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NONEXISTENCE OF GLOBAL SOLUTIONS OF NONLINEAR WAVE EQUATIONS

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Abstract: In this paper the nonexistence of global solutions to wave equations of the type $u_{tt} - \Delta u \pm u_t = \lambda u + |u|^{1+q}$ is considered. We derive, for an averaging of solutions, a nonlinear second differential inequality of the type $w'' \pm w' \geq b\, w + |w|^{1+q}$, and we prove a blowing up phenomenon under some restriction on $u(x,0)$ and $u_t(x,0)$. Similar results are given for other equations.

1 – Introduction

In [2] Glassey proved the non global existence of classical solutions to

$$\text{(1.1)} \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f(u)\,, & (x,t) \in \Omega\times(0,T), \\ \quad u(x,t) = 0\,, & (x,t) \in \partial\Omega\times(0,T)\ , \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary $\partial\Omega$, and f satisfies some growth conditions. Later Souplet [16] studied the equation

$$\text{(1.2)} \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + \dfrac{\partial u}{\partial t} = \lambda\, u + |u|^{1+q}\,, & (x,t) \in \Omega\times(0,T), \\ \quad u(x,t) = 0\,, & (x,t) \in \partial\Omega\times(0,T)\ , \end{cases}$$

where the parameter q is nonnegative. The authors proved that if

$$\text{(1.3)} \qquad \int_\Omega u(x,0)\, \Phi_1\, dx \ > \ 0\ ,$$

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and

$$\int_\Omega u_t(x,0)\,\Phi_1\,dx \;\geq\; 0\;, \tag{1.4}$$

where Φ_1 is the first nonnegative eigenfunction of $-\Delta$ in $H_0^1(\Omega)$, then no global solutions exist for Problems (1.1) and (1.2).

The method used is based on a nonlinear second order differential inequality satisfied by the function

$$w(t) \;:=\; \int_\Omega u(x,t)\,\Phi_1(x)\;dx\;.$$

This approach has been introduced by Kaplan [7] and used successfully by Glassey [2, 3].

In this work we sharpen the results of [2, 16], we shall show that solutions to

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + \dfrac{\partial u}{\partial t} \;=\lambda\,u + |u|^{1+q}\,, & (x,t)\in\Omega\times(0,T),\\ \qquad\qquad u(x,t)\;\geq 0\,, & (x,t)\in\partial\Omega\times(0,T)\;, \end{cases} \tag{1.5}$$

may blow up without conditions (1.4). We prove in particular that for any $u_t(\cdot,0)$, if condition (1.3) holds, then there exists $\lambda^*=\lambda^*(u_0,u_1)$ such that solutions of (1.5) blow-up for any $\lambda\geq\lambda^*$. Using the same method, we can obtain a similar result for the problem

$$\Big(|u_t|^{p-2}\,u_t\Big)_t + \gamma\,|u_t|^{p-2}\,u_t - \Delta u \;=\; \lambda u\;,$$

where $\lambda>\lambda_1$, $\gamma\in\mathbb{R}$ and $1<p\leq 2$. Finally we shall study the problem

$$\begin{cases} \Big(|u_t|^{p-2}u_t\Big)_t + \gamma_1\,|u_t|^{p-1}\,u_t + \gamma_2\,\Delta u \;=\; 0\,, & x\in\Omega,\;\; t>0,\\ \qquad\qquad\qquad\qquad\qquad\qquad u \;=\; 0\,, & x\in\partial\Omega,\;\; t>0\;, \end{cases} \tag{1.6}$$

where $1<p$ and $p<\frac{N+2}{N-2}$ if $N\geq 3$ and $\gamma_1.\gamma_2>0$. This equation is not of type of problems studied by Levine, Park and Serrin [13], in fact we shall construct a global unbounded solution to (1.6). In the oppposite the authors proved in [13] that solutions to

$$\Big(|u_t|^{l-2}\,u_t\Big)_t - a\,\nabla\Big(|\nabla u|^{q-2}\,\nabla u\Big) + b\,|u_t|^{m-2}\,u_t \;=\; c\,|u|^{p-2}\,u\,, \qquad a,b\geq 0,\quad c>0\;,$$

blow up in a finite time.

The plan of the paper is as follows. First we prepare lemmas on an ordinary differential inequality in Section 2. The nonglobal existence is established and proved in Section 3. Some applications are also given.

