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ON CYCLIC SYMMETRIC HEYTING ALGEBRAS

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Abstract: In this work we investigate the variety of k-cyclic symmetric Heyting algebras, k a positive integer. We characterize the simple objects of this variety and we describe the lattice of subvarieties in the linear case. We also give equational bases for each member of this lattice.

1 – Introduction

A symmetric Heyting algebra is an algebra $\langle A, \wedge, \vee, \rightarrow, \sim, 0, 1\rangle$ of type $(2,2,2,1,0,0)$ such that $\langle A, \wedge, \vee, \rightarrow, 0, 1\rangle$ is a Heyting algebra and $\langle A, \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra (see [10]). Let $\mathcal{H}$ denote the variety of all symmetric Heyting algebras.

The study of the variety $\mathcal{H}$ was pioneered by A. Monteiro. It was pursued by, among others, H. Sankappanavar [12] and L. Iturrioz [7, 8]. Later on, A. Monteiro comprehensively investigated the variety of symmetric Heyting algebras and several of its subvarieties in his very important work "Sur les algèbres de Heyting symétriques" [10]. In this work he also proves that these algebras reflect algebraically the properties of modal symmetric propositional calculus in the same way as Heyting algebras are algebraic structures imposed by the study of intuitionistic propositional calculus.

A *k-cyclic symmetric Heyting algebra* is a system $\langle A, T\rangle$ such that A is a symmetric Heyting algebra and T is an *endomorphism* of A such that $T^k(x) = x$

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for every $x \in A$ (observe that T is in fact an automorphism). We will usually use the same notation for a structure as for its universe. The class of k-cyclic symmetric Heyting algebras forms a variety which we denote $\mathcal{H}_k$.

A. Monteiro has investigated both concepts, symmetric Heyting algebras and cyclic operations on an algebra [10]. M. Abad [1] has also investigated the action of a cyclic operation on Łukasiewicz algebras, and L. Iturrioz [7] gave a complete description of the variety of involutive Heyting algebras. Also, the relationship between a cyclic operation T and a monadic operation ∇ has been investigated by A. Monteiro in [11] and by M. Abad in [1]. It is worth noting that if we define $\sigma(x) = \sim T(x)$, then the k-cyclic operation T determines a *correlation* σ on the Heyting algebra A, that is, a unary operation σ satisfying $\sigma(x \wedge y) = \sigma(x) \vee \sigma(y)$, $\sigma(x \vee y) = \sigma(x) \wedge \sigma(y)$ and $\sigma^{2k}(x) = x$ (see [14, 15, 2, 12]).

This paper is devoted to the study of the variety $\mathcal{H}_k$. We characterize the finite simple and subdirectly irreducible objects of this variety in Section 2 and we characterize the subalgebras of a finite simple algebra. In the rest of the paper we consider the particular subvariety $\mathcal{L}_k$ consisting of those algebras satisfying the additional linear condition: $(x \to y) \vee (y \to x) = 1$. We describe the lattice of subvarieties of $\mathcal{L}_k$ in Section 3. The last section is devoted to the determination of equational bases for each member of this lattice.

2 – Simple and subdirectly irreducibles

In this section we characterize the finite simple and subdirectly irreducible objects in the variety $\mathcal{H}_k$. We also give an explicit construction of the subalgebras of a finite simple algebra of the variety. This construction will be used in the determination of the lattice of subvarieties of $\mathcal{L}_k$.

A filter N of a symmetric Heyting algebra A that satisfies: $\neg \sim x \in N$, for every element $x \in N$, where $\neg x = x \to 0$, is called a *kernel* of A [10, 12]. Equivalently, a kernel is a filter N that satisfies the contraposition law: if $x \to y \in N$, then $\sim y \to \sim x \in N$ ([10], Th. 4.11).

Observe that if $A, B \in \mathcal{H}$ and $h \colon A \to B$ is a homomorphism, then $h^{-1}(1) = \mathrm{Ker}(h)$ is a filter with the property that if $x \to y \in \mathrm{Ker}(h)$, then $\sim y \to \sim x \in \mathrm{Ker}(h)$. A. Monteiro [10] proved that if, for a given kernel N in a symmetric Heyting algebra A we define $x \,\theta_N\, y$ if and only if $x \to y \in N$ and $y \to x \in N$, then θ_N is a congruence. Moreover, every congruence on A is determined by a kernel, and the mapping $N \to \theta_N$ is an isomorphism from the lattice of kernels of A onto the lattice of congruences on A.

Consider the following terms: $(\neg\sim)^0 x = x$ and $(\neg\sim)^{n+1}x = \neg\sim(\neg\sim)^n x$. Observe that $\neg\sim(x\wedge y) = \neg\sim x \wedge \neg\sim y$ and $\neg\sim\neg\sim x \leq x$, for all $x, y \in A \in \mathcal{H}$. Thus, for $n \in \omega$, $(\neg\sim)^n(x \wedge \neg\sim x) \geq (\neg\sim)^{n+1}(x \wedge \neg\sim x)$.

We say that $A \in \mathcal{H}$ is *of finite range* if for all $x \in A$ there exists $n \in \omega$ such that $(\neg\sim)^n(x \wedge \neg\sim x) = (\neg\sim)^{n+1}(x \wedge \neg\sim x)$.

Let $\mathrm{Cen}(A) = \{x \in A : \neg\sim x = \sim\neg x\}$ be the center of A, that is, the sublattice of complemented elements of A.

Theorem 2.1 ([12]). *For $A \in \mathcal{H}$ of finite range and $|A| \geq 1$, the following are equivalent:*

(**1**) *A is simple.*

(**2**) *A is subdirectly irreducible.*

(**3**) *A is directly indecomposable.*

(**4**) $\mathrm{Cen}(A) = \{0, 1\}$ *or* $\mathrm{Cen}(A) = \{0, x = \sim x, \neg x, 1\}$. ∎

If A satisfies the Kleene condition ($x \wedge \sim x \leq y \vee \sim y$) then in Theorem 2.1, condition (4) is just $\mathrm{Cen}(A) = \{0, 1\}$.

The variety $\mathcal{H}$ is generated by its finite members ([10]). Thus, if $FSI(\mathcal{K})$ and $FS(\mathcal{K})$ respectively denote the sets of finite subdirectly irreducible algebras and finite simple algebras in a variety $\mathcal{K}$, then

Corollary 2.2. $\mathcal{H} = V(FSI(\mathcal{H})) = V(FS(\mathcal{H}))$. ∎

The following is easily seen.

