PORTUGALIAE MATHEMATICA
Vol. 58 Fasc. 3 – 2001
Nova Série

VARIATIONAL AND TOPOLOGICAL METHODS FOR DIRICHLET PROBLEMS WITH p-LAPLACIAN

G. Dinca, P. Jebelean and J. Mawhin

Presented by L. Sanchez

0 – Introduction

The aim of this paper is to obtain existence results for the Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x,u) \quad \text{in } \Omega , \\ u|_{\partial\Omega} = 0 . \end{cases}$$

Here $\Delta_p = \frac{\partial}{\partial x_i}(|\nabla u|^{p-2}\,\frac{\partial u}{\partial x_i})$, $1<p<\infty$, is the so-called p-Laplacian and $f\colon\ \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function which satisfies some special growth conditions. One of the main ideas is to present the operator $-\Delta_p$ as a duality mapping between $W^{1,p}_0(\Omega)$ and its dual $W^{-1,p'}(\Omega)$, $\frac{1}{p}+\frac{1}{p'}=1$, corresponding to the normalization function $\varphi(t)=t^{p-1}$. This idea, coming from Lions' book [23], proves to be a very fruitful one. The properties of the Nemytskii operator $(N_f\,u)(x)=f(x,u(x))$, generated by the Carathéodory function f, the homotopy invariance of the Leray–Schauder degree (under the form of a priori estimate, uniformly with respect to $\lambda\in[0,1]$, of the solutions set of the equation $u=\lambda(-\Delta_p)^{-1}N_f\,u$ with $(-\Delta_p)^{-1}N_f\colon\ W^{1,p}_0(\Omega)\to W^{1,p}_0(\Omega)$ compact), the well known Mountain Pass Theorem of Ambrosetti and Rabinowitz and the variational characterization of the first eigenvalue of $-\Delta_p$ on $W^{1,p}_0(\Omega)$ are the other essential tools which are also used.

Received: November 24, 2000.

1 – The p-Laplacian as duality mapping

The main idea of this paragraph is to present the operator $-\Delta_p$, $1 < p < \infty$, as duality mapping $J_\varphi\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, corresponding to the normalization function $\varphi(t) = t^{p-1}$.

Originated in the well known book of Lions (see [23]), this presentation has the advantage of allowing to apply the general results known for the duality mapping to the particular case of the p-Laplacian. For example, the surjectivity of the duality mapping (itself an immediate consequence of a well known result of Browder (see e.g. [8])) achieves the existence of the $W_0^{1,p}(\Omega)$-solution for the equation $-\Delta_p u = f$, with $f \in W^{-1,p'}(\Omega)$. Note that if $f \in W^{-1,p'}(\Omega)$ is given, then an element $u \in W_0^{1,p}(\Omega)$ is said to be solution of the Dirichlet problem

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega , \\ u|_{\partial\Omega} = 0 , \end{cases}$$

if the equality $-\Delta_p u = f$ is satisfied in the sense of $W^{-1,p'}(\Omega)$.

For the convenience of the reader we have considered to put away the definitions and the results concerning the duality mapping, which will be used in the sequel. Because these results are already known, the proof is often omitted; however the proof is given when these results achieve specific properties for p-Laplacian.

1.1. Basic results concerning the duality mapping

Below, X always is a real Banach space, X^* stands for its dual and $\langle \cdot, \cdot \rangle$ is the duality between X^* and X. The norm on X and on X^* is denoted by $\| \ \|$.

Given a set valued operator $A\colon X \to \mathcal{P}(X^*)$, *the range of* A is defined to be the set

$$\mathcal{R}(A) \;=\; \bigcup_{x \in D(A)} Ax$$

where $D(A) = \{x \in X \mid Ax \neq \emptyset\}$ is the *domain of* A. The operator A is said to be *monotone* if

$$\langle x_1^* - x_2^*, \, x_1 - x_2 \rangle \;\geq\; 0$$

whenever $x_1, x_2 \in D(A)$ and $x_1^* \in Ax_1$, $x_2^* \in Ax_2$.

A continuous function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is called a *normalization function* if it is strictly increasing, $\varphi(0) = 0$ and $\varphi(r) \to \infty$ with $r \to \infty$.

By *duality mapping corresponding to the normalization function* φ, we mean the set valued operator $J_\varphi\colon X \to \mathcal{P}(X^*)$ as following defined

$$J_\varphi x = \Big\{x^* \in X^* \mid \langle x^*, x\rangle = \varphi(\|x\|)\,\|x\|,\ \|x^*\| = \varphi(\|x\|)\Big\}$$

for $x \in X$.

By the Hahn–Banach theorem, it is easy to check that $D(J_\varphi) = X$.

Some of the main properties of the duality mapping are contained in the following

Theorem 1. *If φ is a normalization function, then:*

(**i**) *for each $x \in X$, $J_\varphi x$ is a bounded, closed and convex subset of X^*;*

(**ii**) *J_φ is monotone:*

$$\langle x_1^* - x_2^*,\, x_1 - x_2\rangle \geq \Big(\varphi(\|x_1\|) - \varphi(\|x_2\|)\Big)\Big(\|x_1\| - \|x_2\|\Big) \geq 0\ ,$$

for each $x_1, x_2 \in X$ and $x_1^* \in J_\varphi x_1$, $x_2^* \in J_\varphi x_2$;

(**iii**) *for each $x \in X$, $J_\varphi x = \partial\psi(x)$, where $\psi(x) = \int_0^{\|x\|} \varphi(t)\,dt$ and $\partial\psi : X \to \mathcal{P}(X^*)$ is the subdifferential of ψ in the sense of convex analysis, i.e.*

$$\partial\psi(x) = \Big\{x^* \in X^* \mid \psi(y) - \psi(x) \geq \langle x^*,\, y - x\rangle \text{ for all } y \in X\Big\}\ .$$

For proof we refer to Beurling and Livingston [5], Browder [8], Lions [23], Ciorãnescu [9].

Remark 1. We recall that a functional $f:\ X \to \mathbb{R}$ is said to be Gâteaux differentiable at $x \in X$ if there exists $f'(x) \in X^*$ such that

$$\lim_{t\to 0} \frac{f(x + t\,h) - f(x)}{t} = \langle f'(x), h\rangle$$

for all $h \in X$.

If the convex function $f:\ X \to \mathbb{R}$ is Gâteaux differentiable at $x \in X$, then it is a simple matter to verify that $\partial f(x)$ consists of a single element, namely $x^* = f'(x)$.

This simple remark will be essentially used in the sequel. □

The geometry of the space X (or X^*) supplies further properties of the duality mapping. That is why we recall the following (see e.g. Diestel [11])

Definition 1. The space X is said to be:

(**a**) uniformly convex if for each $\varepsilon \in (0,2]$, there exists $\delta(\varepsilon) > 0$ such that if $\|x\| = \|y\| = 1$ and $\|x-y\| \geq \varepsilon$ then $\|x+y\| \leq 2\,(1-\delta(\varepsilon))$;

(**b**) locally uniformly convex if from $\|x\| = \|x_n\| = 1$ and $\|x_n + x\| \to 2$ with $n \to \infty$, it results that $x_n \to x$ (strongly in X);

(**c**) strictly convex if for each $x, y \in X$ with $\|x\| = \|y\| = 1$, $x \neq y$ and $\lambda \in (0,1)$, we have $\|\lambda\, x + (1-\lambda)\, y\| < 1$. □

Theorem 2. *The following implications hold:*

X uniformly convex $\Longrightarrow$ X locally uniformly convex $\Longrightarrow$ X strictly convex .

For proof we refer to Diestel [11].

Theorem 3 (Pettis–Milman)**.** *If X is uniformly convex then X is reflexive.*

For proof see e.g. Brézis [6] or Diestel [11] — where the original proof of Pettis is given.

In the sequel, φ will be a normalization function.

Proposition 1.

(**i**) *If X is strictly convex, then J_φ is strictly monotone:*

$$\langle x_1^* - x_2^*,\, x_1 - x_2\rangle \,>\, 0$$

for each $x_1, x_2 \in X$, $x_1 \neq x_2$ and $x_1^* \in J_\varphi x_1$, $x_2^* \in J_\varphi x_2$; in particular, $J_\varphi x_1 \cap J_\varphi x_2 = \phi$ if $x_1 \neq x_2$.

(**ii**) *If X^* is strictly convex, then* $\mathrm{card}(J_\varphi x) = 1$, *for all $x \in X$.*

Proof: (**i**) First, it is easy to check that (see e.g. James [18]) if X is strictly convex, then for each $x^* \in X^* \backslash \{0\}$ there exists at most an element $x \in X$ with $\|x\| = 1$, such that $\langle x^*, x\rangle = \|x^*\|$.

Now, supposing by contradiction that there exist $x_1, x_2 \in X$ with $x_1 \neq x_2$ and $x_1^* \in J_\varphi x_1$, $x_2^* \in J_\varphi x_2$, satisfying

$$\langle x_1^* - x_2^*,\, x_1 - x_2\rangle \,=\, 0$$

we have

$$0 \,=\, \langle x_1^* - x_2^*,\, x_1 - x_2 \rangle \,\geq\, \Big(\varphi(\|x_1\|) - \varphi(\|x_2\|)\Big)\Big(\|x_1\| - \|x_2\|\Big) \,\geq\, 0$$

and so, we get $\|x_1\| = \|x_2\|$.

Remark that $x_1 \neq x_2$, $\|x_1\| = \|x_2\|$ implies $x_1 \neq 0$, $x_2 \neq 0$.

We obtain

$$\begin{aligned} 0 \,&=\, \left\langle x_1^* - x_2^*,\, \frac{x_1}{\|x_1\|} - \frac{x_2}{\|x_2\|} \right\rangle \\ &=\, \left[\varphi(\|x_1\|) - \left\langle x_1^*,\, \frac{x_2}{\|x_2\|}\right\rangle\right] + \left[\varphi(\|x_2\|) - \left\langle x_2^*,\, \frac{x_1}{\|x_1\|}\right\rangle\right] \end{aligned}$$

and both of the brackets being positive, it results

$$\|x_1^*\| \,=\, \varphi(\|x_1\|) \,=\, \left\langle x_1^*, \frac{x_2}{\|x_2\|}\right\rangle$$

which together with

$$\|x_1^*\| \,=\, \left\langle x_1^*, \frac{x_1}{\|x_1\|}\right\rangle$$

yields

$$\left\langle x_1^*, \frac{x_2}{\|x_2\|}\right\rangle \,=\, \|x_1^*\| \,=\, \left\langle x_1^*, \frac{x_1}{\|x_1\|}\right\rangle\,.$$

By the above mentioned result of James, we have $\frac{x_1}{\|x_1\|} = \frac{x_2}{\|x_2\|}$ i.e. $x_1 = x_2$ which is a contradiction.

(ii) It results from the fact that $J_\varphi x$ is a convex part of $\partial B(0, \varphi(\|x\|)) = \{x^* \in X^* \mid\ \|x^*\| = \varphi(\|x\|)\}$. ∎

Proposition 2. *If X is locally uniformly convex and J_φ is single valued ($J_\varphi\colon\, X \to X^*$), then J_φ satisfies the $(\mathcal{S}_+)$ condition: if $x_n \rightharpoonup x$ (weakly in X) and $\limsup\limits_{n\to\infty} \langle J_\varphi x_n,\, x_n - x\rangle \leq 0$ then $x_n \to x$ (strongly in X).*

Proof: It is immediately that from $x_n \rightharpoonup x$ and $\limsup\limits_{n\to\infty} \langle J_\varphi x_n,\, x_n - x\rangle \leq 0$ it results that $\limsup\limits_{n\to\infty} \langle J_\varphi x_n - J_\varphi x,\, x_n - x\rangle \leq 0$.

By

$$0 \,\leq\, \Big(\varphi(\|x_n\|) - \varphi(\|x\|)\Big)\Big(\|x_n\| - \|x\|\Big) \,\leq\, \Big\langle J_\varphi x_n - J_\varphi x,\, x_n - x\Big\rangle$$

it follows

$$\Big(\varphi(\|x_n\|) - \varphi(\|x\|)\Big)\Big(\|x_n\| - \|x\|\Big) \to 0$$

and hence, $\|x_n\| \to \|x\|$.

Now, by a well known result (see e.g. Diestel [11]), in a locally uniformly convex space, from $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ it results $x_n \to x$. ∎

Proposition 3. *If X is reflexive and $J_\varphi\colon X \to X^*$ then J_φ is demicontinuous: if $x_n \to x$ in X then $J_\varphi x_n \rightharpoonup J_\varphi x$ in X^*.*

Proof: By the boundedness of (x_n) it follows that $(J_\varphi x_n)$ is bounded in X^*. Since X^* is also reflexive, in order to prove that $J_\varphi x_n \rightharpoonup J_\varphi x$ it suffices to show that all subsequences of $(J_\varphi x_n)$ which are weakly convergent have the same limit, namely $J_\varphi x$.

