PORTUGALIAE MATHEMATICA
Vol. 58 Fasc. 1 – 2001
Nova Série

HOW MANY INTERVALS COVER A POINT IN RANDOM DYADIC COVERING?

A.H. Fan and J.P. Kahane

Presented by J.P. Dias

Abstract: We consider a random covering determined by a random variable X of the space $\mathbb{D}=\{0,1\}^{\mathbb{N}}$. We are interested in the covering number $N_n(t)$ of a point $t\in\mathbb{D}$ by cylinders of lengths $\geq 2^{-n}$. It is proved that points in $\mathbb{D}$ are differently covered in the sense that the random sets $\{t\in\mathbb{D}\colon\ N_n(t)-b\,n\ \sim\ c\,n^\alpha\}$ are non-empty for a certain range of b, any real number c and any $1/2<\alpha<1$. Actually, the Hausdorff dimensions of these sets are calculated. The method may be applied to the first percolation on an infinite and locally finite tree.

1 – Introduction

We consider the sequence space $\mathbb{D}=\{0,1\}^{\mathbb{N}}$ and a probability distribution represented by a random variable X which takes values in the set of non-negative integers (our methods also apply to the case of an infinite and locally finite tree and a real-valued variable). For any finite sequence $(\epsilon_1,...,\epsilon_n)$ of 0 and 1, we denote by $I(\epsilon_1,...,\epsilon_n)$ the n-cylinder in $\mathbb{D}$ (also called interval of length 2^{-n}) which is defined in the usual way and by $X_{\epsilon_1,...,\epsilon_n}$ a random variable which has the same distribution as X. We consider $X_{\epsilon_1,...,\epsilon_n}$ as the covering number of the cylinder $I(\epsilon_1,...,\epsilon_n)$, that is to say, the cylinder $I(\epsilon_1,...,\epsilon_n)$ is cut off with probability $p_0=P(X=0)$ and is covered m times with probability $p_m=P(X=m)$, $m=1,2,...$. In the sequel, we assume that all variables $X_{\epsilon_1,...,\epsilon_n}$ are independent and they are defined on a probability space $(\Omega,\mathcal{A},P)$.

Received: July 3, 2000.
AMS Subject Classification: 52A22, 52A45, 60D05.
Keywords: Random covering; Hausdorff dimension; Indexed martingale; Peyrière measure.

For $t = (t_n)_{n\geq 1} \in \mathbb{D}$, let

$$N_n(t) = \sum_{k=1}^{n} X_{t_1,\ldots,t_k} .$$

The quantity $N_n(t)$ is called the covering number (or more precisely the n-covering number) of the point t by cylinders of lengths 2^{-k} $(k = 1, 2, ..., n)$. As a consequence of the law of large numbers and Fubini's theorem, we have

$$\lim_{n\to\infty} \frac{N_n(t)}{n} = \mathbb{E}\, X$$

almost surely (a.s.) for almost every point t (with respect to Lebesgue measure on $\mathbb{D}$). It is also well known in the theory of birth processes that a.s. $\lim_{n\to\infty} N_n(t) = \infty$ for *every* $t \in \mathbb{D}$ if and only if

$$p_0 = P(X=0) < \frac{1}{2} .$$

That is to say, a.s. every point is infinitely covered when the above condition is satisfied.

Our aim in this paper is to study the behavior of $N_n(t)$ by considering the random sets

$$E_b = \left\{ t \in \mathbb{D}: \ \lim_{n\to\infty} \frac{N_n(t)}{n} = b \right\}$$

for different $b \in \mathbb{R}$. If $t \in E_b$, we may say that the point t is covered by about $b\,n$ cylinders of lengths $\geq 2^{-n}$ (with the convention that the cylinder $I(\epsilon_1, ..., \epsilon_n)$ is covered m times when $X_{\epsilon_1,\ldots,\epsilon_n} = m$). Actually our method allows us to study the subsets of E_b defined by

$$E_{b,s} = \left\{ t \in \mathbb{D}: \ N_n(t) - b\,n \sim s_n, \ \text{ as } n \to \infty \right\}$$

where $s = \{s_n\}$ is a sequence of real numbers such that $s_n = o(n)$.