Proposition 2.3. *If P is a kernel (prime filter, minimal prime filter, maximal kernel) of an algebra $\langle A, T\rangle \in \mathcal{H}_k$, then $T(P)$ is a kernel (respectively prime filter, minimal prime filter, maximal kernel).* ∎

Definition 2.4. We say that a kernel N of $A \in \mathcal{H}_k$ is a T-kernel if $T(x) \in N$ for every $x \in N$. □

Proposition 2.5. *If N is a maximal T-kernel, there exists a maximal kernel P such that $N = P \cap T(P) \cap T^2(P) \cap \cdots \cap T^{k-1}(P)$.*

Proof: N is a maximal T-kernel, so N is a proper kernel. So there exists a maximal kernel P such that $N \subseteq P$. Since N is a T-kernel, $N \subseteq T^i(P)$ for every $i = 1, 2, ..., k-1$. Thus $N \subseteq N_1 = P \cap T(P) \cap T^2(P) \cap \cdots \cap T^{k-1}(P)$, and necessarily $N = N_1$, as N_1 is a T-kernel. ∎

A maximal kernel P is said to be of *period* d, if d is the least positive integer such that $T^d(P) = P$. In this case we say that the maximal T-kernel $N = P \cap T(P) \cap T^2(P) \cap \cdots \cap T^{d-1}(P)$ is of period d. It is easy to see that if P is of period d, then d is a divisor of k.

Consider now, for a given algebra $\langle A, T\rangle \in \mathcal{H}_k$, the subalgebras $I[A] = \{x\colon T(x) = x\}$ and $K[A] = \{x\colon \neg x = \sim x\}$. The elements of the Boolean algebra $K[A]$ were called *strong Boolean* by A. Monteiro in [10]. If A is a finite algebra in $\mathcal{H}$, z is an atom of $K[A]$ and $F(z)$ denotes the filter generated by z, then $A/F(z)$ is simple as a symmetric Heyting algebra, and $A \cong A/F(z_1) \times \cdots \times A/F(z_n)$, where $z_1, ..., z_n$ are the atoms of $K[A]$ (see [10], p. 91).

Recall that a filter F in an algebra $A \in \mathcal{H}$ is called *strong* (see A. Monteiro, [10]) if for every $x \in F$ there exists a strong Boolean element $a \in F$ such that $a \le x$. Every strong filter is a kernel ([10], Th. 4.26), and we are going to see that in the finite case, the converse also holds.

If P is a prime filter of an algebra $A \in \mathcal{H}$, then $\complement \sim P$ is also a prime filter of A, where $\sim P = \{\sim x,\ x \in P\}$ and $\complement$ is the set-theoretical complementation.

Let A be a finite algebra in $\mathcal{H}$, let Π be the set of join-irreducible elements of A and let $\varphi\colon\ \Pi \to \Pi$ be the mapping defined by $\varphi(p) = q$ if and only if $\complement \sim F(p) = F(q),\ p, q \in \Pi$. The system $\langle \Pi, \varphi\rangle$ characterizes the finite algebra A. We have that for $x \in A$, $\sim x = \bigvee\{p \in \Pi\colon\ \varphi(p) \not\le x\}$ ([10], p. 41).

If $N = F(a)$ is a proper kernel of the finite algebra A, let $\Pi(N) = \{p \in \Pi\colon p \le a\}$. Then $\Pi(N)$ has the following properties:

(**1**) if $p \in \Pi(N)$, $\varphi(p) \in \Pi(N)$,

(**2**) if $p \in \Pi(N)$ and q is a join-irreducible element comparable to p, then $q \in \Pi(N)$,

(**3**) $a = \bigvee\{p \in \Pi(N)\colon\ p \text{ maximal in } \Pi(N)\}$ ([10], p. 85).

Then we have the following result which is not explicitely stated in [10] but is based in Monteiro's ideas.

Proposition 2.6. *If A is a finite algebra in $\mathcal{H}$, then every kernel is a strong filter.*

Proof: If $N = F(a)$ is a kernel of A, $a = \bigvee\{p \in \Pi(N)\colon\ p \text{ maximal in } \Pi(N)\}$, and from the previous remark $\sim a = \bigvee\{p \in \Pi\colon\ \varphi(p) \not\le a\} = \bigvee\{p \in \Pi \setminus \Pi(N)\}$. Thus $a \vee \sim a = 1$ and $a \wedge \sim a = 0$, that is, a is a strong Boolean element of A, and consequently, $N = F(a)$ is a strong filter. ∎

As a consequence, if $A \in \mathcal{H}$, A finite, there exists a one-to-one correspondence between kernels of A and elements of $K[A]$. In particular, A is simple if and only if $K[A] = \{0,1\}$ (compare with Theorem 2.1).

Proposition 2.7. *N is a maximal T-kernel of a finite algebra $\langle A,T\rangle \in \mathcal{H}_k$ if and only if there exists an atom b in the Boolean algebra $K[A] \cap I[A]$ such that $N = F(b)$.*

Proof: Suppose that N is a maximal T-kernel. Then there exists a maximal kernel P such that $N = P \cap T(P) \cap T^2(P) \cap \cdots \cap T^{k-1}(P)$. Since A is finite, there exists $a \neq 0,1$ such that $P = F(a)$ and a is an atom of $K[A]$.

Consider the element $b = a \vee T(a) \vee \cdots \vee T^{k-1}(a)$. It is not hard to prove that b is an atom of $K[A] \cap I[A]$ and $b \in N$.

For $x \in N$, $x \in P \cap T(P) \cap T^2(P) \cap \cdots \cap T^{k-1}(P)$. So $T^i(a) \leq x$, $0 \leq i \leq k-1$, that is, $b \leq x$. Then $N \subseteq F(b)$. Since $F(b)$ is a T-kernel and N is a maximal T-kernel, it follows that $N = F(b)$.

Conversely, if b is an atom of the algebra $K[A] \cap I[A]$, then $F(b)$ is clearly a T-kernel. Let Q be a maximal T-kernel such that $F(b) \subseteq Q$. Then $Q = F(a)$ for some $a \in K[A] \cap I[A]$, $a \neq 0,1$. Then $a \leq b$, which implies $a = b$. Consequently $F(b) = Q$ is a maximal T-kernel of A. ∎

It is easy to see that if $\langle A,T\rangle \in \mathcal{H}_k$ and N is a T-kernel of A, the associated relation θ_N is an $\mathcal{H}_k$-congruence, and every $\mathcal{H}_k$-congruence is determined by a T-kernel. Consequently, an algebra $\langle A,T\rangle \in \mathcal{H}_k$ is simple if and only if the unique T-kernels of A are the trivial ones.

Corollary 2.8. *A finite algebra $\langle A,T\rangle \in \mathcal{H}_k$ is simple if and only if $K[A] \cap I[A] = \{0,1\}$.* ∎

Corollary 2.9. *Every subalgebra of a finite simple algebra is simple.* ∎

An algebra $\langle A,T\rangle \in \mathcal{H}_k$ satisfies the *Stone condition* if $\neg x \vee \neg\neg x = 1$, for every $x \in A$.

Proposition 2.10. *If an algebra $\langle A,T\rangle \in \mathcal{H}_k$ satisfies the Stone condition, then $\langle A,T\rangle$ is simple if and only if $K[A] \cap I[A] = \{0,1\}$.*

Proof: Let $\langle A,T\rangle$ be a simple algebra. For $a \in K[A] \cap I[A]$, $F(a)$ is a T-kernel. So $F(a) = \{1\}$ or $F(a) = A$, that is, $a = 1$ or $a = 0$.