Let $x^* \in X^*$ be the weak limit of a subsequence of $(J_\varphi x_n)$, still denoted by $(J_\varphi x_n)$.

By the weakly lower semicontinuity of the norm, we have:

$$\|x^*\| \le \liminf_{n\to\infty} \|J_\varphi x_n\| = \lim_{n\to\infty} \varphi(\|x_n\|) = \varphi(\|x\|) .$$

On the other hand, from $x_n \to x$ and $J_\varphi x_n \rightharpoonup x^*$, it follows that

$$\langle J_\varphi x_n, x_n\rangle \to \langle x^*, x\rangle .$$

But,

$$\langle J_\varphi x_n, x_n\rangle = \varphi(\|x_n\|)\,\|x_n\| \to \varphi(\|x\|)\,\|x\| .$$

We get $\langle x^*, x\rangle = \varphi(\|x\|)\,\|x\|$ and so, $\varphi(\|x\|) \le \|x^*\|$. Finally $\langle x^*, x\rangle = \varphi(\|x\|)\,\|x\|$ and $\varphi(\|x\|) = \|x^*\|$, which means $x^* = J_\varphi x$. ∎

Theorem 4. *Let X be reflexive and $J_\varphi\colon X \to X^*$. Then $\mathcal{R}(J_\varphi) = X^*$.*

Proof: The result follows from a well known theorem of Browder [8]: if X is reflexive and $T\colon X \to X^*$ is monotone, hemicontinuous and coercive, then T is surjective.

In our case, J_φ is monotone by Theorem 1 (ii). The fact that J_φ is hemicontinuous means:

$$\lim_{t\to 0}\Big\langle J_\varphi(u + t\,v),\, w\Big\rangle = \langle J_\varphi u, w\rangle$$

for $u, v, w \in X$, and it results from the demicontinuity of J_φ (Proposition 3).

Finally,

$$\frac{\langle J_\varphi u, u\rangle}{\|u\|} = \varphi(\|u\|) \to \infty \quad \text{ with } \; \|u\| \to \infty \ ,$$

hence J_φ is coercive. ∎

Theorem 5. *Let X be reflexive, locally uniformly convex and $J_\varphi\colon X \to X^*$.*

Then J_φ is bijective with its inverse J_φ^{-1} bounded, continuous and monotone. Moreover, it holds

$$J_\varphi^{-1} \;=\; \chi^{-1}\, J^*_{\varphi^{-1}}$$

where $\chi\colon X \to X^{**}$, is the canonical isomorphism between X and X^{**} and $J^*_{\varphi^{-1}}\colon X^* \to X^{**}$ is the duality mapping on X^* corresponding to the normalization function φ^{-1}.

Proof: By Theorem 4, J_φ is surjective. The space X being locally uniformly convex, it is strictly convex (Theorem 2) and by Proposition 1 (i) we have that J_φ is injective.

Let, now, χ be the canonical isomorphism between X and X^{**} ($\langle\chi(x), x^*\rangle = \langle x^*, x\rangle$) and let $J^*_{\varphi^{-1}}\colon X^* \to X^{**}$ be the duality mapping corresponding to the normalization function φ^{-1}. It should be noticed that because X is reflexive and locally uniformly convex, so is X^{**}; in particular X^{**} is strictly convex (Theorem 2) and, consequently, $J^*_{\varphi^{-1}}\colon X^* \to X^{**}$ is single valued (Proposition 1 (ii)).

It is easy to see that:

$$J_\varphi^{-1} \;=\; \chi^{-1}\, J^*_{\varphi^{-1}} \ . \tag{1}$$

From (1) and because a duality mapping maps bounded subsets into bounded subsets, it is immediately that J_φ^{-1} is bounded.

To see that J_φ^{-1} is continuous, let $x_n^* \to x^*$ in X^*.

From (1) and by Proposition 3 we have that $J_\varphi^{-1} x_n^* \rightharpoonup J_\varphi^{-1} x^*$. By the definition of the duality mapping J_φ, it is easy to see that $\|J_\varphi^{-1} x_n^*\| \to \|J_\varphi^{-1} x^*\|$. But the space X is assumed to be locally uniformly convex, and so, $J_\varphi^{-1} x_n^* \to J_\varphi^{-1} x^*$.

To prove the monotonicity of J_φ^{-1}, the space X is identified with X^{**} by the canonical isomorphism χ. Then, for $x_1^*, x_2^* \in X^*$, we have:

$$\Big\langle \chi(J_\varphi^{-1} x_1^*) - \chi(J_\varphi^{-1} x_2^*),\ x_1^* - x_2^* \Big\rangle \;=\; \Big\langle J^*_{\varphi^{-1}} x_1^* - J^*_{\varphi^{-1}} x_2^*,\ x_1^* - x_2^* \Big\rangle$$

and we apply Theorem 1 (ii) with φ^{-1} instead of φ and X^* instead of X. ∎

1.2. The functional framework

In the sequel, Ω will be a bounded domain in $\mathbb{R}^N$, $N \geq 2$ with Lipschitz continuous boundary and $p \in (1, \infty)$. We shall use the standard notations:

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \mid \frac{\partial u}{\partial x_i} \in L^p(\Omega),\ i = 1, ..., N \right\}$$

equipped with the norm:

$$\|u\|^p_{W^{1,p}(\Omega)} = \|u\|^p_{0,p} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|^p_{0,p}$$

where $\| \ \|_{0,p}$ is the usual norm on $L^p(\Omega)$.

It is well known that $(W^{1,p}(\Omega), \| \ \|_{W^{1,p}(\Omega)})$ is separable, reflexive and uniformly convex (see e.g. Adams [1, Theorem 3.5]).

We need the space

$$\begin{aligned} W_0^{1,p}(\Omega) &= \text{the closure of } \mathcal{C}_0^\infty(\Omega) \text{ in the space } W^{1,p}(\Omega) \\ &= \left\{ u \in W^{1,p}(\Omega) \mid \ u|_{\partial\Omega} = 0 \right\} \end{aligned}$$

the value of u on $\partial\Omega$ being understood in the sense of the trace: there is a unique linear and continuous operator $\gamma : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega)$ such that γ is surjective and for $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ we have $\gamma u = u|_{\partial\Omega}$. It holds $W_0^{1,p}(\Omega) = \ker \gamma$.

The dual space $(W_0^{1,p}(\Omega))^*$ will be denoted by $W^{-1,p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

For each $u \in W^{1,p}(\Omega)$, we put

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_N} \right), \qquad |\nabla u| = \left(\sum_{i=1}^N \left(\frac{\partial u}{\partial x_i} \right)^2 \right)^{\frac{1}{2}}$$

and let us remark that

$$|\nabla u| \in L^p(\Omega), \qquad |\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \in L^{p'}(\Omega) \quad \text{for } i = 1, ..., N .$$

Therefore, by the theorem concerning the form of the elements of $W^{-1,p'}(\Omega)$ (see Brézis [6] or Lions [23]) it follows that the operator $-\Delta_p$ may be seen acting from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ by

$$\langle -\Delta_p u, v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \, \nabla v \quad \text{for } u, v \in W_0^{1,p}(\Omega) .$$

By virtue of the Poincaré inequality

$$\|u\|_{0,p} \le \mathrm{Const}(\Omega, n)\, \||\nabla u|\|_{0,p} \quad \text{for all} \quad u \in W_0^{1,p}(\Omega)\ ,$$

the functional

$$W_0^{1,p}(\Omega) \ni u \longmapsto \|u\|_{1,p} := \||\nabla u|\|_{0,p}$$

is a norm on $W_0^{1,p}(\Omega)$, equivalent with $\|\ \|_{W^{1,p}(\Omega)}$.

Because the geometrical properties of the space are not automatically maintained by passing to an equivalent norm, we give a direct proof of the following theorem

Theorem 6. *The space $(W_0^{1,p}(\Omega), \|\ \|_{1,p})$ is uniformly convex.*

Proof: First, let $p \in [2, \infty)$. Then (see e.g. Adams [1, pp. 36]) for each $z, w \in \mathbb{R}^N$, it holds:

$$\left|\frac{z+w}{2}\right|^p + \left|\frac{z-w}{2}\right|^p \le \frac{1}{2}\Big(|z|^p + |w|^p\Big)\ .$$

Let $u, v \in W_0^{1,p}$ satisfy $\|u\|_{1,p} = \|v\|_{1,p} = 1$ and $\|u-v\|_{1,p} \ge \varepsilon \in (0,2]$. We have

$$\begin{aligned}\left\|\frac{u+v}{2}\right\|_{1,p}^p + \left\|\frac{u-v}{2}\right\|_{1,p}^p &= \int_\Omega \left(\left|\frac{\nabla u + \nabla v}{2}\right|^p + \left|\frac{\nabla u - \nabla v}{2}\right|^p\right) \\ &\le \frac{1}{2}\int_\Omega \Big(|\nabla u|^p + |\nabla v|^p\Big) = \frac{1}{2}\Big(\|u\|_{1,p}^p + \|v\|_{1,p}^p\Big) = 1\end{aligned}$$

which yields

$$\left\|\frac{u+v}{2}\right\|_{1,p}^p \le 1 - \left(\frac{\varepsilon}{2}\right)^p . \tag{2}$$

If $p \in (1,2)$, then (see e.g. Adams [1, pp. 36]) for each $z, w \in \mathbb{R}^N$, it holds:

$$\left|\frac{z+w}{2}\right|^{p'} + \left|\frac{z-w}{2}\right|^{p'} \le \left[\frac{1}{2}\Big(|z|^p + |w|^p\Big)\right]^{\frac{1}{p-1}} .$$

A straightforward computation shows that if $v \in W_0^{1,p}(\Omega)$ then $|\nabla v|^{p'} \in L^{p-1}(\Omega)$ and $\||\nabla v|^{p'}\|_{0,p-1} = \|v\|_{1,p}^{p'}$.

Let $v_1, v_2 \in W_0^{1,p}(\Omega)$. Then $|\nabla v_1|^{p'}, |\nabla v_2|^{p'} \in L^{p-1}(\Omega)$, with $0 < p-1 < 1$ and, according to Adams [1, pp. 25],

$$\left\| |\nabla v_1|^{p'} + |\nabla v_2|^{p'} \right\|_{0,p-1} \;\geq\; \| |\nabla v_1|^{p'} \|_{0,p-1} + \| |\nabla v_2|^{p'} \|_{0,p-1} \;.$$

Consequently,

$$\begin{aligned}
\left\| \frac{v_1+v_2}{2} \right\|_{1,p}^{p'} + \left\| \frac{v_1-v_2}{2} \right\|_{1,p}^{p'} &= \left\| \left| \nabla\, \frac{v_1+v_2}{2} \right|^{p'} \right\|_{0,p-1} + \left\| \left| \nabla\, \frac{v_1-v_2}{2} \right|^{p'} \right\|_{0,p-1} \\
&\leq \left\| \left| \nabla\, \frac{v_1+v_2}{2} \right|^{p'} + \left| \nabla\, \frac{v_1-v_2}{2} \right|^{p'} \right\|_{0,p-1} \\
&= \left[\int_\Omega \left(\left| \frac{\nabla v_1 + \nabla v_2}{2} \right|^{p'} + \left| \frac{\nabla v_1 - \nabla v_2}{2} \right|^{p'} \right)^{p-1} \right]^{\frac{1}{p-1}} \\
&\leq \left[\frac{1}{2} \int_\Omega \Big(|\nabla v_1|^p + |\nabla v_2|^p \Big) \right]^{\frac{1}{p-1}} \\
&= \left[\frac{1}{2}\, \|v_1\|_{1,p}^p + \frac{1}{2}\, \|v_2\|_{1,p}^p \right]^{\frac{1}{p-1}} .
\end{aligned}$$

For $u, v \in W_0^{1,p}(\Omega)$ with $\|u\|_{1,p} = \|v\|_{1,p} = 1$ and $\|u - v\|_{1,p} \geq \varepsilon \in (0,2]$, we get

$$\left\| \frac{u+v}{2} \right\|_{1,p}^{p'} \leq 1 - \left(\frac{\varepsilon}{2} \right)^{p'} . \tag{3}$$

From (2) and (3), in either case there exists $\delta(\varepsilon) > 0$ such that $\|u + v\|_{1,p} \leq 2\,(1 - \delta(\varepsilon))$. ∎

Below, the space $W_0^{1,p}(\Omega)$ always will be considered to be endowed with the norm $\| \; \|_{1,p}$.