We make the hypothesis that X is not constant and $\mathbb{E}\, e^{tX} < \infty$ for all $t \in \mathbb{R}$ (similar results hold when $\mathbb{E}\, e^{tX} < \infty$ for some interval of t). Let

$$\varphi(u) = \mathbb{E}\, e^{uX} , \qquad c(u) = \log \varphi(u) .$$

The function $c(u)$ is called the free energy of X. Notice that $c(u)$ is strictly increasing and strictly convex. Its Legendre–Fenchel transform is defined by

$$c^*(s) = \sup_{u\in\mathbb{R}} \Big(s\,u - c(u) \Big) .$$

Notice also that c^* is a well defined continuous convex function in the interval $[c'(-\infty), c'(+\infty)]$ which is contained in $[0, +\infty]$, and that it attains its minimal value 0 at $u = \mathbb{E}\,X$.

In the following theorems, dim means the Hausdorff dimension as well as the packing dimension. See [M2] for the definitions of these two notions of dimension. We recall the metric of $\mathbb{D}$, which is defined as $d(t,s) = 2^{-n}$ for $t, s \in \mathbb{D}$ with $n = \sup\{m\colon\ t_j = s_j,\ \forall\, 1 \le j \le m\}$. We denote

$$J = \left\{ b \in [c'(-\infty), c'(+\infty)] \colon\ c^*(b) \le \log 2 \right\} .$$

Theorem 1. *Suppose X is a non-constant random variable taking values in the set of non negative integers such that $\mathbb{E}\, e^{tX} < \infty$ ($\forall\, t \in \mathbb{R}$). Then for any number $b \in J$ and any sequence of positive numbers $s = (s_n)$ such that $s_n - s_{n-1} = o(1)$ and $\sqrt{n \log\log n} = o(s_n)$, we have a.s.*

$$\dim E_{b,s} = \dim E_b = 1 - \frac{c^*(b)}{\log 2} .$$

The proof of Theorem 1 will show that the dimension of the set of points t such that $\liminf n^{-1} N_n(t) \ge b$ is equal to $\dim E_b$ when $b > \mathbb{E}\,X$ and the dimension of the set of points t such that $\limsup n^{-1} N_n(t) \le b$ is equal to $\dim E_b$ when $b < \mathbb{E}\,X$.

What happens for E_b when $b \notin J$? This question is answered by the following theorem.

Theorem 2. *Suppose X satisfies the same condition as in Theorem 1. Let $A = \inf J$ and $B = \sup J$. Then we have a.s.*

$$A \le \liminf_{n\to\infty} \frac{N_n(t)}{n} \le \limsup_{n\to\infty} \frac{N_n(t)}{n} \le B \qquad (\forall\, t \in \mathbb{D}) .$$

If $\lim \frac{N_n(t)}{n}$ doesn't exist, we may say that t is irregularly covered. The following theorem shows that many points are irregularly covered.

Theorem 3. *Suppose X satisfies the same condition as in Theorem 1. Then the set of irregularly covered points is a.s. of Hausdorff dimension* 1.

The present study was partially motivated by Dvoretzky random covering problem on the unit circle [D] (see [S, K2, FK, K5, K6] for the developments of the subject). Recent work on the circle related to ours may be found in [F3], the results are less complete than those for the sequence space $\mathbb{D}$ studied here.

The restriction on $\mathbb{D}$ and the positivity assumption of X are not essential: the above results can be generalized to tree-indexed walks. See [B, L, LP, BP, PP] for related works on tree-indexed walks.

Using results on percolation in [L], Lyons and Pemantle [LP] have obtained the dimension formula for $\dim E_b$, but their method does not give results on $\dim E_{b,s}$.

2 – Preliminaries

Our main tool is multiplicative chaos (for a lower estimate for the dimension). As usual, large deviation is used to get an upper estimate of dimension.

First of all, we recall the notion of the dimension of a measure [F2]. The lower dimension of a measure μ, denoted by $\dim_* \mu$, is the supremum of β's such that $\mu(E) = 0$ for any E with $\dim E < \beta$. The upper dimension of a measure μ, denoted by $\dim^* \mu$, is the infimum of $\dim F$ for F's such that $\mu(F^c) = 0$. It is clear that for a given Borel set A, we have

$$\dim A \ \geq\ \dim_* \mu \quad \text{ if } \ \mu(A) > 0\ .$$

When $\dim_* \mu = \dim^* \mu = \alpha$, we write $\dim \mu = \alpha$.