2 – Preliminaries

For reals α, β, we consider the following ordinary differential inequality

$$u'' + \gamma\, u' \;\geq\; b\, u + |u|^{1+q}\,, \qquad t \in (0,T)\;, \tag{2.1}$$

subject to the condition

$$u(0) = \alpha\,, \qquad u'(0) = \beta\;, \tag{2.2}$$

where $b \geq 0$, $q > 0$, $\gamma = \pm 1$ and $0 < T \leq \infty$.

The goal of this section is to obtain several properties of solutions to (2.1) in terms of α and β. To begin with the case where $\alpha > 0$ and $\beta \geq 0$. The nonglobal existence is obtained in [16] in the case $\gamma = 1$. For completeness we give here the proof. The first simple consequence of the fact that $\beta \geq 0$, is that u is monotone increasing function for small t. The following lemma shows that $u' > 0$ for all $t \in (0,T)$.

Lemma 2.1. *Let u be a function satisfying (2.1)–(2.2) where $\alpha > 0$, $\beta \geq 0$. Then necessarily $T < \infty$ and we have*

$$u(t) > 0 \quad \text{and} \quad u'(t) > 0\;, \tag{2.3}$$

for all $t \in (0,T)$.

Proof: Assume that u has a positive local maximum at t_0. Using (2.1) we arrive at $u''(t_0) \geq b\, u(t_0) + |u(t_0)|^{1+q} > 0$. This is impossible, then $u'(t) \geq 0$, for any t. Suppose now $u'(t_0) = 0$ and $u' > 0$ on $(t_0 - \varepsilon, t_0)$. Using again (2.1) one sees $u''(t_0) > 0$ and then $u'(t_0 - \varepsilon) < u'(t_0) = 0$, a contradiction and (2.3) is done. Thus $\lim_{t\to T} u(t) = c$, exists in $(\alpha, +\infty)$. Assume now, on the contrary that $T = +\infty$ and $c < +\infty$. Therefore $u'(t_n) \to 0$ as $n \to +\infty$, for some sequence (t_n) converging to infinity with n. Integrating (2.1) over $(0, t_n)$ and passing to the limit yield

$$-\beta + \gamma(c - \alpha) \;\geq\; \int_0^\infty u^{1+q}(s)\, ds\;,$$

which implies immediately that u^{q+1} is integrable and then $c = 0$. This is impossible. Therefore $u(t)$ goes to infinity with t. Now as in [16] the function v defined by

$$v(t) = \frac{u^2}{2}\;,$$

satisfies

$$v''(t) \;\geq\; (u')^2 + b\, u^2 + 2^{\frac{2+q}{2}}\, v^{\frac{2+q}{2}} - \gamma\, u\, u'\;.$$

Using Young's inequality we deduce that

$$v''(t) \,\geq\, C\, v^{\frac{2+q}{2}} \;,$$

for t large enough. Therefore, since $v' > 0$, v develops a singularity at a finite time, a contradiction. This ends the proof. ∎

Remark 2.1. Note that, as inequality (2.1) is autonomous, if there exists $t_0 \in (0,T)$ such that $u(t_0) > 0$ and $u'(t_0) \geq 0$ then u cannot be global, since the function $U(t) = u(t+t_0)$ satisfies (2.1)–(2.2). The following result shows that solutions may blow up at a finite point in the case where $u'(0) < 0$. This shows in particular, that the condition $u'(0) \geq 0$ is not essential as it seems to be asserted in [16, Remark 1.2, p. 295]. □

The rest of this section treats the case $\gamma > 0$. For simplification we suppose $\gamma = 1$. A more general inequality (2.1) with $\gamma > 0$ can be transformed to the same inequality where $\gamma = 1$ by introducing a new function $\gamma^{-\frac{2}{q}}\, u(t/\gamma)$ which solves (2.1) with $b\,\gamma^{-2}$ instead of b and 1 instead of γ.

Lemma 2.2. *Let $\gamma = 1$. Assume $0 < -\beta < \alpha$. Then any solution u to (2.1)–(2.2) blows up at a finite time and we have $u(t) \geq \alpha\, e^{-t}$ for all t in the existence interval.*

Proof: We set $w(t) = u(t) - c\, e^{-t}$, where $-\beta < c < \alpha$. As $w(0) > 0$ and $w'(0) > 0$ the function w is positive in a small interval $(0, t_0)$. Next in view of (2.1) we infer

$$w'' + w' \,\geq\, |w + c\, e^{-t}|^{q+1} \;,$$

hence

$$w'' + w' \,\geq\, w^{q+1} \;,$$

for all $t \in (0, t_0)$, and the proof is done thanks to the preceding lemma. ∎

The following result shows that Problem (2.1)–(2.2) cannot have a global solution for a small $|\beta|$.