Theorem 7. *The operator $-\Delta_p \colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is a potential one. More precisely, its potential is the functional $\psi \colon W_0^{1,p}(\Omega) \to \mathbb{R}$, given by*

$$\psi(u) \;=\; \frac{1}{p}\, \|u\|_{1,p}^p$$

and

$$\psi' \;=\; -\Delta_p \;=\; J_\varphi$$

where $J_\varphi \colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is the duality mapping corresponding to the normalization function $\varphi(t) = t^{p-1}$.

Proof: Since $\psi(u) = \int\limits_0^{\|u\|_{1,p}} \varphi(t)\,dt$, it is sufficient to prove that ψ is Gâteaux differentiable and $\psi'(u) = -\Delta_p u$ for all $u \in W_0^{1,p}(\Omega)$ (see Theorem 1 (iii) and Remark 1).

If $u \in W_0^{1,p}(\Omega)$ is such that $|\nabla u| = 0_{L^p(\Omega)}$ (this implies that $\|u\|_{1,p} = 0$ i.e. $u = 0_{W_0^{1,p}(\Omega)}$), then it is immediately that $\langle \psi'(u), h\rangle = 0$ for all $h \in W_0^{1,p}(\Omega)$. Therefore, we may suppose that $|\nabla u| \neq 0_{L^p(\Omega)}$.

It is obvious that ψ can be written as a product $\psi = QP$, where $Q\colon L^p(\Omega) \to \mathbb{R}$ is given by $Q(v) = \frac{1}{p}\,\|v\|_{0,p}^p$ and $P\colon\ W_0^{1,p}(\Omega) \to L^p(\Omega)$ is given by $P(v) = |\nabla v|$.

The functional Q is Gâteaux differentiable (see Vainberg [28]) and

$$\langle Q'(v), h\rangle \;=\; \langle |v|^{p-1}\operatorname{sign} v,\; h\rangle \tag{4}$$

for all $v, h \in L^p(\Omega)$.

Simple computations show that the operator P is Gâteaux differentiable at u and

$$P'(u)\cdot v \;=\; \frac{\nabla u\,\nabla v}{|\nabla u|} \tag{5}$$

for all $v \in W_0^{1,p}(\Omega)$.

Combining (4) and (5), we obtain that ψ is Gâteaux differentiable at u and

$$\begin{aligned}
\langle \psi'(u), v\rangle \;&=\; \Big\langle Q'(P(u)),\; P'(u)\cdot v\Big\rangle\\
&=\; \Big\langle |\nabla u|^{p-1},\; \frac{\nabla u\,\nabla v}{|\nabla u|}\Big\rangle\\
&=\; \int\limits_\Omega |\nabla u|^{p-2}\,\nabla u\,\nabla v \;\;=\;\; \langle -\Delta_p u, v\rangle
\end{aligned}$$

for all $v \in W_0^{1,p}(\Omega)$. ∎

Remark 2. Let $\|\ \|_*$ be the dual norm of $\|\ \|_{1,p}$. Then, we have

$$\|-\Delta_p u\|_* \;=\; \|J_\varphi u\|_* \;=\; \varphi(\|u\|_{1,p}) \;=\; \|u\|_{1,p}^{p-1}\;. \;\square$$

Theorem 8. *The operator $-\Delta_p$ defines a one-to-one correspondence between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$, with inverse $(-\Delta_p)^{-1}$ monotone, bounded and continuous.*

Proof: It is obvious from Theorems 7, 6 and 5. ∎

Remark 3. In fact, the above Theorem 8 asserts that for each $f \in W^{-1,p'}(\Omega)$, the equation $-\Delta_p u = f$ has a unique solution in $W_0^{1,p}(\Omega)$.

The properties of $(-\Delta_p)^{-1}$ show how the solution $u = (-\Delta_p)^{-1} f$ depends on the data f. These properties will be used in the sequel.

Since the elements of $W_0^{1,p}(\Omega)$ vanish on the boundary $\partial\Omega$ in the sense of the trace, it is natural that the unique solution in $W_0^{1,p}(\Omega)$ of the equation $-\Delta_p u = f$ to be called solution of the Dirichlet problem

$$\begin{cases} -\Delta_p u = f \ , \\ u|_{\partial\Omega} = 0 \ . \ \square \end{cases}$$

We shall conclude this section with two technical results which will be useful in the sequel.

We have seen (Theorem 7) that the functional $\psi(u) = \frac{1}{p} \|u\|_{1,p}^p$ is Gâteaux differentiable on $W_0^{1,p}(\Omega)$. Moreover, we have:

Theorem 9. *The functional ψ is continuously Fréchet differentiable on $W_0^{1,p}(\Omega)$.*

For the proof we need the following lemma (see Glowinski and Marrocco [16]).

Lemma 1.

(i) *If $p \in [2, \infty)$ then it holds:*

$$\Big| |z|^{p-2} z - |y|^{p-2}\, y \Big| \ \le\ \beta\, |z-y| \Big(|z| + |y| \Big)^{p-2} \qquad \text{for all} \ \ y, z \in \mathbb{R}^N$$

with β independent of y and z;

(ii) *If $p \in (1, 2]$, then it holds:*

$$\Big| |z|^{p-2} z - |y|^{p-2}\, y \Big| \ \le\ \beta\, |z-y|^{p-1} \qquad \text{for all} \ \ y, z \in \mathbb{R}^N$$

with β independent of y and z.

Proof of Theorem 9: Consider the product space $\mathcal{X} = \prod_{i=1}^N L^{p'}(\Omega)$ endowed with the norm

$$[h]_{0,p'} \ = \ \left(\sum_{i=1}^N \|h_i\|_{0,p'}^{p'} \right)^{\frac{1}{p'}}$$

for $h = (h_1, .., h_N) \in \mathcal{X}$.

We define $g = (g_1, ..., g_N)\colon W_0^{1,p}(\Omega) \to \mathcal{X}$ by

$$g(u) \;=\; |\nabla u|^{p-2}\, \nabla u\;,$$

for $u \in W_0^{1,p}(\Omega)$.

Let us prove that g is continuous.

By the equivalence of the norms on $\mathbb{R}^N$ we can find a constant $C_1 > 0$ such that

$$[h]_{0,p'}^{p'} \;\le\; C_1 \int_\Omega |h|^{p'}\;,$$

for all $h \in \mathcal{X}$.

Let $p \in (2, \infty)$ and $u, v \in W_0^{1,p}(\Omega)$. By Lemma 1 (i) and by the Hölder inequality, we have:

$$\begin{aligned}
\Big[g(u) - g(v)\Big]_{0,p}^{p'} &\le\; C_1 \int_\Omega \Big|g(u) - g(v)\Big|^{p'} \\
&\le\; C_2 \int_\Omega |\nabla u - \nabla v|^{p'} \Big(|\nabla u| + |\nabla v|\Big)^{p'(p-2)} \\
&\le\; C_2\, \|u - v\|_{1,p}^{p'} \Big\||\nabla u| + |\nabla v|\Big\|_{0,p}^{p'(p-2)}
\end{aligned}$$

which yields

$$\Big[g(u) - g(v)\Big]_{0,p} \;\le\; C\, \|u - v\|_{1,p}^{p'} \Big(\|u\|_{1,p} + \|v\|_{1,p}\Big)^{p'(p-2)} \tag{6}$$

with $C > 0$ constant independent of u and v.

If $p \in (1, 2]$ and $u, v \in W_0^{1,p}(\Omega)$, then from Lemma 1 (ii) it follows

$$\Big[g(u) - g(v)\Big]_{0,p'}^{p'} \;\le\; C_2' \int_\Omega |\nabla u - \nabla v|^{p'(p-1)} \;=\; C_2'\, \|u - v\|_{1,p}^{p}$$

or

$$\Big[g(u) - g(v)\Big]_{0,p'} \;\le\; C'\, \|u - v\|_{1,p}^{p-1} \tag{7}$$

with $C' > 0$ constant independent of u and v.

From (6) and (7) the continuity of g is obvious.

On the other hand, it holds

$$\Big\|\psi'(u) - \psi'(v)\Big\|_* \;\le\; K \Big[g(u) - g(v)\Big]_{0,p'} \tag{8}$$

with $K > 0$ constant independent of $u, v \in W_0^{1,p}(\Omega)$.

Indeed, by the Hölder inequality and by the equivalence of the norms on $\mathbb{R}^N$, we successively have:

$$\begin{aligned}
\Big|\Big\langle \psi'(u) - \psi'(v),\, w\Big\rangle\Big| \;&\leq\; \int_\Omega \Big|g(u) - g(v)\Big|\, |\nabla w| \\
&\leq\; \left(\int_\Omega \Big|g(u) - g(v)\Big|^{p'}\right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla w|^p\right)^{\frac{1}{p}} \\
&\leq\; K\left(\sum_{i=1}^N \Big\|g_i(u) - g_i(v)\Big\|_{0,p'}^{p'}\right)^{\frac{1}{p'}} \|w\|_{1,p} \\
&=\; K\Big[g(u) - g(v)\Big]_{0,p'} \|w\|_{1,p}
\end{aligned}$$

for $u, v, w \in W_0^{1,p}(\Omega)$, proving (8).

Now, by the continuity of g and (8), the conclusion of the theorem follows in a standard way: a functional is continuously Fréchet differentiable if and only if it is continuously Gâteaux differentiable. ∎

Remark 4. Naturally, the Fréchet differential of ψ at $u \in W_0^{1,p}(\Omega)$ will be denoted by $\psi'(u)$ and it is clear that $\psi'(u) = -\Delta_p u$. □

Theorem 10. *The operator $-\Delta_p$ satisfies the $(\mathcal{S}_+)$ condition: if $u_n \rightharpoonup u$ (weakly in $W_0^{1,p}(\Omega)$) and $\limsup_{n\to\infty} \langle -\Delta_p u_n,\, u_n - u\rangle \leq 0$, then $u_n \to u$ (strongly in $W_0^{1,p}(\Omega)$).*

Proof: It is a simple consequence of Proposition 2, Theorems 6, 2 and 7. ∎

2 – The problem $-\Delta_p u = f(x,u)$, $u|_{\partial\Omega} = 0$

In this paragraph we are interested about sufficient conditions on the right-hand member f ensuring the existence of some $u \in W_0^{1,p}(\Omega)$ such that the equality $-\Delta_p u = f(x,u)$ holds in the sense of $W^{-1,p'}(\Omega)$. Such an u will be called solution of the Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega\,, \\ u|_{\partial\Omega} = 0\,. \end{cases} \tag{9}$$

Thus, first of all, appropriate conditions on f ensuring that $N_f\, u \in W^{-1,p'}(\Omega)$ must be formulated, N_f being the well known Nemytskii operator defined by f, i.e. $(N_f\, u)(x) = f(x, u(x))$ for $x \in \Omega$. Hence, we are guided to consider some basic results on the Nemytskii operator. Simple proofs of these facts can be found in e.g. de Figueiredo [14] or Kavian [20] (see also Vainberg [28]).

2.1. Detour on the Nemytskii operator

Let Ω be as in the beginning of Section 1.2 and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a *Carathéodory function*, i.e.:

(**i**) for each $s \in \mathbb{R}$, the function $x \mapsto f(x,s)$ is Lebesgue measurable in Ω;

(**ii**) for a.e. $x \in \Omega$, the function $s \mapsto f(x,s)$ is continuous in $\mathbb{R}$.

We make the convention that in the case of a Carathéodory function, the assertion "$x \in \Omega$" to be understood in the sense "a.e. $x \in \Omega$".

Let $\mathcal{M}$ be the set of all measurable function $u \colon \Omega \to \mathbb{R}$.

Proposition 4. *If $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory, then, for each $u \in \mathcal{M}$, the function $N_f\, u \colon \Omega \to \mathbb{R}$ defined by*

$$(N_f\, u)(x) = f(x, u(x)) \quad \textit{for } \ x \in \Omega$$

is measurable in Ω.

In view of this proposition, a Carathéodory function $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ defines an operator $N_f \colon \mathcal{M} \to \mathcal{M}$, which is called *Nemytskii operator*.

The proposition here below states sufficient conditions when a Nemytskii operator maps an L^{p_1} space into another L^{p_2} space.

Proposition 5. *Suppose $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory and the following growth condition is satisfied:*

$$|f(x,s)| \,\le\, C\,|s|^r + b(x) \quad \textit{for } \ x \in \Omega, \ s \in \mathbb{R}\ ,$$

where $C \ge 0$ is constant, $r > 0$ and $b \in L^{q_1}(\Omega)$, $1 \le q_1 < \infty$.

Then $N_f(L^{q_1 r}(\Omega)) \subset L^{q_1}(\Omega)$. Moreover, N_f is continuous from $L^{q_1 r}(\Omega)$ into $L^{q_1}(\Omega)$ and maps bounded sets into bounded sets.