The general theory of multiplicative chaos was developed by the second author in [K3]. We recall it here briefly. The key part for us is the Peyrière probability measure. Let (P_n) be a sequence of non-negative independent random functions defined on $\mathbb{D}$ such that $\mathbb{E}\, P_n(t) = 1$ ($\forall\, t \in \mathbb{D}$). Consider the finite products

$$Q_n(t) \ =\ \prod_{k=1}^{n} P_n(t)\ .$$

We call $Q_n(t)$ an indexed martingale because it is a martingale for each $t \in \mathbb{D}$. It was proved in [K3] that for any Borel probability measure μ on $\mathbb{D}$, a.s. the random measures $Q_n(t)\, d\mu(t)$ converge weakly to a (random) measure that we denote by $Q\mu$. The operator Q is called a multiplicative chaos. If the total mass martingale

$$Y_n \ =\int_{\mathbb{D}} Q_n(t)\, d\mu(t)$$

converges in L^1, the measure $Q\mu$ does not vanish and a probability measure

$\mathcal{Q} = \mathcal{Q}_\mu$ on $\Omega \times \mathbb{D}$, called Peyrière measure, may be defined by the relation

$$\int_{\Omega\times\mathbb{D}} \varphi(\omega,t)\, d\mathcal{Q}_\mu(\omega,t) \;=\; \mathbb{E} \int_{\mathbb{D}} \varphi(\omega,t)\, dQ\mu(t)$$

(for all bounded measurable functions φ). A very useful fact is that if the distribution of the variable $P_n(t)$ is independent of $t \in \mathbb{D}$, then $P_n(t,\omega)$ $(n \geq 1)$ considered as random variables on $\Omega\times\mathbb{D}$ are $\mathcal{Q}$-independent. Furthermore, we have the formula

$$\mathbb{E}_{\mathcal{Q}}\, h(P_n) \;=\; \mathbb{E}\, h(P_n)\, P_n$$

(for any Borel function h).

We shall use a particular class of multiplicative chaos. This corresponds to

$$P_n(t) \;=\; W_{t_1,\dots,t_n}$$

where all the random variables $\{W_{t_1,\dots,t_n}\}$ are independent, non-negative and normalized (i.e. $\mathbb{E}\, W_{t_1,\dots,t_n} = 1$), and for any $n \geq 1$, the subfamily of variables $\{W_{t_1,\dots,t_n}\}$ are identically distributed with common law represented by a variable $\widetilde{W}_n$. The corresponding chaos is called (generalized) random cascades determined by $\widetilde{W}_n$. When $\widetilde{W}_n$ are identically distributed, we recover the classical random cascades, well studied in [KP, M1]. The following lemmas study the random measure $Q\lambda$ determined by a sequence $\{\widetilde{W}_n\}$ and the Lebesgue measure $\lambda = dt$ on $\mathbb{D}$. Recall that Y_n denotes the total mass martingale $\int_{\mathbb{D}} Q_n(t)\, dt$.

Lemma 1. *Suppose that for some* $0 < h < 1$ *we have*

$$\liminf_{n\to\infty} \frac{\mathbb{E}\, \log_2 \widetilde{W}_n^h}{h-1} \;>\; 1\ .$$

Then the martingale Y_n *converges a.s. to zero. Consequently* $Q\lambda = 0$ *a.s..*

Proof: The condition implies that $\mathbb{E}\, \widetilde{W}_j < 2^{h-1}$ for large j. Let B be an arbitrary ball of radius 2^{-n}. We have

$$\mathbb{E} \sup_{t\in B} Q_n(t)^h \;=\; \prod_{j=1}^{n} \mathbb{E}\, \widetilde{W}_j^h \;\leq\; C\, 2^{-n(1-h)}$$

where C is a constant independent of the ball B. We conclude by applying theorem 3 from [K3]. ∎

Lemma 2. *Suppose that for some* $h > 1$ *we have*

$$\limsup_{n\to\infty} \frac{\mathbb{E}\, \log_2 \widetilde{W}_n^h}{h-1} \;<\; 1\ .$$

Then the martingale Y_n converges in L^h. Consequently the Peyrière measure $\mathcal{Q} = \mathcal{Q}_\lambda$ exists.