Lemma 2.3. *Let $\alpha > 0$. Assume*

$$\beta^2 \,\leq\, \frac{2}{q+2}\, \alpha^{2+q} \;.$$

Then any solution to (2.1)–(2.2) is nonnegative and blows up at a finite point.

Proof: The case $\beta+\alpha>0$ was treated in Lemma 2.2. Assume that $\alpha+\beta\le 0$ and let u be a global solution of (2.1)–(2.2). As $\beta<0$, the function u is decreasing for small t. Suppose that there exists $t_0>0$ such that $u(t_0)=0,\ u>0$ on $[0,t_0)$ and then $u'<0$ on $(0,t_0)$, thanks to Remark 2.1. Define

$$H(t) \,=\, \frac{1}{2}\,(u')^2(t) - \frac{b}{2}\,u^2(t) - \frac{1}{q+2}\,|u(t)|^{q+2}\,, \qquad t\in(0,T)\ .$$

Using (2.1) we deduce that

$$H'(t)<0\ ,$$

for any $t\in(0,t_0)$. Therefore $H(0)>H(t_0)$, and thus $\beta^2>\frac{2}{q+2}\,\alpha^{2+q}$, which is impossible. This shows in particular that $u(t)>0$ for all t, and then $u'(t)<0$ (since otherwise $u(t)+u'(t)>0$ for some $t>0$, and then $T<\infty$ by Lemma 2.2). It is clear that u goes to 0 at infinity. Therefore H tends to 0, thanks to the monotonicity of H, hence $H(t)>0$ for all t and then $\beta^2>\frac{2}{q+2}\,\alpha^{2+q}$, we arrive again at a contradiction. ∎

Remark 2.2. We can observe from the last proof that if we suppose

$$b\,\ge\, b_c \,:=\, \frac{\beta^2}{\alpha^2}\ , \tag{2.4}$$

the conclusion of Lemma 2.2 remains true. In the case where $-\beta$ is large enough Problem (2.1)–(2.2) has a positive global solution. To be more precise, set

$$w(t)=\alpha\,\exp\!\left(\frac{\beta}{\alpha}\,t\right), \qquad \alpha>0,\quad \beta<0\ .$$

Thus w satisfies (2.1)–(2.2) provided

$$\frac{\beta^2}{\alpha^2}+\frac{\beta}{\alpha}-b-\alpha^q\,\ge\,0\ ,$$

we then easily obtain

$$\beta\,\le\,-\frac{\alpha}{2}\left[1+\sqrt{1+4\,(b+\alpha^q)}\right]\ .$$

Note that this last condition implies that

$$\beta^2\,>\,\alpha^{2+q}\ .$$

The condition $b\ge b_c$ is not optimal as it is shown in the following lemma. □

Lemma 2.4. *Assume that $\gamma = 1$. Let u be a solution to Problem (2.1)–(2.2) such that $0 < \alpha \le -\beta$. Assume that $b > \frac{\beta^2}{\alpha^2} + \frac{\beta}{\alpha} + 1$, then u is not global.*

Proof: Let $a > -\frac{\beta}{\alpha}$ such that

$$b \,>\, a^2 - a + 1 \,. \tag{2.5}$$

Set

$$w(t) \,=\, \Gamma\Big(u(t) - c\, e^{-at}\Big) \,,$$

where

$$\Gamma(1+c) = 1 \,, \qquad -\frac{\beta}{\alpha} < c < a \,,$$

hence

$$w(0) > 0 \,, \qquad w'(0) > 0 \,. \tag{2.6}$$

On the other hand, due to inequality (2.1) the function w satisfies

$$\begin{aligned} w'' + w' \;&\ge\; b\, w + \Gamma\, c\, e^{-at} + \Gamma\, |u|^{q+1} \,, \\ w'' + w' \;&\ge\; b\, w + \Gamma\, c\, e^{-a(q+1)t} + \Gamma\, |u|^{q+1} \,, \end{aligned}$$

thanks to (2.5). Using the convexity, since $\Gamma c + \Gamma = 1$, we arrive at

$$w'' + w' \,\ge\, bw + |w|^{q+1} \,. \tag{2.7}$$

Finally, by Lemma 2.1 we deduce that w is not global. ∎

Remark 2.3. According to the above results we can conclude that if $\alpha > 0$ any solution blows up at a finite time for a large b. Note that in the case where $\alpha + \beta > 0$ there is no restriction on b. □

Now using the function H we can deduce the following.