Concerning the potentiality of a Nemytskii operator, we have:

Proposition 6. *Suppose $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is Carathéodory and it satisfies the growth condition:*

$$|f(x,s)| \le C\,|s|^{q-1}+b(x) \quad \textit{for } x\in\Omega,\ s\in\mathbb{R}\ ,$$

where $C\ge 0$ is constant, $q>1$, $b\in L^{q'}(\Omega)$, $\frac{1}{q}+\frac{1}{q'}=1$.

Let $F\colon \Omega\times\mathbb{R}\to\mathbb{R}$ be defined by $F(x,s)=\int\limits_0^s f(x,\tau)\,d\tau$.

Then:

(i) *the function F is Carathéodory and there exist $C_1\ge 0$ constant and $c\in L^1(\Omega)$ such that*

$$|F(x,s)| \le C_1\,|s|^q+c(x) \quad \textit{for } x\in\Omega,\ s\in\mathbb{R}\ ;$$

(ii) *the functional $\Phi : L^q(\Omega)\to\mathbb{R}$ defined by $\Phi(u):=\int\limits_\Omega N_F\,u=\int\limits_\Omega F(x,u)$ is continuously Fréchet differentiable and $\Phi'(u)=N_F\,u$ for all $u\in L^q(\Omega)$.*

It should be noticed that, under the conditions of the above Proposition 6, we have $N_f(L^q(\Omega))\subset L^{q'}(\Omega)$, $N_F(L^q(\Omega))\subset L^1(\Omega)$, each of the Nemytskii operators N_f and N_F being continuous and bounded (it is a simple consequence of Proposition 5). It should also be noticed that for each fixed $u\in L^q(\Omega)$, it holds $N_f\,u=\Phi'(u)\in L^{q'}(\Omega)$.

Now, we return to problem (9).

First, let us denote by p^* the Sobolev conjugate exponent of p, i.e.

$$p^* = \begin{cases} \dfrac{N\,p}{N-p} & \text{if } p<N\ , \\ \infty & \text{if } p\ge N\ . \end{cases}$$

Below, the function $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ will be always assumed Carathéodory and satisfying the growth condition

$$|f(x,s)| \le C\,|s|^{q-1}+b(x) \quad \text{for } x\in\Omega,\ s\in\mathbb{R}\ , \tag{10}$$

where $C\ge 0$ is constant, $q\in(1,p^*)$, $b\in L^{q'}(\Omega)$, $\frac{1}{q}+\frac{1}{q'}=1$.

The restriction $q \in (1, p^*)$ ensures that the imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact. Hence, the diagram

$$W_0^{1,p}(\Omega) \overset{I_d}{\hookrightarrow} L^q(\Omega) \overset{N_f}{\to} L^{q'}(\Omega) \overset{I_d^*}{\hookrightarrow} W^{-1,p'}(\Omega)$$

shows that N_f is a compact operator (continuous and maps bounded sets into relatively compact sets) from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$.

An element $u \in W_0^{1,p}(\Omega)$ is said to be *solution* of problem (9) if

$$- \Delta_p u = N_f u \tag{11}$$

in the sense of $W^{-1,p'}(\Omega)$ i.e.

$$\langle -\Delta_p u, v\rangle = \langle N_f u, v\rangle \quad \text{for all } \ v \in W_0^{1,p}(\Omega)$$

or

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v = \int_\Omega f(x,u)\, v \quad \text{for all } \ v \in W_0^{1,p}(\Omega)\ . \tag{12}$$

At this stage, in the approach of problem (9), two strategies appear to be natural.

The first reduces problem (9) to a fixed point problem with compact operator.

Indeed, by Theorem 8, the operator $(-\Delta_p)^{-1} : W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega)$ is bounded and continuous.

Consequently, (11) can be equivalently written

$$u = (-\Delta_p)^{-1} N_f u \tag{13}$$

with $(-\Delta_p)^{-1} N_f \colon W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ a compact operator.

The second is a variational one: the solutions of problem (9) appear as critical points of a C^1 functional on $W_0^{1,p}(\Omega)$.

To see this, we first have that $-\Delta_p = \psi'$, where the functional $\psi(u) = \frac{1}{p}\|u\|_{1,p}^p$ is continuously Fréchet differentiable on $W_0^{1,p}(\Omega)$. On the other hand, under the basic condition (10) and taking into account that the imbedding $W_0^{1,p}(\Omega) \to L^q(\Omega)$ is continuous (in fact, compact), the functional $\Phi\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by $\Phi(u) = \int_\Omega F(x,u)$ with $F(x,s) = \int_0^s f(x,\tau)\, d\tau$, is continuously Fréchet differentiable on $W_0^{1,p}(\Omega)$ and $\Phi'(u) = N_f\, u$.

Consequently, the functional $\mathcal{F}\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{F}(u) \;=\; \psi(u) - \Phi(u) \;=\; \frac{1}{p}\,\|u\|_{1,p}^p - \int_\Omega F(x,u)$$

is C^1 in $W_0^{1,p}(\Omega)$ and

$$\mathcal{F}'(u) \;=\; (-\Delta_p)u - N_f\, u \;.$$

The search for solutions of problem (9) is, now, reduced to the search of critical points of $\mathcal{F}$, i.e. of those $u \in W_0^{1,p}(\Omega)$ such that $\mathcal{F}'(u) = 0$.

2.2. Existence of fixed points for $(-\Delta_p)^{-1}N_f$ via a Leray–Schauder technique

In this section, the "a priori estimate method" will be used in order to establish the existence of fixed points for the compact operator $T = (-\Delta_p)^{-1}N_f\colon W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ (see Dinca and Jebelean [13]).

For it suffices to prove that the set

$$\mathcal{S} \;=\; \Big\{u \in W_0^{1,p}(\Omega) \mid\; u = \alpha\, Tu \;\text{ for some } \alpha \in [0,1]\Big\}$$

is bounded in $W_0^{1,p}(\Omega)$.

By (10), for arbitrary $u \in W_0^{1,p}(\Omega)$, it is obvious that

$$\begin{aligned}
\|Tu\|_{1,p}^p \;&=\; \Big\langle(-\Delta_p)Tu,\, Tu\Big\rangle \;=\; \langle N_f\, u,\, Tu\rangle \;=\; \int_\Omega f(x,u)\, Tu \\
&\le\; \int_\Omega \Big(C\,|u|^{q-1} + b(x)\Big)\,|Tu| \;.
\end{aligned}$$

Furthermore, for $u \in \mathcal{S}$ i.e. $u = \alpha\, Tu$, with some $\alpha \in [0,1]$, we have

$$\begin{aligned}
\|Tu\|_{1,p}^p \;&\le\; C\,\alpha^{q-1}\,\|Tu\|_{0,q}^q + \|b\|_{0,q'}\,\|Tu\|_{0,q} \\
&\le\; C\,\alpha^{q-1}\,C_1^q\,\|Tu\|_{1,p}^q + \|b\|_{0,q'}\,C_1\,\|Tu\|_{1,p} \\
&\le\; C\,C_1^q\,\|Tu\|_{1,p}^q + \|b\|_{0,q'}\,C_1\,\|Tu\|_{1,p}
\end{aligned}$$

the constant C_1 coming from the continuous imbedding $W_0^{1,p}(\Omega) \to L^q(\Omega)$.

Consequently, for each $u \in \mathcal{S}$, it holds

$$\|Tu\|_{1,p}^p - K_1 \|Tu\|_{1,p}^q - K_2 \|Tu\|_{1,p} \leq 0 \tag{14}$$

with $K_1, K_2 \geq 0$ constants.

Remark that if (14) would imply that there is a constant $a \geq 0$ such that $\|Tu\|_{1,p} \leq a$, then the boundedness of $\mathcal{S}$ would be proved, because we would have $\|u\|_{1,p} = \alpha \|Tu\|_{1,p} \leq a$.

But this is obviously true if $q \in (1,p)$.

We have obtained

Theorem 11. *If the Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies (10) with $q \in (1,p)$ then the operator $(-\Delta_p)^{-1} N_f$ has fixed points in $W_0^{1,p}(\Omega)$ or equivalently, problem (9) has solutions. Moreover, the set of all solutions of problem (9) is bounded in the space $W_0^{1,p}(\Omega)$.*

Remark 5. We shall see that if (10) holds with $b \in L^\infty(\Omega)$ then the variational approach allows to weaken the hypotheses of Theorem 11 and problem (9) still has solutions but the boundedness of the set of all solutions will not be ensured. □

Remark 6. The condition $q \in (1,p)$ appear as a technical condition, needed in obtaining the boundedness of $\mathcal{S}$.

It is a natural question if the set $\mathcal{S}$ still remains bounded in case that $q = p$ and it is a simple matter to see that if $q = p$ and $1 - K_1 > 0$ then the above reasoning still works. This means that we are interested to work with "the best constants" C and C_1 such that $1 - C \cdot C_1^p$ be strictly positive.

There are situations when $1 - C \cdot C_1^p > 0$ fails. The example here below shows that then $\mathcal{S}$ can be unbounded.

Let λ be an eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$ and u be a corresponding eigenvector:

$$-\Delta_p u = \lambda |u|^{p-2} u .$$

It is clear that

$$\lambda = \frac{\|u\|_{1,p}^p}{\|u\|_{0,p}^p} . \tag{15}$$

Because $\|v\|_{0,p} \leq C_1 \|v\|_{1,p}$ for all $v \in W_0^{1,p}(\Omega)$, from (15), it results that $1 - \lambda C_1^p \leq 0$.

Consider the Carathéodory function $f(x,s) = \lambda\,|s|^{p-2}\,s$. Clearly, the growth condition (10) is satisfied with $q = p$ and $b = 0$, $c = \lambda$. Consequently, (14) becomes

$$(1 - \lambda\, C_1^p)\,\|Tu\|_{1,p}^p \,\leq\, 0$$

for all $u \in \mathcal{S}$ and no conclusion on the boundedness of $\mathcal{S}$ can be derived.

In fact, $\mathcal{S}$ is unbounded.

Indeed, we have $-\Delta_p(t\,u) = N_f(t\,u)$ i.e. $t\,u = (-\Delta_p)^{-1}N_f(t\,u)$ for all $t \in \mathbb{R}$, which means $\{t\,u \mid t \in \mathbb{R}\} \subset \mathcal{S}$ and so, $\mathcal{S}$ is unbounded. □

Remark 7. In the case $f(x,s) = g(s) + h(x)$ with $g\colon\, \mathbb{R} \to \mathbb{R}$ continuous and $h \in L^\infty(\Omega)$, the homotopy invariance of Leray–Schauder degree (but in a different functional framework) is used by Hachimi and Gossez [17] in order to prove the following result (see [17] Th 1.1):

If

$$\text{(i)}\ \limsup_{s\to\pm\infty} \frac{g(s)}{|s|^{p-2}s} \,\leq\, \lambda_1 \qquad \text{and} \qquad \text{(ii)}\ \limsup_{s\to\pm\infty} \frac{pG(s)}{|s|^p} \,<\, \lambda_1$$

where $G(s) = \int\limits_0^s g(\tau)\,d\tau$ and λ_1 is the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$, then the problem

$$-\Delta_p u \,=\, g(u) + h(x) \quad \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \partial\Omega$$

has a solution in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. □

2.3. Existence results by a direct variational method

As we have already emphasized in Section 2.1, in the variational approach, under the growth condition (10) on f, the solutions of problem (9) are precisely the critical points of the C^1 functional $\mathcal{F}\colon\, W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{F}(u) \,=\, \psi(u) - \Phi(u) \,=\, \frac{1}{p}\,\|u\|_{1,p}^p - \int\limits_\Omega F(x,u)$$

where $F(x,u) = \int\limits_0^s f(x,\tau)\,d\tau$.

Remark that the compact imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ implies that $\mathcal{F}$ is weakly lower semicontinuous in $W_0^{1,p}(\Omega)$.

So, by a standard result, in order to derive sufficient conditions for (9) has solutions, a first suitable way is to ensure the coerciveness of $\mathcal{F}$. Such results were obtained by Anane and Gossez [4] even in more general conditions on f. It is not our aim to detail this direction. However, we depict a few such of results.

First we refer to a result of Anane and Gossez [4].

Let $G\colon \Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function, such that, for any $R>0$,

$$\Omega \ni x \to \sup_{|s|\le R} |G(x,s)| \in L^1(\Omega)\ . \tag{16}$$

We write $G(x,s) = \frac{\lambda_1\,|s|^p}{p} + H(x,s)$, where λ_1 is the first eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ (see e.g. Anane [3], Lindqvist [22]) and let us define $H^\pm(x)$ as the superior limit of $\frac{H(x,s)}{|s|^p}$ as $s\to\pm\infty$ respectively.