Proof: The condition implies that there exists $\epsilon > 0$ such that $2^{1-h}\,\mathbb{E}\,\widetilde{W}_j \le e^{\epsilon(1-h)}$ for large j. By the same calculation as in [K1] (p. 622), we have

$$\mathbb{E}\,Y_{n-1}^h \;\le\; \mathbb{E}\,Y_n^h \;\le\; \mathbb{E}\,Y_{n-1}^h\,\mathbb{E}\,\widetilde{W}_n^h\;2^{1-h}\left(1+\frac{\mathbb{E}^2 Y_{n-1}^{h/2}}{\mathbb{E}\,Y_{n-1}^h}\right)^{h-1}.$$

It follows that for large n, we have $\mathbb{E}\,Y_{n-1}^h \le 1/\epsilon$. Thus the total mass martingale is bounded in L^h. ∎

Lemma 3. *Let $0 < h' < 1 < h''$. Denote*

$$D_- \;=\; 1-\liminf_{n\to\infty}\frac{\mathbb{E}\,\log_2\widetilde{W}_n^{h'}}{h'-1}\,,\qquad D_+ \;=\; 1-\limsup_{n\to\infty}\frac{\mathbb{E}\,\log_2\widetilde{W}_n^{h''}}{h''-1}\;.$$

Then $D_+ \le \dim_* Q\lambda \le \dim^* Q\lambda \le D_-$ a.s..

Proof: We follow [K4] using a result from [F1]. For $\beta > 0$, let W_β be the variable such that $P(W_\beta = 2^\beta) = 2^{-\beta} = 1 - P(W_\beta = 0)$. The random cascades determined by W_β (called β-model) gives rise to a multiplicative chaos Q_β. We construct Q_β independent of Q. The product $Q_\beta\,Q$ is the chaos defined by $\{W_\beta\,\widetilde{W}_n\}$. A simple calculation gives

$$\frac{\log_2(\widetilde{W}_n\,W_\beta)^h}{h-1} \;=\; \frac{\log_2\widetilde{W}_n^h}{h-1}+\beta \qquad (\forall\, h\neq 1)\;.$$

Take $0 < \beta < D_+$ (there is nothing to do if D_+ is negative). We have

$$\limsup_{n\to\infty}\frac{\log_2(\widetilde{W}_n\,W_\beta)^{h''}}{h''-1} \;=\; 1-D_+ +\beta \;<\; 1\;.$$

By Lemma 2 and the main result of [F1], we get $\dim_* Q\lambda \ge \beta$ a.s..

Take $\beta > D_-$. We have

$$\liminf_{n\to\infty}\frac{\log_2(\widetilde{W}_n\,W_\beta)^{h''}}{h''-1} \;=\; 1-D_- +\beta \;>\; 1\;.$$

By Lemma 1 and the result of [F1], we get $\dim_* Q\lambda \le \beta$ a.s.. ∎

Suppose $c'(\lambda_b) = b$. Take $\xi_n = \lambda_b + \eta_n$ with $\eta_n \to 0$ as $n\to\infty$. Consider

$$\widetilde{W}_n \;=\; \frac{e^{\xi_n X}}{\varphi(\xi_n)}$$

where X is the variable determining our covering. For different choices $\{\eta_n\}$, the corresponding random measure $Q\lambda$ may be singular each other, but they have the same dimension.

Lemma 4. *Let $Q\lambda$ be the random measure defined by the above sequence $\{\widetilde{W}_n\}$. Then*

$$\dim Q\lambda \;=\; 1 - \frac{c^*(b)}{\log 2} \qquad \text{a.s..}$$

Proof: Since $\xi_n \to \lambda_b$ and

$$\frac{\log \mathbb{E}\, \widetilde{W}_n^h}{h-1} \;=\; \frac{c(\xi_n\, h) - c(\xi_n)}{h-1} - c(\xi_n) \ ,$$

we have

$$\lim_{n\to\infty} \frac{\log \mathbb{E}\, \widetilde{W}_n^h}{h-1} \;=\; \frac{c(\lambda_b\, h) - c(\lambda_b)}{h-1} - c(\lambda_b) \ .$$