Lemma 2.5. *Let u be a global nonnegative solution to (2.1)–(2.2) then*

$$u(t) \le \alpha\, e^{-\sqrt{b}t} \,, \qquad \forall\, t \ge 0 \,. \;∎$$

Remark 2.4. For the problem ($\gamma = 0,\ b = 0$);

$$u'' = |u|^{1+q} \,, \qquad u(0) = \alpha > 0 \,,$$

it is easy to see that there exists exactly one global solution defined by

$$u(t) \,=\, \frac{\alpha}{\Big(1 + \frac{q}{2}\sqrt{\frac{2}{q+2}}\ \alpha^{\frac{q}{2}}\, t\Big)^{\frac{2}{q}}} \;. \;□$$

3 – Blow–up results for nonlinear wave equations and applications

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary $\partial\Omega$. Consider the following nonlinear wave equation with damping and source terms

$$(3.1)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + \dfrac{\partial u}{\partial t} = \lambda\, u + |u|^{1+q}\,, & (x,t) \in \Omega\times(0,T)\ , \\ \qquad\qquad u(x,t) \geq 0\,, & (x,t) \in \partial\Omega\times(0,T)\ , \end{cases}$$

where $q > 0$. Let us denote by $\lambda_1 = \lambda_1(\Omega)$ the first eigenvalue of the problem

$$(3.2)\qquad \begin{cases} \Delta\Phi + \lambda\Phi \,=\, 0, & \text{in } \ \Omega, \\ \qquad\quad \Phi \,=\, 0, & \text{on } \ \partial\Omega\ , \end{cases}$$

and by Φ_1 the first eigenfunction which is positive. It is known that $\dfrac{\partial\Phi_1}{\partial\nu} < 0$ on $\partial\Omega$, where ν is the outward normal. Assume that

$$\int_\Omega \Phi_1(x)\, dx \,=\, 1\ .$$

For any $\alpha > 0$ and $\beta \in \mathbb{R}$, set

$$(3.3)\qquad \lambda^\star(\alpha,\beta) \,=\, \begin{cases} \lambda_1\,, & \text{if } \ \alpha+\beta > 0, \\ \lambda_1 + \dfrac{\beta^2}{\alpha^2} + \dfrac{\beta}{\alpha} + 1\,, & \text{otherwise}\ . \end{cases}$$

Theorem 3.1. *Let $\lambda > \lambda^\star(\alpha,\beta)$. There is no global solution, $u \in C^2$, to (3.1) such that*

$$\int_\Omega u(x,0)\, \Phi_1(x)\, dx \,=\, \alpha \,>\, 0\ ,$$

and

$$\int_\Omega u_t(x,0)\, \Phi_1(x)\, dx \,=\, \beta\ .$$

Proof: Let

$$w(t) \,=\, \int_\Omega u(x,t)\, \Phi_1(x)\, dx\ .$$

Using (3.1) we obtain

$$w'' + w' \,\geq\, (\lambda-\lambda_1)\, w + \int_\Omega |u|^{q+1}\, \Phi_1(x)\; dx\ .$$

By the Jensen inequality we get

$$w'' + w' \ \geq \ b\,w + |w|^{q+1} \ ,$$

where

$$b := \lambda - \lambda_1 \, , \qquad w(0) > 0 \ .$$

According to Lemmas 2.1, 2.2 and 2.4, the function $w(t)$ is nonnegative and goes to infinity at a finite time. ∎

By Lemma 2.5 it is easy to obtain the asymptotic behavior of global nonnegative solutions.