It holds (see Proposition 2.1 in Anane–Gossez [4]):

Theorem 12. *Assume (16) and*

(**i**) $H^\pm(x)\le 0$ *a.e. uniformly in* x*;*

(**ii**) $H^+(x)<0$ *on* Ω^+ *and* $H^-(x)<0$ *on* Ω^- *for subsets* $\Omega^\pm$ *of positive measure.*

Then

$$\mathcal{G}(u) \,=\, \frac{1}{p}\,\|u\|_{1,p}^p - \int_\Omega G(x,u)$$

is well defined on $W_0^{1,p}(\Omega)$*, takes values in* $]-\infty,+\infty]$*, is weakly lower semicontinuous and coercive.*

We now return to problem (9).

By virtue of Proposition 6 (i), we have for any $R>0$

$$\sup_{|s|\le R} |F(x,s)| \,\le\, C_1\,R^q + c(x) \,\in\, L^1(\Omega)$$

showing that (16) is fulfilled with $G(x,s)=F(x,s)$.

Clearly, in this case $H(x,s) = F(x,s) - \frac{\lambda_1|s|^p}{p}$.

In order to extend a result of Mawhin–Ward–Willem [25] for the particular case $p=2$ to the general case $p\in(1,\infty)$, suppose that there exists a function $\alpha(x)\in L^\infty(\Omega)$ with $\alpha(x)<\lambda_1$, on a set of positive measure, such that

$$\limsup_{s\to\pm\infty} \frac{p\,F(x,s)}{|s|^p} \,\le\, \alpha(x) \,\le\, \lambda_1 \quad \text{uniformly in } \ \Omega\ . \tag{17}$$

We obtain

$$\begin{aligned} H^{\pm}(x) &= \limsup_{s\to\pm\infty} \frac{H(x,s)}{|s|^p} = \limsup_{s\to\pm\infty} \left(\frac{F(x,s)}{|s|^p} - \frac{\lambda_1}{p}\right) = \\ &= \limsup_{s\to\pm\infty} \frac{F(x,s)}{|s|^p} - \frac{\lambda_1}{p} \le \frac{\alpha(x)-\lambda_1}{p} \end{aligned}$$

which yields $H^{\pm}(x) \le 0$ uniformly in Ω, i.e. (i) in Theorem 12.

On the other hand, it is clear that

$$H^{\pm}(x) \le \frac{\alpha(x)-\lambda_1}{p} < 0$$

on the set of positive measure $\Omega_1 = \{x \in \Omega \mid \alpha(x) < \lambda_1\}$ and (ii) in Theorem 12 is checked.

We have obtained

Theorem 13. *Let $f\colon \Omega\times\mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the growth condition (10). Suppose that there exists $\alpha(x) \in L^\infty(\Omega)$ with $\alpha(x) < \lambda_1$ on a set of positive measure such that (17) holds.*

Then $\mathcal{F}$ is coercive; consequently problem (9) has solutions.

A **direct proof** of Theorem 13 can be given as it follows.

Define $\mathcal{N}\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\mathcal{N}(v) = \|v\|_{1,p}^p - \int_\Omega \alpha(x)\,|v|^p$$

and let us prove that there exists $\varepsilon_0 > 0$ such that

$$\mathcal{N}(v) \ge \varepsilon_0 \quad \text{for all } v \in W_0^{1,p}(\Omega) \text{ with } \|v\|_{1,p} = 1\ . \tag{18}$$

For, let us recall (see e.g. Anane [3]) that

$$\lambda_1 = \inf\left\{ \frac{\|v\|_{1,p}^p}{\|v\|_{0,p}^p} \mid v \in W_0^{1,p}(\Omega)\backslash\{0\}\right\} \tag{19}$$

the infimum being attained exactly when v is multiple of some function $u_1 > 0$.

By (17) and (19) it follows that $\mathcal{N}(v) \ge 0$ for all $v \in W_0^{1,p}(\Omega)$.

Supposing, by contradiction, that there is a sequence (v_n) in $W_0^{1,p}(\Omega)$ with $\|v_n\|_{1,p} = 1$ and $\mathcal{N}(v_n) \to 0$, we can find a subsequence of (v_n), still denoted by (v_n), and some $v_0 \in W_0^{1,p}(\Omega)$ with $v_n \rightharpoonup v_0$, weakly in $W_0^{1,p}(\Omega)$ and $v_n \to v_0$, strongly in $L^p(\Omega)$.

The functional $v \mapsto \int\limits_\Omega \alpha(x)\,|v|^p$ is continuous on $L^p(\Omega)$ and weakly continuous on $W_0^{1,p}(\Omega)$.

By the weakly lower semicontinuity of $\mathcal{N}$ on $W_0^{1,p}(\Omega)$, we infer

$$0 \;\le\; \|v_0\|_{1,p}^p - \int\limits_\Omega \alpha(x)\,|v_0|^p \;\le\; \liminf_{n\to\infty} \mathcal{N}(v_n) \;=\; 0$$

and so, $\|v_0\|_{1,p}^p = \int\limits_\Omega \alpha(x)\,|v_0|^p$. But $\mathcal{N}(v_n) \to 1 - \int\limits_\Omega \alpha(x)\,|v_0|^p$, hence

$$\|v_0\|_{1,p}^p \;=\; \int\limits_\Omega \alpha(x)\,|v_0|^p \;=\; 1$$

which yields $v_0 \neq 0$.

Then, by (17) and (19) we have

$$\lambda_1\,\|v_0\|_{0,p}^p \;\le\; \|v_0\|_{1,p}^p \;=\; \int\limits_\Omega \alpha(x)\,|v_0|^p \;\le\; \lambda_1\,\|v_0\|_{0,p}^p \tag{20}$$

which implies that $\lambda_1 = \frac{\|v_0\|_{1,p}^p}{\|v_0\|_{0,p}^p}$.

It results that v_0 is a nonzero multiple of u_1.

Consequently, $|v_0(x)| > 0$ a.e. in Ω.

But, then, denoting $\Omega_1 := \{x \in \Omega \mid \alpha(x) < \lambda_1\}$, because $\mathrm{meas}(\Omega_1) > 0$, we get

$$\int\limits_\Omega \alpha(x)\,|v_0|^p \;=\; \int\limits_{\Omega_1} \alpha(x)\,|v_0|^p \;+\; \int\limits_{\Omega\setminus\Omega_1} \alpha(x)\,|v_0|^p \;<\; \lambda_1\,\|v_0\|_{0,p}^p$$

contradicting (20). So, (18) is proved.

Obviously, from (18) we have

$$\|v\|_{1,p}^p - \int\limits_\Omega \alpha(x)\,|v_0|^p \;\ge\; \varepsilon_0\,\|v\|_{1,p}^p \quad \text{for all } \ v \in W_0^{1,p}(\Omega)\ . \tag{21}$$

Let $\varepsilon > 0$ be such that $\varepsilon < \lambda_1\,\varepsilon_0$.

Using (17) and by Proposition 6 (i) a straightforward computation shows that there exists a constant $k = k(\varepsilon)$ such that

$$F(x,s) \;\le\; \frac{\alpha(x)+\varepsilon}{p}\,|s|^p + k + c(x) \quad \text{for } \ x \in \Omega, \quad s \in \mathbb{R}\ . \tag{22}$$

Now, by (21) and (22) we estimate $\mathcal{F}$ as it follows

$$\begin{aligned}\mathcal{F}(v) &\geq \frac{1}{p}\left(\varepsilon_0 \|v\|_{1,p}^p - \varepsilon \|v\|_{0,p}^p\right) - k_1 \\ &\geq \frac{\lambda_1 \varepsilon_0 - \varepsilon}{p} \|v\|_{1,p}^p - k_1 \;\to\; \infty\end{aligned}$$

as $\|v\|_{1,p} \to \infty$. ∎

Remark 8.

(**i**) If in (10) the function b is required to be in $L^\infty(\Omega)$ then it is easy to check that if $q \in (1,p)$ then (17) holds with $\alpha \equiv 0$ and so, problem (9) has solutions. But, as we have already remarked (see Remark 5), if $b \in L^\infty(\Omega)$ and only (17) is required then the boundedness of the set of solutions (as in Theorem 11) is not stated.

(**ii**) The idea of the above direct proof of Theorem 13 is a suitable one in proving the existence of solutions for a multivalued variant of problem (9) (see Proposition 4.1 in Jebelean [19]). □

2.4. Using the Mountain Pass Theorem (cf. Dinca, Jebelean and Mawhin [12])

Again, the Carathéodory function $f:\ \Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to satisfy the growth condition (10).

The existence of nontrivial critical points of the C^1 functional $\mathcal{F}\colon\ W_0^{1,p}(\Omega)\to\mathbb{R}$ means that the Dirichlet problem (9) has nontrivial solutions.

This section is devoted to formulate supplementary conditions on f and F ensuring the existence of such nontrivial critical points for $\mathcal{F}$.

The main tool in this direction is the well known "Mountain Pass Theorem" of Ambrosetti and Rabinowitz [2] which we recall here in a useful and popular form (see e.g. Theorem 2.2 in Rabinowitz [26]).

Theorem 14. *Let X be a real Banach space and $I \in C^1(X,\mathbb{R})$ satisfying the Palais–Smale (PS) condition. Suppose $I(0) = 0$ and*

($\mathbf{I}_1$) *there are constants $\rho, \alpha > 0$ such that $I|_{\|x\|=\rho} \geq \alpha$;*

($\mathbf{I}_2$) *there is an element $e \in X$, $\|e\| > \rho$ such that $I(e) \leq 0$.*

Then I possesses a critical value $c \geq \alpha$. Moreover c can be characterized as

$$c = \inf_{g\in\Gamma} \max_{u\in g([0,1])} I(u)$$

where

$$\Gamma = \Big\{g \in C([0,1], X) \,|\; g(0) = 0,\; g(1) = e\Big\} .$$

It is obvious that each critical point u at level c ($I'(u) = 0,\ I(u) = c$) is a nontrivial one. Consequently, if the hypothesis of Theorem 14 are satisfied with $X = W_0^{1,p}(\Omega)$ and $I = \mathcal{F}$ then the existence of nontrivial solutions for problem (9) is ensured.

We first deal with the (PS) condition for $\mathcal{F}$.

Recall that $\mathcal{F}$ is said to satisfy the *(PS) condition* if any sequence $(u_n) \subset W_0^{1,p}(\Omega)$ for which $\mathcal{F}(u_n)$ is bounded and $\mathcal{F}'(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence.

Lemma 2. *If $(u_n) \subset W_0^{1,p}(\Omega)$ is bounded and $\mathcal{F}'(u_n) \to 0$ as $n \to \infty$, then (u_n) has a convergent subsequence.*

Proof: One can extract a subsequence (u_{n_k}) of (u_n), weakly convergent to some $u \in W_0^{1,p}(\Omega)$. As $\mathcal{F}'(u_{n_k}) \to 0$, we infer

$$\Big\langle \mathcal{F}'(u_{n_k}),\, u_{n_k} - u\Big\rangle = \Big\langle -\Delta_p u_{n_k} - N_f\, u_{n_k},\, u_{n_k} - u\Big\rangle \to 0 . \tag{23}$$

But

$$\langle N_f\, u_{n_k},\, u_{n_k} - u\rangle \to 0$$

because of

$$\Big|\langle N_f\, u_{n_k},\, u_{n_k} - u\rangle\Big| \leq \|N_f\, u_{n_k}\|_{o,q'}\, \|u_{n_k} - u\|_{0,q}$$

and, by $u_{n_k} \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and by the compact imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, we get $u_{n_k} \to u$ strongly in $L^q(\Omega)$. Notice that $(N_f\, u_{n_k})$ is bounded in $L^{q'}(\Omega)$.

By (23) we obtain

$$\langle -\Delta_p u_{n_k},\, u_{n_k} - u\rangle \to 0$$

which, together with Theorem 10 shows that $u_{n_k} \to u$ strongly in $W_0^{1,p}(\Omega)$. ∎

Theorem 15. *If there exist $\theta > p$ and $s_0 > 0$ such that*

$$\theta\, F(x,s) \leq s\, f(x,s) \quad \textit{for}\;\; x \in \Omega,\;\; |s| \geq s_0 , \tag{24}$$

then $\mathcal{F}$ satisfies the (PS) condition.

Remark 9. It is worth noticing that (24) extends the well known condition

$$\text{there are } \theta > 2 \text{ and } s_0 > 0 \text{ such that}$$
$$0 < \theta F(x,s) \le s\, f(x,s) \quad \text{for } x \in \Omega, \ |s| \ge s_0$$

which was first formulated by Ambrosetti and Rabinowitz [2] as a sufficient condition to ensure that $\mathcal{F}$ satisfies (PS) in the particular case $p = 2$. □

Proof of Theorem 15: It suffices to show that any sequence $(u_n) \subset W_0^{1,p}(\Omega)$ for which $(\mathcal{F}(u_n))$ is bounded and $\mathcal{F}'(u_n) \to 0$, is bounded. Then Lemma 2 will accomplish the proof.