Conversely, suppose that $K[A] \cap I[A] = \{0,1\}$ and let N be a proper T-kernel of $\langle A,T\rangle$. Let $a \in N$, $a \neq 0$. Let $b = a \wedge T(a) \wedge \cdots \wedge T^{k-1}(a)$. Then $b \in I[A]$. The element $x = \neg\sim b \wedge \neg\sim\neg\sim b$ is such that $x \in N$ and $x \in I[A]$. In addition, as $\sim x = \neg x$ (see [10], p. 103), so $x \in K[A]$. So, $x \in K[A] \cap I[A]$. Hence, $x = 1$ or $x = 0$. Since N is a proper T-kernel of A, it follows that $x = 1$, that is, $b = a = 1$, and consequently, $\langle A,T\rangle$ is simple. ∎

Our next objective is to determine the finite simple algebras of $\mathcal{H}_k$. Starting from an $\mathcal{H}$-simple algebra (simple in the variety $\mathcal{H}$) A we can construct $\mathcal{H}_k$-simple algebras in two ways. Let $\mathcal{D}(k)$ denote the set of all positive divisors of k.

Theorem 2.11. *Let A be an $\mathcal{H}$-simple finite algebra and let $d \in \mathcal{D}(k)$.*

- **(i)** *For any automorphism τ of A such that $\tau^d = Id$, the algebra $A^{(d)} = \langle A,\tau\rangle$ is $\mathcal{H}_k$-simple.*
- **(ii)** *The algebra $A_d = \langle A^d, T\rangle$, where $A^d = \{f\colon \{1,2,....,d\} \to A\}$, with the pointwise defined operations $\wedge$, $\vee$, $\to$ and $\sim$, and for every $f \in A^d$,*

$$T(f)(i) = \begin{cases} f(i-1) & \text{if } i \neq 1 \\ f(d) & \text{if } i = 1 \end{cases}$$

is $\mathcal{H}_k$-simple.
- **(iii)** *Every finite simple algebra in the variety $\mathcal{H}_k$ is isomorphic to an algebra of the form* (i) *or* (ii).

Proof: (i) and (ii) are immediate consequences of Corollary 2.8 and the fact that finite simple algebras A in $\mathcal{H}$ verify the condition $K[A] = \{0,1\}$. Let us see (iii). Let M be a maximal T-kernel of period d of the k-cyclic symmetric Heyting algebra $\langle A,T\rangle$, $M = P \cap T(P) \cap \cdots \cap T^{d-1}(P)$, $d|k$, P maximal kernel of A.

If $d = 1$, then $M = P$, and A/M is an $\mathcal{H}$-simple and $\mathcal{H}_k$-simple algebra, and consequently, A/M is of the form (i).

Suppose that $d > 1$. Consider the simple k-cyclic symmetric Heyting algebra A/M and the symmetric Heyting algebra $A/P \times A/T(P) \times \cdots \times A/T^{d-1}(P)$. Since P is maximal, it is clear that $A/M \cong_{\mathcal{H}} A/P \times A/T(P) \times \cdots \times A/T^{d-1}(P)$ (see [4], Th. I.7.5), where the isomorphism is the map $\alpha\colon A/M \to A/P \times A/T(P) \times \cdots \times A/T^{d-1}(P)$, $|a|_M \mapsto (|a|_P, |a|_{T(P)}, ..., |a|_{T^{d-1}(P)})$.

If we define on $\prod_{0\le i<d} A/T^i(P)$,

$$T'\Big(|a|_P, |a|_{T(P)}, ..., |a|_{T^{d-1}(P)}\Big) = \Big(|T(a)|_P, |T(a)|_{T(P)}, ..., |T(a)|_{T^{d-1}(P)}\Big)$$

then $\langle \prod A/T^i(P), T'\rangle$ is a k-cyclic symmetric Heyting algebra, and $\alpha(|T(a)|_M) =$

$T'(\alpha(|a|_M))$. So A/M and $\prod_{0\le i<d} A/T^i(P)$ are isomorphic as k-cyclic symmetric Heyting algebras.

Observe that $A/T^i(P)$ and A/P are isomorphic as symmetric Heyting algebras for every i, where the isomorphism is $|a|_{T^i(P)} \mapsto |T^{-i}(a)|_P = |T^{d-i}(a)|_P$. Hence $A/M \cong_{\mathcal{H}} (A/P)^d$. By this identification,

$$T'\Big(|a|_P, |T^{d-1}(a)|_P, ..., |T(a)|_P\Big) = \Big(|T(a)|_P, |a|_P, |T^{d-1}(a)|_P, ..., |T^2(a)|_P\Big)\ .$$

Hence $A/M \cong_{\mathcal{H}_k} \langle (A/P)^d, T'\rangle$, where A/P is simple in $\mathcal{H}$ and T' is defined by $T'(x_1, ..., x_d) = (x_d, x_1, ..., x_{d-1})$. Thus A/M is of the form (ii). ∎

We call $\mathcal{O}(a) = \{b \in A\colon\ b = T^n(a),\ n \in N\}$ the orbit of a under T. We say that T *acts transitively* on the atoms of the Boolean algebra $K[A]$ if for each atom a the orbit of a contains all atoms of A.

In what follows, d will always be a positive divisor of k.

Theorem 2.12. *Any finite subdirectly irreducible algebra $\langle A, T\rangle$ in $\mathcal{H}_k$ is simple.*

Proof: Since $\langle A, T\rangle$ is subdirectly irreducible, it follows that $\langle A, T\rangle$ is (directly) indecomposable in $\mathcal{H}_k$. If A is $\mathcal{H}$-indecomposable, as A is finite, then by Theorem 2.1 A is $\mathcal{H}$-simple, and from the previous remarks, $\langle A, T\rangle$ is an $\mathcal{H}_k$-simple algebra of the form (i).

If A is $\mathcal{H}$-decomposable, then the set $K[A] = \{x\colon\ \neg x = \sim x\}$ has at least two non-trivial elements (see [11], p. 90). Let $\{p_1, p_2, ..., p_d\}$ be the set of atoms of $K[A]$. Since $\langle A, T\rangle$ is $\mathcal{H}_k$-indecomposable, it is clear that T acts transitively on the atoms of $K[A]$, that is, $\{p_1, T(p_1), T^2(p_1), ..., T^{d-1}(p_1)\} = \{p_1, p_2, ..., p_d\}$. Consequently, the $\mathcal{H}$-simple algebras $A_i = A/F(p_i)$ are all isomorphic and $\langle A, T\rangle$ is of the form $C_d = \langle C^d, T\rangle$, that is, $\langle A, T\rangle$ is an $\mathcal{H}_k$-simple algebra of the form (ii). ∎

Next we characterize the subalgebras of a finite simple algebra in $\mathcal{H}_k$. This will be used when dealing with the lattice of subvarieties of $\mathcal{H}_k$.