Theorem 3.2. *Assume $\lambda \geq \lambda_1$ and let $u \in C^2$ be a nonnegative global solution to (3.1). Then*

$$\int_\Omega u(x,t)\, \Phi_1(x) \ dx \ \leq \ e^{-\sqrt{\lambda-\lambda_1}\,t} \int_\Omega u(x,0)\, \Phi_1(x) \ dx \ ,$$

for any $t \geq 0$. ∎

Remark 3.1. In fact we can deduce from Lemma 2.1 that if u is a global positive solution, then we have necessarily

$$\int_\Omega u(x,t)\, \Phi_1(x)\, dx + \int_\Omega u_t(x,t)\, \Phi_1(x)\, dx \ < \ 0 \ ,$$

for all $t \geq 0$. Therefore the function $t \to e^t \int_\Omega u(x,t)\, \Phi_1(x)\, dx$ is positive, global and decreasing. Hence the limit

$$\lim_{t\to\infty} e^t \int_\Omega u(x,t)\, \Phi_1(x) \ dx$$

exists. And if, in addition, $\lambda > \lambda_1 + 1$ this limit is zero. □

Let us now give examples of applications of our results

Example 3.1. Consider the equation

$$\left\{ \begin{array}{lll} \dfrac{\partial^2 u}{\partial t^2} + \gamma\,\dfrac{\partial u}{\partial t} & = \ -\Delta |u|^{1+q}\,, & t > 0, \quad x \in \Omega, \\ u(x,t) & = \ 0\,, & t > 0, \quad x \in \partial\Omega\ , \end{array} \right. \tag{3.4}$$

where $q > 0$, subject to the initial condition

$$u(x,0) = u_0(x) \geq 0\,, \qquad u_t(x,0) = u_1(x) \geq 0 \ . \ \square \tag{3.5}$$

Theorem 3.3. *There is no global solution, $u \in C^2$, to (3.4)–(3.5) such that*

$$\int_\Omega u_0(x)\,\Phi_1(x)\,dx \;:=\; \alpha \;>\; 0 \quad \textit{and} \quad \int_\Omega u_1(x)\,\Phi_1(x)\,dx \;:=\; \beta \;>\; 0\;.$$

Proof: We multiply the equation of u by Φ_1 and integrate over Ω. We obtain

$$w'' + \gamma\,w' \;\geq\; \lambda_1\,w^{q+1}\;,$$

where

$$w(t) \;=\; \int_\Omega u(x,t)\,\Phi_1(x)\,dx\;.$$

Therefore we use Section 2 to conclude. ∎

This result allows us to consider

$$\Big(|u_t|^{p-2}\,u_t\Big)_t + \gamma\,|u_t|^{p-2}\,u_t - \Delta u \;=\; \lambda\,u\,, \qquad (x,t) \in \Omega{\times}(0,T)\;,$$

where $\gamma \in \mathbb{R}$, $1 < p < 2$ and $\lambda > \lambda_1$. Setting $v = |u_t|^{p-2}\,u_t$ yields

$$v_{tt} + \gamma\,v_t - \Delta|v|^q\,v \;=\; \lambda\,|v|^q\,v\,, \qquad q = \frac{2-p}{p-1}\;,$$

which is of the type (3.4) if $v \geq 0$.

Example 3.2. A similar result can be obtained if we consider Problem (3.1) with the term $h(t,x)\,|u|^{q+1}$ instead of $|u|^{q+1}$ where the function h satisfies $h(x,t) \geq c > 0$ for all $t > 0$ and $x \in \overline{\Omega}$. Now we study, in $\Omega{\times}(0,T)$, the equation

$$u_{tt} - \Delta u + \gamma\,u_t \;=\; \lambda_1\,u + \Phi_1^{-q}\,|u|^{1+q}\;, \tag{3.6}$$

with the condition

$$u(x,0) = 0, \quad x \in \overline{\Omega}\,, \qquad u(x,t) = 0, \quad x \in \partial\Omega\;,$$

and

$$u_t(x,0) \geq 0\;.$$

It is clear that this problem has in the addition to the trivial solution $u \equiv 0$, the solution defined by

$$u(x,t) \;=\; w(t)\,\Phi_1(x)\;,$$

where the function w is a solution to

$$w'' + \gamma\,w' \;=\; |w|^{q+1}\,, \qquad w(0) = 0\;.$$

Thus if $w'(0) > 0$, the function w is not global. Note that in the case where $\gamma \le 0$, the blow-up takes place in the interval $(0, T_0)$, where

$$T_0 = \int_0^\infty \frac{ds}{\sqrt{w'(0) + \frac{2}{q+2}\, s^{q+2}}} \ . \ \square$$