Let $d \in \mathbb{R}$ be such that $\mathcal{F}(u_n) \le d$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we denote

$$\Omega_n = \Big\{x \in \Omega \,|\, |u_n(x)| \ge s_0\Big\}, \qquad \Omega_n' = \Omega \backslash \Omega_n \ .$$

We have

$$\frac{1}{p}\,\|u_n\|_{1,p}^p - \Big(\int_{\Omega_n} F(x,u_n) + \int_{\Omega_n'} F(x,u_n)\Big) \le d \ . \tag{25}$$

We proceed with obtaining estimations independent of n for the integrals in (25).

Let $n \in \mathbb{N}$ be arbitrary chosen.

If $x \in \Omega_n'$, then $|u_n(x)| < s_0$ and by Proposition 6 (i), it follows

$$F(x,u_n) \le C_1\,|u_n(x)|^q + c(x) \le C_1\, s_0^q + c(x)$$

and hence

$$\int_{\Omega_n'} F(x,u_n) \le C_1\, s_0^q \cdot \mathrm{meas}(\Omega) + \int_\Omega c(x) = K_1 \ . \tag{26}$$

If $x \in \Omega_n$, then $|u_n(x)| \ge s_0$ and by (24) it holds

$$F(x,u_n) \le \frac{1}{\theta}\, f(x,u_n(x))\, u_n(x)$$

which gives

$$\int_{\Omega_n} F(x,u_n) \le \int_{\Omega_n} \frac{1}{\theta}\, f(x,u_n)\, u_n = \frac{1}{\theta}\Big(\int_\Omega f(x,u_n)\, u_n - \int_{\Omega_n'} f(x,u_n)\, u_n\Big) \ . \tag{27}$$

By the growth condition (10), we deduce

$$\begin{aligned} \Big|\int_{\Omega_n'} f(x,u_n)\, u_n\Big| &\le \int_{\Omega_n'} \Big(C\,|u_n|^q + b(x)\,|u_n|\Big) \\ &\le C\, s_0^q \cdot \mathrm{meas}(\Omega) + s_0 \int_\Omega b(x) = K_2 \end{aligned}$$

which yields

$$-\frac{1}{\theta}\int_{\Omega_n'} f(x,u_n)\,u_n \;\le\; \frac{K_2}{\theta}\;. \tag{28}$$

Finally, by (25), (26), (27) and (28) we get

$$\frac{1}{p}\,\|u_n\|_{1,p}^p - \frac{1}{\theta}\int_\Omega f(x,u_n)\,u_n \;\le\; d+K_1+\frac{K_2}{\theta} \;=\; K\;,$$

$$\frac{1}{p}\,\|u_n\|_{1,p}^p - \frac{1}{\theta}\,\langle N_f\,u_n,\,u_n\rangle \;\le\; K\;. \tag{29}$$

On the other hand, because $\mathcal{F}'(u_n)\to 0$ as $n\to\infty$, there is $n_0\in\mathbb{N}$ such that $|\langle\mathcal{F}'(u_n),u_n\rangle|\le\|u_n\|_{1.p}$ for $n\ge n_0$. Consequently, for all $n\ge n_0$, we have

$$\Big|\langle-\Delta_p u_n,\,u_n\rangle - \langle N_f\,u_n,\,u_n\rangle\Big| \;\le\; \|u_n\|_{1,p}$$

or

$$\Big|\|u_n\|_{1.p}^p - \langle N_f\,u_n,\,u_n\rangle\Big| \;\le\; \|u_n\|_{1,p}$$

which gives

$$-\frac{1}{\theta}\,\|u_n\|_{1,p}^p - \frac{1}{\theta}\,\|u_n\|_{1,p} \;\le\; -\frac{1}{\theta}\,\langle N_f\,u_n,\,u_n\rangle\;. \tag{30}$$

Now, from (29) and (30) it results

$$\left(\frac{1}{p}-\frac{1}{\theta}\right)\|u_n\|_{1,p}^p - \frac{1}{\theta}\,\|u_n\|_{1,p} \;\le\; K$$

and taking into account that $\theta>p$, we conclude that (u_n) is bounded. ∎

Now, viewing (I_2) in Theorem 14, the next step is to obtain sufficient conditions for $\mathcal{F}$ be unbounded below in $W_0^{1,p}(\Omega)$. The following lemma will throw light in the role of this unboundedness.

Lemma 3. *The functional $\mathcal{F}$ has the properties:*

(i) $\mathcal{F}(0)=0$;

(ii) *$\mathcal{F}$ maps bounded sets into bounded sets.*

Proof: (**i**) Obvious.

(**ii**) Because

$$\mathcal{F}'(u) \;=\; -\Delta_p u - N_f\,u$$

it is clear that

$$\begin{aligned}\|\mathcal{F}'(u)\|_* &\le \|-\Delta_p u\|_* + \|N_f\, u\|_* \\ &\le \|u\|_{1,p}^{p-1} + K\,\|N_f\, u\|_{0,q'}\end{aligned}$$

for all $u \in W_0^{1,p}(\Omega)$.

Furthermore, by the (compact) imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ and by virtue of the fact that N_f maps bounded sets in $L^q(\Omega)$ into bounded sets in $L^{q'}(\Omega)$, we conclude that $\mathcal{F}'$ maps bounded sets in $W_0^{1,p}(\Omega)$ into bounded sets in $W^{-1,p'}(\Omega)$.

Let v be arbitrary chosen in $W_0^{1,p}(\Omega)$. We have:

$$|\mathcal{F}(v)| = |\mathcal{F}(v) - \mathcal{F}(0)| = |\langle \mathcal{F}'(\xi v), v\rangle| \le \|\mathcal{F}'(\xi v)\|_*\, \|v\|_{1,p}$$

with $\xi \in (0,1)$. Then (ii) follows by the above conclusion on $\mathcal{F}'$. ∎

Remark 10. Actually, (ii) in the above Lemma 3 is a consequence of the growth condition (10) and of the fact that $\|-\Delta_p u\|_* = \|u\|_{1,p}^{p-1}$. □

Remark 11. Assume $\mathcal{F}$ is unbounded from below. Then, for any $\rho > 0$ there is an element $e \in W_0^{1,p}(\Omega)$ with $\|e\|_{1,p} \ge \rho$, such that $\mathcal{F}(e) \le 0$.

Indeed, suppose by contradiction that there is some $\rho > 0$ such that for all $u \in W_0^{1,p}(\Omega)$ with $\|u\| \ge \rho$, it holds $\mathcal{F}(u) \ge 0$. Then by Lemma 3 (ii), the set $\{\mathcal{F}(u) \mid \|u\|_{1,p} < \rho\}$ is bounded. It results that $\mathcal{F}$ is bounded from below, which is a contradiction. □

Theorem 16. *If either*

(i) *there are numbers* $\theta > p$ *and* $s_1 > 0$ *such that*

$$0 < \theta\, F(x,s) \le s\, f(x,s) \quad \textit{for } x \in \Omega, \quad s \ge s_1\,, \tag{31}$$

or

(ii) *there are numbers* $\theta > p$ *and* $s_1 < 0$ *such that*

$$0 < \theta\, F(x,s) \le s\, f(x,s) \quad \textit{for } x \in \Omega, \quad s \le s_1\,, \tag{32}$$

then $\mathcal{F}$ *is unbounded from below.*

Proof: We shall prove the sufficiency of condition (i) (similar argument if (ii) holds).

More precisely, we'll show that if $u \in W_0^{1,p}(\Omega)$, $u > 0$ is such that $\mathrm{meas}(M_1(u)) > 0$ holds, with

$$M_1(u) = \Big\{ x \in \Omega \mid u(x) \geq s_1 \Big\} ,$$

then $\mathcal{F}(\lambda u) \to -\infty$ as $\lambda \to \infty$.

First, for $\lambda \geq 1$, let us denote

$$M_\lambda(u) = \Big\{ x \in \Omega \mid \lambda\, u(x) \geq s_1 \Big\}$$

and let us remark that $M_1(u) \subset M_\lambda(u)$, and hence $\mathrm{meas}(M_\lambda(u)) > 0$.

On the other hand, there is a function $\gamma \in L^1(\Omega)$, $\gamma > 0$ such that

$$F(x,s) \geq \gamma(x)\, s^\theta \quad \text{for } x \in \Omega, \quad s \geq s_1 \; . \tag{33}$$

Indeed, for $x \in \Omega$ and $\tau \geq s_1$, by (31) we have

$$\frac{\theta}{\tau} \leq \frac{f(x,\tau)}{F(x,\tau)} = \frac{F'_\tau(x,\tau)}{F(x,\tau)}$$

and integrating from s_1 to s we get

$$\ln\left(\frac{s}{s_1}\right)^\theta \leq \ln F(x,s) - \ln F(x,s_1)$$

which implies (33) with $\gamma(x) = \frac{F(x,s_1)}{s_1^\theta} > 0$.

Now, let $\lambda \geq 1$. Clearly,

$$\mathcal{F}(\lambda u) = \frac{\lambda^p}{p}\, \|u\|_{1,p}^p - \bigg(\int\limits_{M_\lambda(u)} F(x,\lambda u) + \int\limits_{\Omega\setminus M_\lambda(u)} F(x,\lambda u) \bigg) \; . \tag{34}$$

If $x \in M_\lambda(u)$ then $\lambda u(x) \geq s_1$, and by (33)

$$F(x,\lambda u(x)) \geq \gamma(x)\, \lambda^\theta\, u^\theta \; .$$

Therefore,

$$\int\limits_{M_\lambda(u)} F(x,\lambda u) \geq \lambda^\theta \int\limits_{M_\lambda(u)} \gamma(x)\, u^\theta \geq \lambda^\theta \int\limits_{M_1(u)} \gamma(x)\, u^\theta = \lambda^\theta\, K_1(u) \; , \tag{35}$$

with $K_1(u) > 0$.

If $x \in \Omega \setminus M_\lambda(u)$ then $\lambda u(x) < s_1$, and by virtue of Proposition 6 (i), we obtain

$$|F(x,\lambda u(x))| \leq C_1\, \lambda^q\, u^q + c(x) \leq C_1\, s_1^q + c(x) \; .$$

Therefore,

$$\left| \int_{\Omega \setminus M_\lambda(u)} F(x, \lambda u) \right| \le C_1 \, s_1^q \cdot \mathrm{meas}(\Omega) + \int_\Omega c(x) = K_2 \,. \tag{36}$$

From (34), (35) and (36) it results

$$\mathcal{F}(\lambda u) \le \frac{\lambda^p}{p} \, \|u\|_{1,p}^p - \lambda^\theta \, K_1(u) + K_2 \;\to\; -\infty \qquad \text{as} \;\; \lambda \to \infty$$

and the proof is complete. ∎

Concerning condition (I_1) in Theorem 14, we have the following

Theorem 17. *Suppose the Carathéodory function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies*

(i) *there is $q \in (1, p^*)$ such that*

$$|f(x,s)| \le C(|s|^{q-1} + 1) \qquad \textit{for} \;\; x \in \Omega, \quad s \in \mathbb{R} \,,$$

with $C \ge 0$ constant;

(ii)

$$\limsup_{s \to 0} \frac{f(x,s)}{|s|^{p-2}\, s} < \lambda_1 \qquad \textit{uniformly with} \;\; x \in \Omega$$

where λ_1 is the first eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\Omega)$.

Then there are constants ρ, $\alpha > 0$ such that $\mathcal{F}|_{\|u\|_{1,p}=\rho} \ge \alpha$.

Proof: We define $h\colon \Omega \to \mathbb{R}$ by putting

$$h(x) = \limsup_{s \to 0} \frac{f(x,s)}{|s|^{p-2}\, s} \,.$$

By (ii) we can find $\mu \in (0, \lambda_1)$ such that $h(x) < \mu$ uniformly with $x \in \Omega$. Therefore, there is some $\delta_\mu > 0$ such that

$$\frac{f(x,s)}{|s|^{p-2}\, s} \le \mu \qquad \text{for} \;\; x \in \Omega, \quad 0 < |s| < \delta_\mu \,,$$

or

$$f(x,s) \le \mu \, s^{p-1} \qquad \text{for} \;\; x \in \Omega, \quad s \in (0, \delta_\mu) \,, \tag{37}$$

$$-\mu \, |s|^{p-1} \le f(x,s) \qquad \text{for} \;\; x \in \Omega, \quad s \in (-\delta_\mu, 0) \,. \tag{38}$$

Remark that the Carathéodory function f satisfies $f(x,0) = 0$ for $x \in \Omega$.