Let $\mathbf{2}_d = \langle \mathbf{2}^d, T\rangle$ be the Boolean k-cyclic simple algebra with d atoms (that is, the Boolean algebra with d atoms with an automorphism which acts transitively on the atoms). A. Monteiro proved in [10] (see also [2]) that the family of subalgebras of $\mathbf{2}_d$ ordered by inclusion is isomorphic to $\mathcal{D}(d)$ ordered by division, where for each $t \in \mathcal{D}(d)$, the subalgebra $\mathbf{2}_t$ of $\mathbf{2}_d$ associated to t is the unique subalgebra of $\mathbf{2}_d$ characterized by $\mathbf{2}_t = \{g \in \mathbf{2}_d\colon\ T^t(g) = g\}$. The atoms of $\mathbf{2}_t$

are the elements g_i^*, $1 \le i \le t$, of the form

$$g_i^*(j) = \begin{cases} 1 \text{ if } j \equiv i \pmod t \\ 0 \text{ otherwise} \end{cases}$$

and we have $T(g_1^*) = g_2^*$, $T(g_2^*) = g_3^*$, ..., $T(g_t^*) = g_1^*$.

If $\langle A, T\rangle$ is a simple algebra of type (i), then $\langle A', T\rangle$ is an $\mathcal{H}_k$-subalgebra of $\langle A, T\rangle$ if and only if A' is an $\mathcal{H}$-subalgebra of A with $T(A') = A'$.

The following theorem gives us the subalgebras of the finite simple algebras of type (ii). Recall that we denote $A^{(d)}$ the simple algebras of type (i) and A_d the simple algebras of type (ii). Let X be a subalgebra of A, $t \in \mathcal{D}(k)$ and τ an automorphism of A of period t, provided it exists. Let $S(X,t) = \{f \in A_d \colon f(i) \in X$ and $f(i) = f(i')$ if $i \equiv i' \pmod t\}$ and $X^{(t)} = (X, \tau)$.

Theorem 2.13. *For every $\mathcal{H}$-simple finite algebra A, the subalgebras of A_d are the algebras $S(X,t)$ and $X^{(t)}$. In addition, $S(X,t)$ is isomorphic to X_t.*

Proof: It is easy to prove that $S(X,t)$ is a subalgebra of A_d isomorphic to the simple algebra X_t. On the other hand, if there exists an automorphism τ of X such that $\tau^t = Id$, the mapping $X^{(t)} \to A_d$, $x \to f_x$, $f_x \in A_d$, $f_x(i) = \tau^{t-i}(x)$, $1 \le i \le t$, $f_x(i) = f_x(i')$, $i \equiv i' \pmod t$ establishes an isomorphism between $X^{(t)}$ and a subalgebra of A_d.

Let us see that the only subalgebras of A_d are the ones indicated above.

Let S be a subalgebra of A_d. For each i, $1 \le i \le d$,

$$X_i = \Big\{a \in A\colon \text{ there exists } f \in S \text{ such that } a = f(i)\Big\}$$

is a subalgebra of A. In addition, $X_i = X_j$ for every i, j, $1 \le i, j \le d$. Indeed, suppose that $i < j$ and let $a \in X_i$. Then there exists $f \in S$ such that $a = f(i)$. So $T^{j-i}(f)(j) = f(j-(j-i)) = f(i) = a$. Since S is a subalgebra, $T^{j-i}(f) \in S$, and consequently, $a \in X_j$, that is, $X_i \subseteq X_j$. Conversely, if $a \in X_j$ there exists $f \in S$ such that $f(j) = a$. Then $T^{d-(j-i)}(f)(i) = T^{d-j+i}(f)(i) = f(i-(d-j+i)+d) = f(j) = a$, being that $d-j+i \ge i$. So $a \in X_i$, and then $X_j \subseteq X_i$.

Now let us put $X = X_1 = X_2 = \cdots = X_d$. For $f \in S$, $f(i) \in X$ for every i, $1 \le i \le d$. Let t be the least positive integer such that $T^t(f) = f$ for every $f \in S$. Then $t \,|\, d$. Suppose that $f \in S$, $i \equiv i' \pmod t$ and $i > i'$. Then $i = q\cdot t + i'$, with $q \ge 1$. So $f(i) = f(q\cdot t+i') = T^{q\cdot t}(f)(q\cdot t+i') = f(q\cdot t+i'-q\cdot t) = f(i')$. Thus, if $f \in S$, $f(i) \in X$, $1 \le i \le d$, and $f(i) = f(i')$ for $i \equiv i' \pmod t$. Therefore $S \subseteq S(X,t)$.

Observe that the Boolean algebra $K[A_d]$ is the simple cyclic Boolean algebra $\mathbf{2}_d$ with d atoms g_i, $1 \le i \le d$, where $g_i(j) = 1$ for $i = j$ and $g_i(j) = 0$ for $i \ne j$ [10]. T acts transitively on the set $\{g_1, g_2, ..., g_d\}$.

Now, suppose that $K[S] = K[A_d] \cap S \ne \{0,1\}$, and t is the least positive integer such that $T^t(f) = f$ for every $f \in S$. Then $K[S] = \mathbf{2}_t$, $t \ne 1$. To see this it is enough to prove that if $K[S] \ne \{0,1\}$, t is the least positive integer such that $T^t(g) = g$ for every $g \in K[S]$. Suppose that there exists $t' < t$ such that for every $g \in K[S]$, $T^{t'}(g) = g$. Since $K[S] \ne \{0,1\}$, $t' \ne 1$. Then $K[S] = \mathbf{2}_{t'}$ and if b_1, b_2, ..., $b_{t'}$ are the atoms of $K[S]$, then S is isomorphic to a product $\prod S/F(b_i)$, with T acting transitively on the components. So $T^{t'}(x) = x$ for every $x \in S$, a contradiction. Let us see that, in this case, $S(X,t) \subseteq S$. Let $f \in S(X,t)$. For each i, $1 \le i \le d$, $f(i) \in X$, so there exists $h_i \in S$ such that $h_i(i) = f(i)$. Since $S \subseteq S(X,t)$, we have $h_i \in S(X,t)$ and consequently $h_i(j) = f(i)$ if $j \equiv i \pmod t$. On the other hand, the elements g_i^*, $1 \le i \le t$ belong to S, as they are the atoms of the Boolean algebra $\mathbf{2}_t = K[S]$. Then $h_i \wedge g_i^* \in S$ for every i, $1 \le i \le t$, and

$$(h_i \wedge g_i^*)(j) \;=\begin{cases} f(i) & \text{if } j \equiv i \pmod t \\ 0 & \text{otherwise} \end{cases}$$

so $f = (h_1 \wedge g_1^*) \vee (h_2 \wedge g_2^*) \vee \cdots \vee (h_t \wedge g_t^*)$, and consequently, $f \in S$. Hence $S(X,t) \subseteq S$. Therefore $S = S(X,t)$.