Example 3.3. By using similar arguments, we can prove the nonexistence of global solution to the nonlinear hyperbolic inequation

$$\text{(3.7)} \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + \dfrac{\partial u}{\partial t} \ \ge\ \lambda\, u + |u|^{1+q}\,, & (x,t) \in \Omega{\times}(0,T)\ , \\ \qquad\qquad u(x,t) \ \ge\ 0\,, & (x,t) \in \partial\Omega{\times}(0,T)\ , \end{cases}$$

where $q > 0$. $\square$

Example 3.4. We finish this section by the problem

$$\text{(3.8)} \qquad \begin{cases} \Big(|u_t|^{p-2}\, u_t\Big)_t + \gamma_1\, |u_t|^{p-1}\, u_t + \gamma_2\, \Delta u \ =\ 0\,, & x \in \Omega,\ \ t > 0\ , \\ \qquad\qquad\qquad\qquad\qquad u \ =\ 0\,, & x \in \partial\Omega,\ \ t > 0\ , \end{cases}$$

where $\gamma_1 \gamma_2 > 0$ and $p > 1$. It is clear that we are not in the situation of the precedent section. In fact we shall show that the problem has at least one global solution. $\square$

By a similar argument due to Haraux [6] we obtain the following.

Theorem 3.4. *Assume $p > 1$ and $p < \frac{N+2}{N-2}$ if $N \ge 3$. Let $u_1 \in H^1_0(\Omega)$ be a solution to*

$$-\Delta u_1 = \frac{\gamma_1}{\gamma_2} \Big(\frac{p}{p-1}\Big)^p\, u_1^p\,, \qquad u_1 > 0 \ \ in\ \Omega\ ,$$

where $\gamma_1 . \gamma_2 > 0$, and $u_0 \in H^1_0(\Omega)$ be the unique solution to

$$-\gamma_2\, \Delta u_0 = u_1^{p-1} \qquad in\ \ \Omega\ ,$$

then the function

$$u(x,t) \ =\ t^{\frac{p}{p-1}}\, u_1 + \Big(\frac{p}{p-1}\Big)^{p-1} u_0\ ,$$

is a global unbounded solution to (3.8).

Proof: It is known that u_1 exists. Using the definition of u we have

$$u_t = \frac{p}{p-1}\, t^{\frac{1}{p-1}}\, u_1\,, \qquad \Big(|u_t|^{p-2}\, u_t\Big)_t = \Big(\frac{p}{p-1}\Big)^{p-1} u_1^{p-1}\ .$$

Hence

$$\Big(|u_t|^{p-2}\,u_t\Big)_t + \gamma_1 |u_t|^{p-1}\,u_t + \gamma_2\,\Delta u \;=$$

$$= \left(\frac{p}{p-1}\right)^{p-1} u_1^{p-1} + \gamma_2\; t^{\frac{p}{p-1}}\,\Delta u_1 + \gamma_2 \left(\frac{p}{p-1}\right)^{p-1} \Delta u_0 + \gamma_1 \left(\frac{p}{p-1}\right)^{p} t^{\frac{p}{p-1}}\,u_1^p \;.$$

Using the definitions of u_0 and u_1 we deduce that u satisfies (3.8). This ends the proof. ∎

Remark 3.2. If we look for solution to (3.8) independent of x, $u(x,t) = u(t)$, such that $u(0) = 0$ and $u'(0) = \beta > 0$, we find that

$$u(t) \;=\; u_\beta(t) \;=\; \frac{p-1}{\gamma_1}\;\log\left(1 + \frac{\gamma_1\,\beta}{p-1}\;t\right) \;.$$

It is clear that if $\gamma_1 < 0$, u_β tends to infinity as t approaches $T(\beta) = -\frac{p-1}{\gamma_1 \beta}$. Note that the existence time goes to 0 as β tends to infinity.

It may be of interest to note that, in the case where $\gamma_2 > 0$, Equation (3.8) is of elliptic type, and Theorem 3.4 gives a solution to the problem

$$\Big(|u_{x_1}|^{p-2}\,u_{x_1}\Big)_{x_1} + \gamma_1\,|u_{x_1}|^{p-1}\,u_{x_1} + \gamma_2\,\Delta_y u \;=\; 0\,, \qquad (x_1, y) \in \Sigma\;,$$

where Σ is an infinite cylindrical domain $\Sigma = \mathbb{R}^+ \times \Omega$. We can also study the nonexistence of global solutions, in Σ, of

$$u_{tt} + \gamma\,u_t - |u|^p + \Delta |u|^q \;\geq\; 0\;,$$

where

$$\max\{p, q\} > 1\;. \;\square$$

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