The function $c(\cdot)$ being strictly convex, we have

$$\frac{c(\lambda_b\, h) - c(\lambda_b)}{h-1} - c(\lambda_b) \;>\; c'(\lambda_b)\,\lambda_b - c(\lambda_b) \;=\; c^*(b) \qquad \text{if} \quad h > 1 \ ;$$

$$\frac{c(\lambda_b\, h) - c(\lambda_b)}{h-1} - c(\lambda_b) \;<\; c'(\lambda_b)\,\lambda_b - c(\lambda_b) \;=\; c^*(b) \qquad \text{if} \quad h < 1 \ .$$

Now we can apply Lemma 3. ∎

Now let us recall some properties of the free energy function of X and of its Legendre–Fenchel transform. Let $x_{\max}$ (resp. $x_{\min}$) be the essential upper (resp. lower) bound of the variable X. Then let

$$p_{\min} = P(X = x_{\min}) \,, \qquad p_{\max} = P(X = x_{\max}) \ .$$

We first claim that (assuming $p_{\min} > 0$)

$$c'(-\infty) = x_{\min} \,, \qquad c^*(x_{\min}) = \log \frac{1}{p_{\min}} \ .$$

In fact, since

$$\mathbb{E}\, e^{tX} \;=\; p_{\min}\, e^{t x_{\min}} (1 + O(e^t)) \qquad (t \to -\infty) \ ,$$

$$\mathbb{E}\, X e^{tX} \;=\; p_{\min}\, x_{\min}\, e^{t x_{\min}} (1 + O(e^t)) \qquad (t \to -\infty) \ ,$$

we have

$$c'(t) \;=\; \frac{\mathbb{E} X e^{tX}}{\mathbb{E} e^{tX}} \;=\; x_{\min} + O(e^t) \qquad (t \to -\infty) \ ,$$

$$c^*(c(t)) \;=\; t\, c'(t) - c(t) \;=\; p_{\min}\, x_{\min}\, e^{t x_{\min}} (1 + O(t e^t)) \qquad (t \to -\infty) \ .$$

We also claim that if $x_{\max} < \infty$

$$c'(+\infty) = x_{\max}\,, \qquad c^*(x_{\max}) = \log\frac{1}{p_{\max}}\ .$$

In fact, the proof is the same as above because, assuming $p_{\max} > 0$,

$$\begin{aligned}\mathbb{E}\, e^{tX} &= p_{\max}\, e^{tx_{\max}}(1+O(e^{-t})) \qquad (t\to+\infty)\ ,\\ \mathbb{E}\, Xe^{tX} &= p_{\max}\, x_{\max}\, e^{tx_{\max}}(1+O(e^{-t})) \qquad (t\to+\infty)\ .\end{aligned}$$

Consequently, we have $c^*(x_{\min}) \le \log 2$ if and only if $p_{\min} \ge \frac{1}{2}$. If $p_{\min} < \frac{1}{2}$, there is a point $0 < c_0 < \mathbb{E}\, X$ such that $c^*(c_0) = \log 2$. Also if $p_{\max} < \frac{1}{2}$, there is a point $b_0 > \mathbb{E}\, X$ such that $c^*(b_0) = \log 2$.

3 – Proof of Theorem 1

Let us first prove $\dim_P E_b \le 1 - c^*(b)/\log 2$ where $\dim_P$ denotes the packing dimension. Notice that the interval J contains $\mathbb{E}\, X$ because $\mathbb{E}\, X = c'(0)$ and $c^*(c'(0)) = 0$. Suppose $b > \mathbb{E}\, X$. (The case $b < \mathbb{E}\, X$ may be similarly treated). Fix a small $\delta > 0$. Let

$$\mathcal{C}_n = \left\{ I(t_1, ..., t_n) \colon \ \sum_{k=1}^{n} X_{t_1,\ldots,t_k} > (b-\delta)\, n \right\} .$$

Let G_n be the union set of all cylinders in $\mathcal{C}_n$. It is clear that

$$E_b \subset \bigcup_{\ell=1}^{\infty} \bigcap_{n=\ell}^{\infty} G_n\ .$$

Then

$$\dim_P E_b \le \sup_{\ell\ge 1} \dim_P \bigcap_{n=\ell}^{\infty} G_n \le \sup_{\ell\ge 1} \overline{\dim}_B \bigcap_{n=\ell}^{\infty} G_n$$

where $\overline{\dim}_B$ denotes the upper box dimension. Remark that when $n \ge \ell$, $\mathcal{C}_n$ is a cover of $\bigcap_{n=\ell}^{\infty} G_n$ by cylinders of length 2^{-n}. Thus we have

$$\overline{\dim}_B \bigcap_{n=\ell}^{\infty} G_n \le \limsup_{n\to\infty} \frac{\text{Card}\ \mathcal{C}_n}{\log 2^n}\ .$$