If $K[S] = \{0,1\}$, then S is indecomposable (simple) as a symmetric Heyting algebra, and so, the projections are isomorphisms in $\mathcal{H}$, that is, $S \cong_{\mathcal{H}} X$, and the operation T on S can be translated to X by means of this isomorphism. So $S \cong X_t$, where t is the least positive integer such that $T^t(x) = x$ for every $x \in S$. Observe that if $t = 1$, then $S \cong S_1 \cong S(X,1)$, and S is a subalgebra of the diagonal of A_d. ∎

3 – The linear case

A. Monteiro introduced in [10] the subvariety $\mathcal{L}$ of $\mathcal{H}$ of *linear* symmetric Heyting algebras, that is, symmetric Heyting algebras satisfying the condition

$$(L) \qquad (x \to y) \vee (y \to x) = 1\ .$$

The corresponding k-cyclic subvariety is the variety $\mathcal{L}_k$ of k-cyclic linear symmetric Heyting algebras, that is, k-cyclic symmetric Heyting algebras satisfying (L). In this section we consider the lattice of subvarieties of $\mathcal{L}_k$. We describe the structure of the poset of its join-irreducible elements and we find equational bases for each subvariety of $\mathcal{L}_k$.

We denote by C_n, $n \geq 2$, the Heyting algebra of all fractions $\dfrac{i}{n-1}$, $i = 0, 1, ..., n-1$ ([10], p. 136), with $\sim x = 1 - x$, and by D_n the Heyting algebra $C_n \times C_n$, with $\sim(x, y) = (1-y, 1-x)$. C_n and D_n are linear symmetric Heyting algebras.

A. Monteiro proved that the subdirectly irreducible (simple) finitely generated algebras in $\mathcal{L}$ are the algebras C_n and D_n, for some n. Then, for a given r, there are only finitely many subdirectly irreducible algebras with $m \leq r$ generators, and consequently, $\mathcal{L}$ is locally finite.

We recall the characterization of subalgebras of the algebras C_n and D_n given in [13]. This characterization will be useful in the sequel.

Let $n \geq 2$. If n is even, then the subalgebras of C_n are the algebras C_m, $m \leq n$, m even. If n is odd, then C_k is a subalgebra of C_n for every $k \leq n$.

Let $S_Y = C_n - Y$, where $Y \subseteq C_n - \{0, 1\}$. Let $S_{\mathcal{H}}$ be the set of Heyting subalgebras of C_n. Then $S_{\mathcal{H}} = \{S_Y \colon Y \subseteq C_n - \{0, 1\}\}$. For every j, $2 \leq j \leq n$, let $S_Y \in S_{\mathcal{H}}$ be such that $|Y| = n - j$. Then $A = S_Y \times S_{\sim Y}$ is a subalgebra of D_n isomorphic to D_j. In addition, $D_i \subseteq D_j$ if and only if $i \leq j$. If A is a subalgebra of D_n and A is not isomorphic to D_k, for any k, then $A \cong C_t$, for $t \leq n$. We have that $A = \{(x, \alpha(x)),\ x \in p_1(A)\}$, where α is an isomorphism from $p_1(A)$ onto $p_2(A)$, p_1, p_2 the projections in $D_n = C_n \times C_n$.

Observe that $\mathrm{Aut}_{\mathcal{H}}(C_n) = \{Id\}$ and $\mathrm{Aut}_{\mathcal{H}}(D_n) = \{Id, T\}$, where $T(x, y) = (y, x)$.

From the results of Section 2, we have the following.

Theorem 3.1. *For k odd, the finite subdirectly irreducible algebras in $\mathcal{L}_k$ are the algebras $(C_n)_d$ and $(D_n)_d$, $n \geq 2$, $d \in \mathcal{D}(k)$. The finite subdirectly irreducible algebras in $\mathcal{L}_k$, k even, are the algebras $(C_n)_d$, $(D_n)_d$ and the algebras $D_n^{(2)}$, $n \geq 2$, $d \in \mathcal{D}(k)$.* ∎

If K is a class of algebras, $SI(K)$ and $FSI(K)$ respectively denote the sets of one representative of each class of subdirectly irreducible and finite subdirectly irreducible isomorphic algebras in K. If $A \in K$, $H(A)$ denotes the set of homomorphic images of A, and $S(A)$ denotes the set of subalgebras of A.

Since $\mathcal{L}$ is locally finite, then $\mathcal{L}_k$ is clearly locally finite. In addition, $\mathcal{L}_k$ is congruence distributive, so we can apply Jónsson and Davey's results to find the lattice $\Lambda(\mathcal{L}_k)$ of subvarieties of $\mathcal{L}_k$. Jónsson's Theorem can be stated as follows.

Theorem 3.2 ([9]). *Let $V = V(K)$ be a congruence distributive variety generated by a finite set K of finite algebras. If we order $SI(V)$ by*

$$A \leq B \iff A \in H(S(B)) ,$$

then the lattice $\Lambda(V)$ of subvarieties of V is a finite distributive lattice isomorphic to $\mathcal{O}(SI(V))$, the lattice of order ideals of the ordered set $SI(V)$. In addition, a subvariety $X \in \Lambda(V)$ is join-irreducible if and only if $X = V(A)$, for some subdirectly irreducible algebra A. ∎

Davey proved the following generalization.

Theorem 3.3 ([5]). *Let V be a congruence distributive locally finite variety. Then $\Lambda(V)$ is a completely distributive lattice isomorphic to $\mathcal{O}(FSI(V))$.* ∎

Observe that if A is a finite subdirectly irreducible algebra in $\mathcal{L}_k$, then identifying isomorphic algebras, $H(S(A)) \setminus \{\text{the trivial algebra}\} = S(A)$.

From Theorem 2.13 it follows that the subalgebras of finite simple algebras in $\mathcal{L}_k$ are the following, where $\mathcal{H}_d$ denotes the variety of d-cyclic symmetric Heyting algebras:

1. $S((C_n)_d) = \{(C_m)_t, \text{ with } C_m \in S(C_n) \text{ and } t \in \mathcal{D}(d)\}$.