From (37), (38) and by the definition of F, we infer

$$F(x,s) \le \frac{\mu}{p}\,|s|^p \quad \text{for } \; x \in \Omega, \quad |s| < \delta_\mu \;. \tag{39}$$

Taking into account (i), it is easy to see that F satisfies

$$|F(x,s)| \;\le\; C_1(|s|^q + 1) \quad \text{for } \; x \in \Omega, \quad s \in \mathbb{R} \tag{40}$$

with $C_1 \ge 0$ constant.

Choose $q_1 \in (\max\{p,q\},\, p^*)$. Then by (40), there is a constant $C_2 \ge 0$ such that

$$|F(x,s)| \;\le\; C_2\,|s|^{q_1} \quad \text{for } \; x \in \Omega, \quad |s| \ge \delta_\mu \;. \tag{41}$$

From (39) and (41), we have

$$F(x,s) \;\le\; \frac{\mu}{p}\,|s|^p + C_2\,|s|^{q_1} \quad \text{for } \; x \in \Omega, \quad s \in \mathbb{R} \;. \tag{42}$$

Now, by the variational characterization of the first eigenvalue λ_1 (see (19)), by the estimate (42) and by the imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^{q_1}(\Omega)$, it results

$$\begin{aligned}
\mathcal{F}(u) \;&=\; \frac{1}{p}\,\|u\|_{1,p}^p \,-\, \int_\Omega F(x,u) \\
&\ge\; \frac{1}{p}\,\|u\|_{1,p}^p \,-\, \frac{\mu}{p}\int_\Omega |u|^p \,-\, C_2\int_\Omega |u|^{q_1} \\
&\ge\; \frac{1}{p}\,\|u\|_{1,p}^p - \frac{\mu}{p}\,\|u\|_{0,p}^p - C_3\,\|u\|_{1,p}^{q_1} \\
&=\; \|u\|_{1,p}^p \left[\frac{1}{p}\left(1 - \mu\,\frac{\|u\|_{0,p}^p}{\|u\|_{1,p}^p}\right) \,-\, C_3\,\|u\|_{1,p}^{q_1-p}\right] \\
&\ge\; \|u\|_{1,p}^p \left[\frac{1}{p}\left(1 - \frac{\mu}{\lambda_1}\right) - C_3\,\|u\|_{1,p}^{q_1-p}\right] \;\ge\; \alpha \;>\; 0 \;,
\end{aligned}$$

provided $\|u\|_{1,p} = \rho$ is sufficiently small. ∎

The following lemma will be needed in the sequel.

Lemma 4.

(i) *If $u \in W_0^{1,p}(\Omega)$ is a solution of problem (9) with $f(x,s) \ge 0$ for $x \in \Omega$ and $s \le 0$, then $u \ge 0$.*

(ii) *If $u \in W_0^{1,p}(\Omega)$ is a solution of problem (9) with $f(x,s) \le 0$ for $x \in \Omega$ and $s \ge 0$, then $u \le 0$.*

Proof: We shall prove (i) (similar argument for (ii)).

Let $u \in W_0^{1,p}(\Omega)$ be a solution of problem (9) and let us denote $\Omega_- = \{x \in \Omega \mid u(x) < 0\}$. We define $u_- = \max\{-u, 0\}$. By Theorem A.1 in Kinderlehrer and Stampacchia [21], it is known that $u_- \in W_0^{1,p}(\Omega)$ and it is obvious that

$$\nabla u_- = \begin{cases} -\nabla u & \text{in } \Omega_- , \\ 0 & \text{in } \Omega \backslash \Omega_- . \end{cases}$$

From

$$\int_\Omega |\nabla u|^{p-2} \, \nabla u \, \nabla u_- = \int_\Omega f(x,u) \, u_-$$

we obtain

$$-\int_{\Omega_-} |\nabla u|^p = -\int_{\Omega_-} f(x,u) \, u \geq 0 .$$

Thus, $\nabla u = 0$ a.e. in Ω_-, consequently $\nabla u_- = 0$ a.e. in Ω. Therefore, $\|u_-\|_{1,p} = 0$ or $u_- = 0$ a.e. in Ω.

We conclude that $\operatorname{meas}(\Omega_-) = 0$, i.e. $u \geq 0$ a.e. in Ω. ∎

At this stage we are in position to prove the main result of this section.

Theorem 18. *Suppose $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory and it satisfies:*

(i) *there is $q \in (1, p^*)$ such that*

$$|f(x,s)| \leq C(|s|^{q-1} + 1) \quad \textit{for } x \in \Omega, \quad s \in \mathbb{R} ,$$

with $C \geq 0$ constant;

(ii)

$$\limsup_{s \to 0} \frac{f(x,s)}{|s|^{p-2} \, s} < \lambda_1 \quad \textit{uniformly with } x \in \Omega$$

where λ_1 is the first eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\Omega)$;

(iii) *there are constants $\theta > p$ and $s_0 > 0$ such that*

$$0 < \theta \, F(x,s) \leq s \, f(x,s) \quad \textit{for } x \in \Omega, \quad |s| \geq s_0 .$$

Then problem (9) has nontrivial solutions $u_- \leq 0 \leq u_+$.

Proof: We shall prove that (9) has a nontrivial solution $u_+ \geq 0$ (similar argument for the existence of u_-).

We define $f_+ \colon \Omega \times \mathbb{R} \to \mathbb{R}$ by $f_+(x,s) = f(x, \frac{s+|s|}{2})$ i.e.

$$f_+(x,s) \;=\; \begin{cases} 0 & \text{if } s \le 0 \;, \\ f(x,s) & \text{if } s > 0 \;, \end{cases}$$

and let $F_+ \colon \Omega \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F_+(x,s) \;=\; \int_0^s f_+(x,\tau)\, d\tau \;.$$

The following assertions are true:

(i)$_+$ the function f_+ is Carathéodory and it satisfies

$$|f_+(x,s)| \;\le\; C(|s|^{q-1}+1) \qquad \text{for} \;\; x \in \Omega, \quad s \in \mathbb{R} \;;$$

(ii)$_+$ $\limsup\limits_{s\to 0} \frac{f_+(x,s)}{|s|^{p-2}\, s} < \lambda_1$ uniformly with $x \in \Omega$;

(iii)$_+$ $\theta\, F_+(x,s) \le s\, f_+(x,s)$ for $x \in \Omega$, $|s| \ge s_0$;

(iv)$_+$ $0 < \theta\, F_+(x,s) \le s\, f_+(x,s)$ for $x \in \Omega$, $s \ge s_0$.

Indeed, (i)$_+$, (iii)$_+$ and (iv)$_+$ are easily seen.

To see (ii)$_+$, we have

$$\begin{aligned} \limsup_{s\to 0} \frac{f_+(x,s)}{|s|^{p-2}\, s} \;&=\; \max\left\{ \limsup_{s\nearrow 0} \frac{f_+(x,s)}{|s|^{p-2}\, s},\; \limsup_{s\searrow 0} \frac{f_+(x,s)}{|s|^{p-2}\, s} \right\} \\ &=\; \max\left\{ 0,\; \limsup_{s\searrow 0} \frac{f(x,s)}{|s|^{p-2}\, s} \right\} < \lambda_1 \qquad \text{uniformly with} \;\; x \in \Omega \;. \end{aligned}$$

From (i)$_+$–(iv)$_+$ we infer that the C^1 functional $\mathcal{F}_+ \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{F}_+(u) \;=\; \frac{1}{p}\, \|u\|_{1,p}^p - \int_\Omega F_+(x,u)$$

has a nontrivial critical point $u_+ \in W_0^{1,p}(\Omega)$.

To see this, we apply Theorem 14 with $I = \mathcal{F}_+$.

To this end, first of all, it will be remarked that viewing (i)$_+$, the results concerning $\mathcal{F}$ remain valid for $\mathcal{F}_+$, with f_+ instead of f.

Clearly, $\mathcal{F}_+(0) = 0$.

By (i)$_+$, (ii)$_+$ and Theorem 17, there are constants $\alpha, \rho > 0$ such that $\mathcal{F}_+|_{\|u\|_{1,p}=\rho} \ge \alpha$.

Furthermore, by $(\mathrm{iv})_+$, Theorem 16 (i) and Lemma 3 (ii) (also see Remark 11), there is an element $e \in W_0^{1,p}(\Omega)$ with $\|e\|_{1,p} \geq \rho$, such that $\mathcal{F}_+(e) \leq 0$.

Finally, by $(\mathrm{iii})_+$ and Theorem 15, $\mathcal{F}_+$ satisfies the (PS) condition.

The nontrivial critical point $u_+ \in W_0^{1,p}(\Omega)$, whose existence is ensured by Theorem 14, satisfies

$$\int_\Omega |\nabla u_+|^{p-2}\, \nabla u_+\, \nabla v \;=\; \int_\Omega f_+(x, u_+)\, v \qquad \text{for all} \;\; v \in W_0^{1,p}(\Omega)\ . \tag{43}$$

As $f_+(x,s) = 0$ for $x \in \Omega$, $s \leq 0$, Lemma 4 (i) shows that $u_+ \geq 0$.

Now, by the definition of f_+, (43) becomes

$$\int_\Omega |\nabla u_+|^{p-2}\, \nabla u_+\, \nabla v \;=\; \int_\Omega f(x, u_+)\, v \qquad \text{for all} \;\; v \in W_0^{1,p}(\Omega)$$

and the proof is complete. ∎

Remark 12. Theorem 18 was originally stated by Ambrosetti and Rabinowitz (see Corollary 3.11 in Ambrosetti–Rabinowitz [2]), in the case $p = 2$. After, their result became frequently cited as a typical existence result for nonlinear Dirichlet problems with right-hand member having a superlinear growth (see e.g. Corollary 2.23 in Rabinowitz [26], Theorem 6.9 in de Figueiredo [14], Theorem 6.2 in Struwe [27], et. al.).

In this context, Theorem 18 can be seen as a model of existence result for Dirichlet problems with p-Laplacian having the right-hand member a function with “super $p-1$ polynomial” growth, condition (iii) implying

$$\lim_{|s|\to\infty} \frac{f(x,s)}{|s|^{p-2}\, s} \;=\; +\infty\ . \tag{44}$$

Moreover, (44) shows that the generality of Theorem 18 is not lost if in (i) q is required to be in (p, p^*) instead of $(1, p^*)$.

On the other hand, a reasoning similar to that in the proof of Theorem 16 shows that conditions (iii) and (i) in Theorem 18 yield the existence of some $\gamma \in L^\infty(\Omega)$, $\gamma > 0$, such that $F(x,s) \geq \gamma(x)\, |s|^\theta$ for $x \in \Omega$, and $|s| \geq s_0$ (also see the proof of Proposition 7 bellow). This shows that the potential F grows faster than $|s|^p$ with $|s| \to \infty$. For an existence result allowing F to grow faster than $|s|^p$ or slower than $|s|^p$ we refer the reader to Costa and Magalhaes [10]. □

2.5. Multiple solutions

Taking into account the minimax methods in critical point theory, invoking the "Mountain Pass Theorem" in order to prove existence of nontrivial solutions for problem (9), make natural the question: what about multiple solutions?

More precisely, following the particular case $p = 2$, it would be expected that under the basic hypothesis of Theorem 18, the oddness of f be sufficient to guarantee the existence of an unbounded sequence of solutions for problem (9).

Such a result will conclude the paper.

We need the following

Proposition 7. *Suppose the Carathéodory function* $f\colon \Omega\times\mathbb{R} \to \mathbb{R}$ *satisfies*

(i) *there is* $q \in (1, p^*)$ *such that*

$$|f(x,s)| \;\le\; C(|s|^{q-1}+1) \quad \textit{for } \; x\in\Omega, \;\; s\in\mathbb{R}\;,$$

with $C \ge 0$ *constant;*

(ii) *there are numbers* $\theta > \rho$ *and* $s_0 > 0$ *such that*

$$0 \;<\; \theta\, F(x,s) \;\le\; s\, f(x,s) \quad \textit{for } \; x\in\Omega, \;\; |s|\ge s_0\;.$$

Then, if X_1 *is a finite dimensional subspace of* $W_0^{1,p}(\Omega)$, *the set* $S = \{v \in X_1 \mid \mathcal{F}(v) \ge 0\}$ *is bounded in* $W_0^{1,p}(\Omega)$.

Proof: From (i), F satisfies

$$|F(x,s)| \;\le\; C_1(|s|^q+1) \quad \text{for } \; x\in\Omega, \;\; s\in\mathbb{R}\;, \tag{45}$$

with $C_1 \ge 0$ constant.

We claim that there is $\gamma \in L^\infty(\Omega)$, $\gamma > 0$ on Ω, such that

$$F(x,s) \;\ge\; \gamma(x)\,|s|^\theta \quad \text{for } \; x\in\Omega, \;\; |s|\ge s_0\;. \tag{46}$$

Indeed, as in the proof of Theorem 16, we have

$$F(x,s) \ge \gamma_1(x)\, s^\theta \quad \text{for } \; x\in\Omega, \;\; s\ge s_0\;, \tag{47}$$

where $\gamma_1(x) = \frac{F(x,s_0)}{s_0^\theta}$. By virtue of (45), it is obvious that $\gamma_1 \in L^\infty(\Omega)$ and (ii) yields $\gamma_1 > 0$ on Ω.