Now estimate the random variable Card $\mathcal{C}_n$. It is obvious that

$$\mathbb{E}\ \text{Card}\ \mathcal{C}_n = \sum_{t_1,\ldots t_n} P\left(\sum_{k=1}^{n} X_{t_1,\ldots,t_k} > (b-\delta)\, n \right) .$$

However, by the theorem of large deviation [E] (p. 230), we have

$$P\left(\sum_{k=1}^{n} X_{t_1,\dots,t_k} > (b-\delta)\,n\right) \le e^{-nc^*(b-\delta)} = 2^{-n\frac{c^*(b-\delta)}{\log 2}} .$$

This, together with the preceding equality, gives us

$$\mathbb{E}\ \mathrm{Card}\ \mathcal{C}_n \le 2^{n(1-\frac{c^*(b-\delta)}{\log 2})} .$$

Then

$$\mathbb{E}\sum_{n=1}^{\infty} n^{-2}\, 2^{-n(1-\frac{c^*(b-\delta)}{\log 2})}\ \mathrm{Card}\ \mathcal{C}_n < \infty .$$

It follows that almost surely we have

$$\mathrm{Card}\ \mathcal{C}_n = O\Big(n^2\ 2^{n(1-\frac{c^*(b-\delta)}{\log 2})}\Big) .$$

Therefore

$$\limsup_{n\to\infty} \frac{\mathrm{Card}\ \mathcal{C}_n}{\log 2^n} \le 1 - \frac{c^*(b-\delta)}{\log 2} .$$

Letting $\delta \to 0$, we obtain the desired upper bound.

Suppose b is an interior point of J. In order to prove $\dim_H E_{b,s} \ge 1 - c^*(b)/\log 2$, we consider the random measure $Q\lambda$ determined by

$$\widetilde{W}_n = \frac{e^{\xi_n X}}{\varphi(\xi_n)}$$

where $\xi_n \in \mathbb{R}$ is the solution of $c'(\xi_n) = b+(s_n - s_{n-1})$. By Lemma 2, the Peyrière measure $\mathcal{Q}$ is well defined. We have

$$\mathbb{E}_{\mathcal{Q}} X_{t_1,\dots,t_n} = \frac{\mathbb{E}\, X\, e^{\xi_n X}}{\varphi(\xi_n)} = c'(\xi_n) ,$$

$$\mathbb{E}_{\mathcal{Q}} X^2_{t_1,\dots,t_n} = \frac{\mathbb{E}\, X^2\, e^{\xi_n X}}{\varphi(\xi_n)} = \frac{\varphi''(\xi_n)}{\varphi(\xi_n)} ,$$

$$\mathrm{Var}_{\mathcal{Q}} X_{t_1,\dots,t_n} = c''(\xi_n) .$$

Notice that the variables $X_{t_1,\dots,t_n}$ $(n=1,2,\dots)$ are $\mathcal{Q}$-independent. Then by the law of the iterated logarithm, $\mathcal{Q}$-almost surely

$$\sum_{k=1}^{n} X_{t_1,\dots,t_k} - bn - s_n = O\Big(\sqrt{n\log\log n}\Big) .$$

Since $\sqrt{n \log\log n} = o(s_n)$, $Q\lambda(E^c_{b,s}) = 0$ a.s.. Then

$$\dim_H E_{b,s} \;\geq\; \dim_* Q\lambda \quad \text{a.s..}$$

On the other hand, by Lemma 4,

$$\dim Q\lambda \;=\; 1 - \frac{c^*(b)}{\log 2} \quad \text{a.s..}$$

Thus the formula is proved when b is in the interior of J.