2. Since $\mathrm{Aut}_{\mathcal{H}}(D_n) = \{Id, T\}$ where $T(x,y) = (y,x)$, it follows that

 (**i**) For d odd, $\mathrm{Aut}_{\mathcal{H}_d}(D_n) = \{Id\}$ being that $T^2 = Id$. Consequently,

$$S((D_n)_d) = \Big\{(D_m)_t,\ (C_m)_t,\ m \leq n,\ t \in \mathcal{D}(d)\Big\} .$$

 (**ii**) For d even, $\mathrm{Aut}_{\mathcal{H}_d}(D_n) = \mathrm{Aut}_{\mathcal{H}}(D_n)$, so D_n can be endowed with two cyclic structures, namely, $D_n^{(1)} = \langle D_n, Id\rangle$ and $D_n^{(2)} = \langle D_n, T\rangle$. We have $S(D_n^{(1)}) = S((D_n)_1)$, and

$$S(D_n^{(2)}) = \begin{cases} \Big\{D_m^{(2)},\ C_m,\ m \leq n,\ m \text{ even}\Big\} & \text{for } n \text{ even} \\ \Big\{D_m^{(2)},\ C_m,\ m \leq n\Big\} & \text{for } n \text{ odd} . \end{cases}$$

 In addition,

$$S((D_n)_d) = \Big\{(D_m)_t,\ (C_m)_t,\ D_m^{(2)},\ m \leq n,\ t \in \mathcal{D}(d)\Big\} .$$

Part of the poset $FSI(\mathcal{L}_k)$, k even, can be visualized in Fig. 1. For the whole poset $FSI(\mathcal{L}_k)$, k even, the left part of the figure, that is, the part corresponding to the algebras $(C_2)_1$, $(C_3)_1$, ..., $(D_2)_1$, $(D_3)_1$ is repeated so many times as the elements of the lattice of divisors d of k, replacing the subscript 1 by d, and with the order given above. The right part of the Fig. 1, that is, the one corresponding to the algebras $D_n^{(2)}$, has to be deleted when k is odd. This figure will provide us with useful information for an easier understanding of the statements and proofs of the rest of the paper.

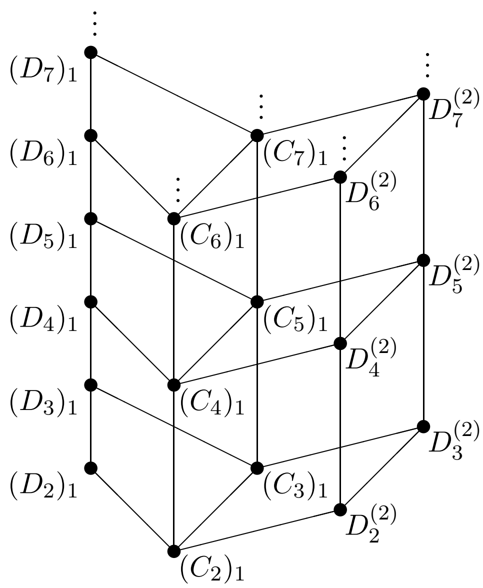

Fig. 1

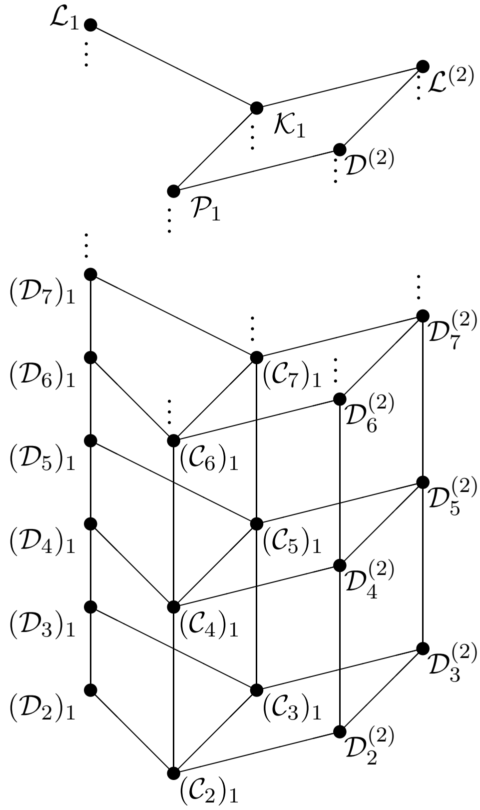

Fig. 2

Consider now the subvarieties (Fig 2) $\mathcal{D}_n^{(2)} = V(D_n^{(2)}),\ n \geq 2$,

$$\mathcal{D}^{(2)} = V\Big(\{D_{2n}^{(2)},\ n \geq 1\}\Big) = \bigvee_{n\geq 1} \mathcal{D}_{2n}^{(2)}\,, \qquad \mathcal{L}^{(2)} = V\Big(\{D_n^{(2)},\ n \geq 2\}\Big) = \bigvee_{n\geq 2} \mathcal{D}_n^{(2)}\,,$$

and, for each divisor d of k, $(\mathcal{C}_n)_d = V((C_n)_d)$, $(\mathcal{D}_n)_d = V((D_n)_d)$, $n \geq 2$,

$$\mathcal{P}_d = V\Big(\{(C_{2n})_d,\ n \geq 1\}\Big) = \bigvee_{n\geq 1} (\mathcal{C}_{2n})_d\ ,$$

$$\mathcal{K}_d = V\Big(\{(C_n)_d,\ n \geq 2\}\Big) = \bigvee_{n\geq 2} (\mathcal{C}_n)_d\ ,$$

and

$$\mathcal{L}_d = V\Big(\{(D_n)_d, n \geq 2\}\Big) = \bigvee_{n\geq 2} (\mathcal{D}_n)_d\ .$$

In what follows, all the results are to be stated for k even. Ignoring the varieties $\mathcal{D}_n^{(2)}$, $\mathcal{D}^{(2)}$ and $\mathcal{L}^{(2)}$ we obtain the corresponding results for k odd.

It is known that the varieties $\mathcal{D}_n^{(2)}$, $(\mathcal{C}_n)_d$ and $(\mathcal{D}_n)_d$ are join-irreducible.

Theorem 3.4. *The subvarieties $\mathcal{D}^{(2)}$, $\mathcal{L}^{(2)}$, and for every d divisor of k, $\mathcal{P}_d$, $\mathcal{K}_d$ and $\mathcal{L}_d$, are join-irreducible in $\Lambda(\mathcal{L}_k)$.*

Proof: Suppose that $\mathcal{P}_d = V_1 \vee V_2$ and let $I_1 = \{i \in \mathbb{N}\colon (C_{2i})_d \in V_1\}$ and $I_2 = \{i \in \mathbb{N}\colon (C_{2i})_d \in V_2\}$. If I_1 and I_2 are finite, then there exists m such that $(C_{2m})_d \notin V_1$ and $(C_{2m})_d \notin V_2$, which is a contradiction. So I_l is not bounded for some $l = 1, 2$. Suppose that I_1 is infinite. Then $(C_{2i})_d \in V_1$, for every $i \in \mathbb{N}$, which implies that $\mathcal{P}_d \subseteq V_1$. Thus $V_1 = \mathcal{P}_d$.

Suppose that $\mathcal{K}_d = V_1 \vee V_2$ and let $I_l = \{i \in \mathbb{N}\colon (C_{2i+1})_d \in V_l\}$, for $l = 1, 2$. Arguing as before and taking into account that $(C_{2i})_d$ is a subalgebra of $(C_{2i+1})_d$, we prove that $\mathcal{K}_d$ is join-irreducible.

A similar argument shows that $\mathcal{D}^{(2)}$, $\mathcal{L}^{(2)}$ and $\mathcal{L}_d$ are join-irreducible. ∎

It is long but computational to prove the following theorem. The demonstration is not included since the argument is an adaptation of a similar result that can be found in [3].