A similar reasoning shows that

$$F(x,s) \ge \gamma_2(x)\,|s|^\theta \quad \text{for } \; x\in\Omega, \;\; s\le -s_0\;, \tag{48}$$

where $\gamma_2(x) = \frac{F(x,-s_0)}{s_0^\theta}$. Again $\gamma_2 \in L^\infty(\Omega)$ and $\gamma_2 > 0$ on Ω.

Therefore, (46) holds with $\gamma(x) = \min\{\gamma_1(x), \gamma_2(x)\}$ for $x \in \Omega$, as claimed.

We shall prove that $\mathcal{F}$ satisfies

$$\mathcal{F}(v) \leq \frac{1}{p}\|v\|_{1,p}^p - \int_\Omega \gamma(x)\,|v|^\theta - K \quad \text{for all } v \in W_0^{1,p}(\Omega) \tag{49}$$

with $K \geq 0$ constant.

Let v be arbitrary chosen in $W_0^{1,p}(\Omega)$ and let us denote $\Omega_< = \{x \in \Omega \mid |v(x)| < s_0\}$.

By (45) we have

$$\int_{\Omega_<} F(x,v) \geq -C_1 \int_{\Omega_<} (|v|^q + 1) \geq -C_1 \int_\Omega (s_0^q + 1) =$$
$$= -C_1(s_0^q + 1)\cdot \mathrm{meas}(\Omega) = K_1$$

and by (46) it holds

$$\int_{\Omega\setminus\Omega_<} F(x,v) \geq \int_{\Omega\setminus\Omega_<} \gamma(x)\,|v|^\theta\,.$$

Then

$$\begin{aligned}
\mathcal{F}(v) &= \frac{1}{p}\|v\|_{1,p}^p - \Big(\int_{\Omega_<} F(x,v) + \int_{\Omega\setminus\Omega_<} F(x,v)\Big)\\
&\leq \frac{1}{p}\|v\|_{1,p}^p - \int_{\Omega\setminus\Omega_<} \gamma(x)\,|v|^\theta - K_1\\
&= \frac{1}{p}\|v\|_{1,p}^p - \int_\Omega \gamma(x)\,|v|^\theta + \int_{\Omega_<} \gamma(x)\,|v|^\theta - K_1\\
&\leq \frac{1}{p}\|v\|_{1,p}^p - \int_\Omega \gamma(x)\,|v|^\theta + K\,,
\end{aligned}$$

where $K = \|\gamma\|_{0,\infty}\, s_0^q \cdot \mathrm{meas}(\Omega) - K_1$, and (49) is proved.

The functional $\|\;\|_\gamma\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\|v\|_\gamma = \Big(\int_\Omega \gamma(x)\,|v|^\theta\Big)^{\frac{1}{\theta}}$$

is a norm on $W_0^{1,p}(\Omega)$. On the finite dimensional subspace X_1 the norms $\|\;\|_{1,p}$ and $\|\;\|_\gamma$ being equivalent, there is a constant $\widetilde{K} = \widetilde{K}(X_1) > 0$ such that

$$\|v\|_{1,p} \leq \widetilde{K}\Big(\int_\Omega \gamma(x)\,|v|^\theta\Big)^{\frac{1}{\theta}} \quad \text{for all } v \in X_1\,.$$

Consequently, by (49), on X_1 it holds:

$$\begin{aligned}\mathcal{F}(v) \;&\leq\; \frac{1}{p}\,\widetilde{K}^p \Big(\int\limits_\Omega \gamma(x)\,|v|^\theta\Big)^{\frac{p}{\theta}} - \int\limits_\Omega \gamma(x)\,|v|^\theta - K \\ &=\; \frac{1}{p}\,\widetilde{K}^p\,\|v\|_\gamma^p - \|v\|_\gamma^\theta - K\ .\end{aligned}$$

Therefore

$$\frac{1}{p}\,\widetilde{K}^p\,\|v\|_\gamma^p - \|v\|_\gamma^\theta - K \;\geq\; 0 \qquad \text{for all} \;\; v \in S$$

and taking into account $\theta > p$, we conclude that S is bounded. ∎

We also need the following $\mathbb{Z}_2$ symmetric version of the "Mountain Pass Theorem" (see e.g. Theorem 9.12 in Rabinowitz [26]).

Theorem 19. *Let X be an infinite dimensional real Banach space and let $I \in C^1(X,\mathbb{R})$ be even, satisfy (PS) condition and $I(0) = 0$. If:*

($\mathbf{I}_1$) *there are constants ρ, $\alpha > 0$ such that $I|_{\|x\|=\rho} \geq \alpha$;*

($\mathbf{I}_2'$) *for each finite dimensional subspace X_1 of X the set $\{x \in X \mid I(x) \geq 0\}$ is bounded,*

then I possesses an unbounded sequence of critical values.

Now, we can state

Theorem 20. *Suppose the Carathéodory function $f:\,\Omega\times\mathbb{R}\to\mathbb{R}$ is odd in the second argument: $f(x,s) = -\,f(x,-s)$. If conditions (i), (ii), (iii) of Theorem 18 are satisfied, that is*

(**i**) *there is $q \in (1,p^*)$ such that*

$$|f(x,s)| \;\leq\; C(|s|^{q-1}+1) \qquad \textit{for} \;\; x\in\Omega, \quad s\in\mathbb{R}\ ,$$

with $C \geq 0$ constant;

(**ii**)

$$\limsup_{s\to 0} \frac{f(x,s)}{|s|^{p-2}\,s} \;<\; \lambda_1 \qquad \textit{uniformly with} \;\; x\in\Omega$$

where λ_1 is the first eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\Omega)$;

(**iii**) *there are constants $\theta > p$ and $s_0 > 0$ such that*

$$0 \;<\; \theta\,F(x,s) \;\leq\; s\,f(x,s) \qquad \textit{for} \;\; x\in\Omega, \quad |s| \geq s_0\ ,$$

then problem (9) has an unbounded sequence of solutions.

Proof: The function f being odd, the functional $\mathcal{F}$ is even. It is obvious that $\mathcal{F}(0) = 0$.

From (iii) by Theorem 15, $\mathcal{F}$ satisfies the (PS) condition.

By (i), (ii) and Theorem 17, there are constants $\alpha, \rho > 0$ such that $\mathcal{F}|_{\|u\|_{1,p}=\rho} \geq \alpha$.

Proposition 7 and (i), (iii) show that the set $\{v \in X_1 \mid \mathcal{F}(u) \geq 0\}$ is bounded in $W_0^{1,p}(\Omega)$, whenever X_1 is a finite dimensional subspace of $W_0^{1,p}(\Omega)$.

Theorem 19 applies with $X = W_0^{1,p}(\Omega)$ and $I = \mathcal{F}$. ∎

Remark 13. In Proposition 7 by condition (ii), the exponent q in the growth condition (i) is forced to be in the interval (p, p^*) (see Remark 12). Therefore, as in the case of Theorem 18, the generality of Theorem 20 is not lost if q in (i) is required to be in the interval (p, p^*) instead of $(1, p^*)$. □

Remark 14. In the particular case $p = 2$ the symmetry assumption on f allows to remove condition (ii) in Theorem 20 (see e.g. Theorem 9.38 in Rabinowitz [26], Theorem 6.6 in Struwe [27]). □

REFERENCES

[1] Adams, R.A. – *Sobolev Spaces*, Academic Press, 1975.

[2] Ambrosetti, A. and Rabinowitz, P.H. – Dual variational methods in critical points theory and applications, *J. Funct. Anal.,* 14 (1973), 349–381.

[3] Anane, A. – Simplicité et isolation de la première valeur propre du p-laplacien avec poids, *C. R. Acad. Sci. Paris,* 305 (1987), 725–728.

[4] Anane, A. and Gossez, J.P. – Strongly nonlinear elliptic problems near resonance: a variational approach, *Comm. Partial Differential Equations,* 15(8) (1990), 1141–1159.

[5] Beurling, A. and Livingston, A.E. – A theorem on duality mappings in Banach Spaces, *Ark. Mat.,* 4 (1961), 405–411.

[6] Brezis, H. – *Analyse Functionelle*, Masson, Paris, 1983.

[7] Brezis, H. and Nirenberg, L. – *Remarks on finding critical points*, Publ. du Lab. d'Analyse Numerique, Univ. Pierre et Marie Curie, R91006, 1991.

[8] Browder, F.E. – *Problemes Non-Lineaires*, Les Presses de l'Université de Montréal, 1964.

[9] Cioranescu, I. – *Duality Mappings in Non Linear Functional Analysis*, Romanian Academy, Bucharest, 1974 (in romanian).

[10] Costa, D.G. and Magalhaes, C.A. – Existence results for perturbations of the p-Laplacian, *J. Nonlinear Analysis TMA,* 24(3) (1995), 409–418.

[11] Diestel, J. – *Geometry of Banach Spaces-Selected Topics*, Springer-Verlag, 1975.

[12] Dinca, G.; Jebelean, P. and Mawhin, J. – *A result of Ambrosetti–Rabinowitz type for p-Laplacian*, in "Qualitative Problems for Differential Equations and Control Theory" (C. Corduneanu, Ed.), World Scientific, Singapore, 1995.

[13] Dinca, G. and Jebelean, P. – Une méthode de point fixe pour le p-Laplacien, *C. R. Acad. Sci. Paris,* t. 324, Série I (1997), 165–168.

[14] Figueiredo, D.G. – *Lectures on Ekeland Variational Principle with Applications and Detours*, Tata Inst. of Fundamental Research, Springer-Verlag, 1989.

[15] Figueiredo, D.G. – On superlinear elliptic problems with nonlinearities interacting with higher eigenvalues, *Rocky Mountain J. of Math.,* 18(2) (1988), 287–303.

[16] Glowinski, R. and Marroco, A. – *Sur l'approximation par éléments finis d'ordre un, et la résolution, par penalisation dualité, d'une classe de problèmes de Dirichlet non linéaires*, Univ. Paris VI et Centre National de la Recherche Scientifique, 189, nr. 74023.

[17] Hachimi, A. and Gossez, J.P. – *On a nonresonance condition near the first eigenvalue for a quasilinear elliptic problem*, Partial Differential Equations (Han-sur-Lesse, 1993), 144–151, Math. Res. 82, Akademie-Verlag, Berlin.

[18] James, R. – Reflexivity and the sup of linear functionals, *Israel J. of Math.,* 13(3–4) (1972), 289–301.

[19] Jebelean, P. – Finite-dimensional approximation in a multivalued problem with p-Laplacian, *East-West J. Numerical Math.,* (to appear).

[20] Kavian, O. – *Introduction à La Théorie des Points Critiques et Applications aux Problemes Elliptiques*, Springer-Verlag, 1993.

[21] Kinderlehrer, D. and Stampacchia, G. – *An Introduction to Variational Inequalities and Their Applications*, Academic Press, 1980.

[22] Lindqvist, P. – *On a nonlinear eigenvalue problem*, Fall School Analysis (Jyväskylä, 1994), 33–54 Report, 68, Univ. Jyväskylä, 1995.

[23] Lions, J.-L. – *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod–Gauthier Villars, Paris, 1969.

[24] Mawhin, J. and Willem, M. – *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag, 1989.

[25] Mawhin, J.; Ward, J.R. and Willem, M. – Variational methods and semi-linear elliptic equations, *Arch. Rat. Mech. Anal.,* 95 (1986), 269–277.

[26] Rabinowitz, P.H. – *Minimax methods in critical point theory with applications to differential equations*, Conference Board of the Math. Sci. Regional Conference Series in Math, no. 65, Amer. Math. Soc., Providence, R. I., 1986.

[27] Struwe, M. – *Variational Methods; Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer-Verlag, 1990.

[28] Vainberg, M.M. – *Variational Methods for the Study of Nonlinear Operators*, Holden-Day, Inc., San Francisco, 1964.

[29] Vainberg, M.M. – *Variational Methods and Methods of Monotone Operators in Theories of Nonlinear Equations*, John Wiley, New York, 1973.

G. Dinca,
University of Bucharest, Faculty of Mathematics,
Str. Academiei 14, 70109, Bucharest – ROMANIA

and

P. Jebelean,
West University of Timisoara, Faculty of Mathematics,
Bv. V. Pârvan 4, 1900 Timisoara – ROMANIA

and

J. Mawhin,
Institut de Mathématique Pure et Appliquée,
Bât. Marc de Hemptinne, Chemin du Cyclotron 2, B-1348, Louvain-la-Neuve – BELGIQUE