extension of $C^G$ with Galois group $G|_C \cong G$ with a Galois system $\{e_l, \frac{1}{6}e_l |\ l = 1,2,3,4,5,6\}$ where $e_l$ is the element in C with $l^{\text{th}}$ component 1 and elsewhere 0.

(**6**) By (4) and (5), $B$ is a DeMeyer–Kanzaki Galois extension of $B^G$ with Galois group $G$.

(**7**) The nontrivial subgroups of $G$ are $K_1 = \{1, g^3\}$ and $K_2 = \{1, g^2, g^4\}$.

(**8**) $B^{K_1} = C^{K_1}[i,j,k]$ where $C^{K_1} = \{(a_1,a_2,a_3,a_1,a_2,a_3) |\ a_1,a_2,a_3 \in Q\}$ and $B^{K_2} = C^{K_2}[i,j,k]$ where $C^{K_2} = \{(a_1,a_2,a_1,a_2,a_1,a_2) |\ a_1,a_2 \in Q\}$.

(**9**) $J_g^{(B^{K_1})} = J_{g^2}^{(B^{K_1})} = J_{g^4}^{(B^{K_1})} = J_{g^5}^{(B^{K_1})} = B^{K_1} = C^{K_1}[i,j,k]$ are faithful $C$-modules. $J_g^{(B^{K_2})} = J_{g^3}^{(B^{K_2})} = J_{g^5}^{(B^{K_2})} = \{(b,-b,b,-b,b,-b) |\ b \in Q[i,j,k]\}$ are faithful $C$-modules.

(**10**) Let $S = \{(b_1,b_1,b_2,b_1,b_1,b_2) |\ b \in Q[i,j,k]\}$. Then $S (\cong (Q \oplus Q)[i,j,k])$ is separable over $B^G (\cong Q[i,j,k])$, $K = \{g \in G \mid g(s) = s \text{ for all } s \in S\} = K_1 = \{1, g^3\}$, and $S \neq B^{K_1}$ and $J_g^{(S)} = \{(b,0,-b,b,0,-b) |\ b \in Q[i,j,k]\}$ is not a faithful $C$-module. $\square$

**Example 2.** Let $Q$, $C$, and $G$ acts on $C$ as given in Example 1. Let $A_2(Q) = \{\left(\begin{smallmatrix} q_1 & q_2 \\ 0 & q_3 \end{smallmatrix}\right) |\ q_1,q_2,q_3 \in Q\}$, the ring of all 2 by 2 upper triangular matrices over $Q$, $B = \{\left(\begin{smallmatrix} c_1 & c_2 \\ 0 & c_3 \end{smallmatrix}\right) |\ c_1,c_2,c_3 \in C\}$ $(\cong A_2(Q) \otimes_Q C)$, and $g\left(\begin{smallmatrix} c_1 & c_2 \\ 0 & c_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} g(c_1) & g(c_2) \\ 0 & g(c_3) \end{smallmatrix}\right)$

for all $\left(\begin{smallmatrix} c_1 & c_2 \\ 0 & c_3 \end{smallmatrix}\right) \in B$. Then

(**1**) The center of $B$ is $\{\left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right) \mid c \in C\} \cong C$.

(**2**) $C^G = \{(a,a,a,a,a,a) |\ a \in Q\} \cong Q$ as given in Example 1-(2).

(**3**) $C$ is a Galois extension of $C^G$ with Galois group $G|_C \cong G$ as shown in Example 1-(5). Hence $B$ is a center Galois extension.

(**4**) By the argument in Example 4.3-(8) in [6], $B$ is not an Azumaya $C$-algebra. Hence $B$ is not a DeMeyer–Kanzaki Galois extension of $B^G$ with Galois group $G$.

(**5**) $B^G = \{\left(\begin{smallmatrix} c_1 & c_2 \\ 0 & c_3 \end{smallmatrix}\right) |\ c_1, c_2, c_3 \in C^G\} \cong A_2(Q)$.

(**6**) Let $S_C = \{(q_1, 2q_1, q_2, q_1, 2q_1, q_2) |\ q_1, q_2 \in Q\}$ $(\cong Q \oplus Q)$ and $S = \{\left(\begin{smallmatrix} c_1 & c_2 \\ 0 & c_3 \end{smallmatrix}\right) |\ c_1, c_2, c_3 \in S_C\}$ $(\cong A_2(Q) \otimes_Q (Q \oplus Q))$. Then $S$ is separable over $B^G$, $K = \{g \in G \mid g(s) = s \text{ for all } s \in S\} = \{1, g^3\}$, $S \neq B^K = \{\left(\begin{smallmatrix} c_1 & c_2 \\ 0 & c_3 \end{smallmatrix}\right) |\ c_1, c_2, c_3 \in C^K\}$ $(\cong A_2(Q) \otimes_Q (Q \oplus Q \oplus Q))$ where $C^K = \{(q_1, q_2, q_3, q_1, q_2, q_3) |\ q_1, q_2, q_3 \in Q\} \cong (Q \oplus Q \oplus Q)$. But $J_g^{(S)} = J_{g^2}^{(S)} = J_{g^4}^{(S)} = J_{g^5}^{(S)} = \{\left(\begin{smallmatrix} c_1 & c_2 \\ 0 & c_3 \end{smallmatrix}\right) |\ c_1, c_2, c_3 \in J_C\}$ where $J_C = \{(a, b, -a-b, a, b, -a-b) |\ a, b \in Q\}$, so $CJ_g^{(S)} = CJ_{g^2}^{(S)} = CJ_{g^4}^{(S)} = CJ_{g^5}^{(S)} = B$ are faithful $C$-modules even through $S \neq B^K$. Hence Theorem 4 does not hold for a center Galois extension $B$ in general. □

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## REFERENCES

**[1]** DeMeyer, F.R. and Ingraham, E. – Separable algebras over commutative rings, *Volume 181, Springer Verlag, Berlin, Heidelberg, New York, 1971.*

**[2]** DeMeyer, F.R. – Some notes on the general Galois theory of rings, *Osaka J. Math.,* 2 (1965), 117–127.

**[3]** Sugano, K. – On centralizers in separable extensions II, *Osaka J. Math.,* 8 (1971), 465–469.

**[4]** Sugano, K. – On a special type of Galois extensions, *Hokkaido J. Math.,* 9 (1980), 123–128.

**[5]** Szeto, G. and Xue, L. – On Characterizations of a Center Galois Extension; with George Szeto, *International Journal of Mathematics and Mathematical Sciences,* 23(11) (2000), 753–758.

**[6]** Szeto, G. and Xue, L. – On Central Commutator Galois Extensions of Rings; with George Szeto, *International Journal of Mathematics and Mathematical Sciences,* 24(5) (2000), 289–294.

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