Theorem 3.5.

$$\mathcal{J}(\Lambda(\mathcal{L}_k)) = \{(\mathcal{C}_n)_d,\ (\mathcal{D}_n)_d,\ \mathcal{D}_n^{(2)}\}_{n\geq 2, d\in\mathcal{D}(k)} \cup \{\mathcal{P}_d,\ \mathcal{K}_d,\ \mathcal{L}_d\}_{d\in\mathcal{D}(k)} \cup \{\mathcal{D}^{(2)},\ \mathcal{L}^{(2)}\}\ .$$

Every variety $V \in \Lambda(\mathcal{L}_k)$ is a finite join of varieties in $\mathcal{J}(\Lambda(\mathcal{L}_k))$. ∎

4 – Equational bases

In this section we will give equational bases for each subvariety of $\mathcal{L}_k$.

Observe that if $d, d' \in \mathcal{D}(k)$ are such that d' is not a divisor of d, then for every integer $n \geq 2$, the equation $T^d(x) = x$ holds in $(D_n)_d$ and does not hold in $(D_n)_{d'}$. We denote by $\gamma_d(x)$ the term

$$\gamma_d(x) = (T^d(x) \to x) \wedge (x \to T^d(x)) \ .$$

Then we have the following theorem.

Theorem 4.1. *The equation $\gamma_d(x) = 1$ characterizes the variety $\mathcal{L}_d$, within $\mathcal{L}_k$, for each $d \in \mathcal{D}(k)$.* ∎

Let $\gamma_{\mathcal{K}}(x) = \neg\, x \to \sim x$. Observe that the chains C_n in $\mathcal{H}$ satisfy the Kleene equation

$$\gamma_{\mathcal{K}}(x) = 1 \ ,$$

and that this equation is not satisfied in the algebras D_n (it is enough to check in the Boolean elements of D_n). Since the operations $\to$ and $\sim$ are componentwise defined in $(C_n)_d$, these algebras satisfy the equation $\gamma_{\mathcal{K}}(x) = 1$, while the algebras $(D_n)_d$ and $D_n^{(2)}$, don't.

Let $\gamma_{\mathcal{P}}(x) = \neg \sim(\sim x \to x) \to \neg\neg\sim(x \to \sim x)$ and consider now the identity $\gamma_{\mathcal{P}}(x) = 1$ [3].

If $x \in C_{2n}$ and $x > \sim x$, then $x \to \sim x = \sim x$ and $\sim x \to x = 1$. Thus

$$\gamma_{\mathcal{P}}(x) = \neg \sim 1 \ \to \ \neg\neg \sim\sim x = 1 \ \to \ \neg\neg x \ .$$

As $x \neq 0$, $\neg\neg x = 1$ and consequently $\gamma_{\mathcal{P}}(x) = 1$. If $x < \sim x$ we also obtain $\gamma_{\mathcal{P}}(x) = 1$. Hence the equation $\gamma_{\mathcal{P}}(x) = 1$ holds in C_{2n}. If we choose the element $c \in C_{2n+1}$ such that $\sim c = c$, then $\gamma_{\mathcal{P}}(c) = 0$, and consequently the equation $\gamma_{\mathcal{P}}(x) = 1$ does not hold in C_{2n+1}. Again, since the operations $\to$ and $\sim$ are componentwise defined in $(C_n)_d$, the algebras $(C_{2n})_d$ satisfy the equation $\gamma_{\mathcal{P}}(x) = 1$, while the algebras $(C_{2n+1})_d$ don't. Then we have the following.

Theorem 4.2. *For each $d \in \mathcal{D}(k)$, the equation $\gamma_d(x) \wedge \gamma_{\mathcal{K}}(x) = 1$ characterizes the variety $\mathcal{K}_d$ and the equation $\gamma_d(x) \wedge \gamma_{\mathcal{K}}(x) \wedge \gamma_{\mathcal{P}}(x) = 1$ characterizes the variety $\mathcal{P}_d$.* ∎

The following term gives the form of the elements fixed by T of an algebra in $\mathcal{L}_k$. Consider ([2], [14])

$$\gamma_T^k(x) \ = \ x \wedge T(x) \wedge T^2(x) \wedge \cdots \wedge T^{k-1}(x) \ .$$

Observe that $T(\gamma_T^k(x)) = \gamma_T^k(x)$ and that if x is an element fixed by T, then $\gamma_T^k(x) = x$. Thus, if $A \in \mathcal{L}_k$ and $\gamma_T(A)$ denotes the set $\{\gamma_T(x) \colon x \in A\}$, then $\gamma_T(A) = \{x \in A : T(x) = x\}$. This term will play an important role in the equational description of $\mathcal{L}^{(2)}$ and $\mathcal{D}^{(2)}$. For this, let

$$\gamma^{(2)}(x) = \Big(T(\sim\neg x) \to \neg\neg x\Big) \wedge \Big(\neg\neg x \to T(\sim\neg x)\Big)$$

and consider the equation $\gamma^{(2)}(x) = 1$.

Observe that if $A \in FSI(\mathcal{L}_2)$, then, for $a \in A$, $\neg a$ is Boolean, and then, if $a \in \{0,1\}$, we have $\gamma^{(2)}(a) = 1$. If $A \notin \mathcal{L}^{(2)}$, then there exists a Boolean element $a \in A$ such that $T(\sim\neg a) = \neg a \neq \neg\neg a$, and consequently, $\gamma^{(2)}(a) \neq 1$, that is, $\gamma^{(2)}(x) = 1$ does not hold in A. If $A \in \mathcal{L}^{(2)}$ it is easy to check that A satifies $\gamma^{(2)}(x) = 1$.

Let $\gamma_{\mathcal{L}^{(2)}}(x) = \gamma_2(x) \wedge \gamma^{(2)}(x)$ and $\gamma_{\mathcal{D}^{(2)}}(x) = \gamma_{\mathcal{L}^{(2)}}(x) \wedge \gamma_{\mathcal{P}}(\gamma_T^2(x))$.

Theorem 4.3. *The equation $\gamma_{\mathcal{L}^{(2)}}(x) = 1$ characterizes the variety $\mathcal{L}^{(2)}$, and the equation $\gamma_{\mathcal{D}^{(2)}}(x) = 1$ characterizes the variety $\mathcal{D}^{(2)}$.*

Proof: If $A \in FSI(\mathcal{D}^{(2)})$, then $\gamma_T^2(A)$ is the diagonal of A, which is an n-element chain, n even, and consequently $\gamma_{\mathcal{P}}(\gamma_T^2(x)) = 1$ holds. The rest of the proof follows from the above remarks. ∎

Let

$$\gamma_n(x_0, x_1, ..., x_n) = \bigvee_{i=0}^{n-1} (x_i \to x_{i+1})$$

and consider now the equation $\gamma_n(x_0, x_1, ..., x_n) = 1$ given by Hecht and Katrinák [6]. This equation holds in $(C_n)_d$ and does not hold in $(C_{n+1})_d$. To see this, it is sufficient to consider C_n and C_{n+1}. If $b_0, b_1, ..., b_{n-1}, b_n$ are $n+1$ elements in the n-element chain C_n, for some j, $0 \leq j \leq n$, we have $b_j \leq b_{j+1}$, and the equation $\gamma_n(x_0, x_1, ..., x_n) = 1$ holds. For C_{n+1}, let $x_i = \frac{n-i}{n}$, $0 \leq i \leq n$. Then

$$\bigvee_{i=0}^{n-1} (x_i \to x_{i+1}) = \bigvee_{i=0}^{n-1} \frac{n-i-1}{n} = \frac{n-1}{n} \neq 1 .$$

So the equation does not hold in C_{n+1}.