Let A be the left end point of J. If $c^*(A) = \log 2$, the proof of the upper bound shows that $\dim E_A = 0$ then $\dim E_{A,s} = 0$. Suppose $c^*(A) < \log 2$. This implies $A = x_{\min} = c'(-\infty) > -\infty$ (see the definition of J and the strict convexity of c^*). In order to prove $\dim E_{x_{\min},s} = 1 - \log_2 \frac{1}{p_{\min}}$, it suffices to consider the random cascade by choosing ξ_n tending to $-\infty$ such that $c'(x_n) = c'(-\infty) + (s_n - s_{n-1})$ to get the lower bound (the upper bound is proved as above).

Let $B < +\infty$ be the right end point of J. As for the left end point, if $c^*(B) = \log 2$, we have $\dim E_B = \dim E_{B,s} = 0$. Suppose $c^*(B) < \log 2$. This implies $x_{\max} = c'(+\infty) = B < \infty$. In order to prove $\dim E_{x_{\max},s} = 1 - \log_2 \frac{1}{p_{\max}}$, it suffices to consider the random cascade by choosing ξ_n tending to $+\infty$ such that $c'(x_n) = c'(+\infty) + (s_n - s_{n-1})$ to get the lower bound. ∎

4 – Proof of Theorem 2

Notice that

$$x_{\min} \;\leq\; \frac{N_n(t)}{n} \;\leq\; x_{\max} \;.$$

So, there is nothing to prove for $\limsup \frac{N_n(t)}{n} \leq \sup J$ when $\sup J = x_{\max}$ and nothing to prove for $\liminf \frac{N_n(t)}{n} \geq \inf J$ when $\inf J = x_{\min}$.

Suppose that $\sup J < x_{\max}$. That implies $c^*(c'(\infty)) > \log 2$. Denote by $[x]$ the integral part of a real number x. Let $\gamma > 0$ be a large number. For $j \geq 4$, introduce the following notation

$$S_j(t) \;=\; \sum_{[\gamma(j-1)\log(j-1)] \leq k < [\gamma j \log j]} X_{t_1,\ldots,t_k}$$

$$U_j \;=\; \max_{t\in\mathbb{D}} S_j(t)\,, \qquad V_j \;=\; \min_{t\in\mathbb{D}} S_j(t) \;.$$

Suppose $U_j = S_j(t_0)$ for some point t_0. It is clear that $U_j \leq S_j(t)$ for all t in the $[\gamma \log j]$-cylinder containing t_0. It follows that for any $\lambda > 0$, we have

$$e^{\lambda U_j} \;\leq\; 2^{\gamma \log j} \int_{\mathbb{D}} e^{\lambda S_j(t)}\, dt \;.$$

Taking expectation gives us

$$\mathbb{E}\, e^{\lambda U_j} \;\leq\; 2^{\gamma \log j}\, (\mathbb{E}\, e^{\lambda X})^{\gamma \log j} \;.$$

Take $B' > B$. Then $c^*(B') > \log 2$. By using Chebyshev's inequality, we get

$$P(U_j \geq B'\gamma \log j) \;\leq\; j^{\gamma\{\log 2 - (B'\lambda - c(\lambda))\}} \;.$$

Take $\lambda > 0$ such that $c'(\lambda) = B'$ and $B'\lambda - c(\lambda) = \sup_t (B't - c(t)) = c^*(B')$. Such a $\lambda > 0$ does exist because $c^*(c'(\infty)) > \log 2$. Then

$$P(U_j \geq B'\gamma \log j) \;=\; O(j^{\gamma(\log 2 - c^*(B'))}) \;.$$

Since $\log 2 - c^*(B')$ is strictly negative, the series $\sum_j j^{\gamma(\log 2 - c^*(B'))}$ converges if γ is sufficiently large. According to the Borel–Cantelli lemma, almost surely for large j

$$U_j \;\leq\; B'\, \gamma\, \log j \;.$$

Thus we have

$$\sum_{j=1}^{J} U_j \;\leq\; B'\, \gamma \sum_{j=1}^{J} \log j \,+\, O(1) \;\leq\; B'\, \gamma\, J\, \log J \,+\, O(1) \;.$$

For any $n \geq 1$, there is a K such that $[\gamma(K-1)\log(K-1)] \leq n < [\gamma\, K\, \log K]$. Then

$$\sum_{k=1}^{n} X_{\epsilon_1,\ldots,\epsilon_k} \;\leq\; \sum_{j=1}^{K} U_j \;\leq\; B'\, \gamma\, K\, \log K + O(1) \;\sim\; B'\, n \;.$$