We will often abbreviate $(x_0, x_1, ..., x_n) = \vec{x}$.

Theorem 4.4. *The equation $\gamma_n(\vec{x}) \wedge \gamma_d(x) = 1$ characterizes the variety $(\mathcal{D}_n)_d$. The equation $\gamma_n(\vec{x}) \wedge \gamma_d(x) \wedge \gamma_{\mathcal{K}}(x) = 1$ characterizes the variety $(\mathcal{C}_n)_d$ for n odd. The equation $\gamma_n(\vec{x}) \wedge \gamma_d(x) \wedge \gamma_{\mathcal{K}}(x) \wedge \gamma_{\mathcal{P}}(x) = 1$ characterizes the variety $(\mathcal{C}_n)_d$ for n even.* ∎

We have given an equational basis for every join-irreducible variety in $\Lambda(\mathcal{L}_k)$. Our aim now is to find equational bases for the remaining varieties, that is, for join-reducible varieties. By Theorem 3.5, it suffices to give an equation for the finite join of varieties in $\mathcal{J}(\Lambda(\mathcal{L}_k))$.

Let $\mathbf{V}$ be a join-reducible variety in $\Lambda(\mathcal{L}_k)$. Then $\mathbf{V} = \bigvee_{i=1}^n V_i$, with $V_i \in \mathcal{J}(\Lambda(\mathcal{L}_k))$. Let $\gamma_{V_i}(\vec{x}) = 1$ be the characteristic equation of V_i,

$$\gamma_{\mathbf{V}}(\vec{y}) = \bigvee_{i=1}^{n} \Big(\gamma_T^k(\gamma_{V_i}(\vec{y})) \leftrightarrow 1\Big) \wedge \Big(\sim \gamma_T^k(\gamma_{V_i}(\vec{y})) \leftrightarrow 0\Big)$$

and consider the identity $\gamma_{\mathbf{V}}(\vec{y}) = 1$, with $\vec{y} = (y_1, ..., y_r)$ and r as needed.

We claim that $\gamma_{\mathbf{V}}(\vec{y}) = 1$ is the characteristic equation for $\mathbf{V}$.

If $A \in FSI(\mathbf{V})$, then $A \in FSI(V_i)$ for some i. So the equation $\gamma_T^k(\gamma_{V_i}(\vec{y})) = 1$, and hence the equation $\sim \gamma_T^k(\gamma_{V_i}(\vec{y})) = 0$, holds in A. Thus A satisfies the equation $\gamma_{\mathbf{V}}(\vec{y}) = 1$. Since $\mathcal{L}_k$ is locally finite, it follows that $\mathbf{V}$ satisfies $\gamma_{\mathbf{V}}(\vec{y}) = 1$.

Suppose now that $(D_n)_d \notin \mathbf{V}$. Then $(D_n)_d \notin V_i$, for every $i = 1, ..., n$. So there exists $\vec{y} \in (D_n)_d$ such that $a = \gamma_T^k(\gamma_{V_i}(\vec{y})) \neq 1$, for $i = 1, ..., n$. Observe that a is an element of $(D_n)_d$ fixed by T, and that the set of elements in $(D_n)_d$ fixed by T is isomorphic to D_n. So, by the definition of D_n we can put $a = (a_1, a_2)$ with $a_i \in C_n$.

Since $a = (a_1, a_2) \neq (1, 1)$, then $a_1 \neq 1$ or $a_2 \neq 1$. If $a_1 \neq 1$, then

$$\gamma_T^k(\gamma_{V_i}(\vec{y})) \leftrightarrow 1 = a \leftrightarrow 1 = a = (a_1, a_2)$$

and

$$\sim \gamma_T^k(\gamma_{V_i}(\vec{y})) \leftrightarrow 0 = \sim a \leftrightarrow 0 = (\sim a_2, \sim a_1) \leftrightarrow 0 = (\neg \sim a_2, \neg \sim a_1)\ .$$

So

$$\Big(\gamma_T^k(\gamma_{V_i}(\vec{y})) \leftrightarrow 1\Big) \wedge \Big(\sim \gamma_T^k(\gamma_{V_i}(\vec{y})) \leftrightarrow 0\Big) = \Big(\neg \sim a_2 \wedge a_1,\ \neg \sim a_1 \wedge a_2\Big)\ .$$

If $a_1 = 0$, $(\neg \sim a_2 \wedge a_1, \neg \sim a_1 \wedge a_2) = (0, 0)$. If $a_1 \neq 0$, (and $a_1 \neq 1$) we have that $\sim a_1 \neq 0, 1$, and consequently, $(\neg \sim a_2 \wedge a_1, \neg \sim a_1 \wedge a_2) \leq (a_1, 0) \leq (b, 0)$, where b is the antiatom of C_n.

In a similar way it can be seen that if $a_1 = 1$ and $a_2 \neq 1$ then $(\neg\sim a_2 \wedge a_1, \neg\sim a_1 \wedge a_2) \leq (0, a_2) \leq (0, b)$.

Hence

$$\Big(\gamma_T^k(\gamma_{V_i}(\vec{y})) \leftrightarrow 1\Big) \wedge \Big(\sim\gamma_T^k(\gamma_{V_i}(\vec{y})) \leftrightarrow 0\Big) \;\leq\; (b, b)$$

for every $i = 1, ..., n$, and then $\gamma_{\mathbf{V}}(\vec{y}) \leq (b, b) \neq 1$, that is, $\gamma_{\mathbf{V}}(\vec{y}) = 1$ does not hold in $(D_n)_d$ for $(D_n)_d \notin \mathbf{V}$.

A similar argument proves that if $(C_n)_d$, $D_n^{(2)} \notin \mathbf{V}$ then $(C_n)_d$, $D_n^{(2)}$ do not satisfy $\gamma_{\mathbf{V}}(\vec{y}) = 1$. Therefore we have proved the following.

Theorem 4.5. *Let* $\mathbf{V}$ *be a join-reducible variety in* $\Lambda(\mathcal{L}_k)$*, with* $\mathbf{V} = \bigvee_{i=1}^n V_i$, $V_i \in \mathcal{J}(\Lambda(\mathcal{L}_k))$*. Then* $\gamma_{\mathbf{V}}(\vec{y}) = 1$ *is a characteristic equation for* $\mathbf{V}$. ∎

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