Thus we have proved

$$\limsup_{n\to\infty} \frac{N_n(t)}{n} \leq B' \qquad (\forall\, t \in \mathbb{D}) \;.$$

Since B' is an arbitrary number such that $B' > B$, it follows from the last inequality that

$$\limsup_{n\to\infty} \frac{N_n(t)}{n} \leq B \qquad (\forall\, t \in \mathbb{D}) \;.$$

The proof of the lower estimate is similar. We just point out what should be changed. If $V_j = S_j(t_0)$ for some point t_0, then $V_j \geq S_j(t)$ for all t in the $[\gamma\, \log j]$-cylinder containing t_0. This allows us to get that for any $\lambda > 0$,

$$\mathbb{E}\, e^{-\lambda V_j} \;\leq\; 2^{\gamma \log j}\, (\mathbb{E}\, e^{-\lambda X})^{\gamma \log j} \;.$$

By using Chebyshev's inequality, for $A' < A$ we get

$$P(V_j \le A'\gamma \log j) \;\le\; j^{\gamma\{\log 2 - (-A'\lambda - c(-\lambda))\}} \;.$$

Take $\lambda > 0$ such that $A'(-\lambda) - c(-\lambda) = \sup_t(A't - c(t)) = c^*(A')$. Then

$$P(V_j \le A'\gamma \log j) \;\le\; j^{\gamma(\log 2 - c^*(A'))} \;. \blacksquare$$

5 – Proof of Theorem 3

Let (ξ_n) be a sequence of positive numbers such that $0 < a \le \xi_n \le b < \infty$ which will be determined later. Consider the random measure $Q\lambda$ defined by

$$\widetilde{W}_n \;=\; \frac{e^{\xi_n X}}{\varphi(\xi_n)} \;.$$

By Lemma 2, if b is small enough, the Peyrière measure $\mathcal{Q}$ exists. From now on, we assume that b is small. Since

$$\mathbb{E}_{\mathcal{Q}} X_{\epsilon_1,\dots,\epsilon_k} \;=\; c'(\xi_k)\,, \qquad \mathrm{Var}_{\mathcal{Q}}(X_{\epsilon_1,\dots,\epsilon_k}) \;=\; c''(\xi_k) \;,$$

by the law of the iterated logarithm, $\mathcal{Q}$-almost everywhere

$$\sum_{k=1}^n X_{\epsilon_1,\dots,\epsilon_k} - \sum_{k=1}^n c'(\xi_k) \;=\; O\Big(\sqrt{\sigma_n^2 \log\log \sigma_n^2}\,\Big)$$

where

$$\sigma_n^2 \;=\; \sum_{k=1}^n c''(\xi_k) \;\approx\; n \;.$$

Thus a.s. Q-almost everywhere we have the equivalence

$$N_n(t) \;\sim\; \sum_{k=1}^n c'(\xi_k) \;.$$

Take a rapidly increasing sequence of positive integers (n_k) such that

$$\lim_{k\to\infty} \frac{n_k}{n_1 + \dots + n_k} \;=\; 1 \;.$$

For any two given small numbers $0 < a < b$, define the sequence (ξ_j) in the following way

$$\xi_j \;=\; a \qquad \text{if} \quad n_{2k} \le j < n_{2k+1} \;,$$

$$\xi_j \;=\; b \qquad \text{if} \quad n_{2k+1} \le j < n_{2k+2} \;.$$

Then we have

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} c'(\xi_k) \;\le\; c'(a) \;<\; c'(b) \;\le\; \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} c'(\xi_k) \;.$$

Let E be the set of points $t \in \mathbb{D}$ such that $\lim \frac{N_n(t)}{n}$ doesn't exists. Then almost surely $Q\lambda(E^c) = 0$. It follows that almost surely $\dim E \ge \dim Q\lambda$. By Lemma 3, if $b \to 0$ then $\dim Q \to 1$. We get $\dim E = 1$. ∎

6 – Poisson covering and Bernoulli covering

We look at two examples.

Example 1. Suppose X is a Poisson variable with parameter $a > 0$ (i.e. $P(X = k) = e^{-a} \frac{a^k}{k!}$). Then

$$\varphi(t) = e^{-a(1-e^t)}\,, \qquad c(t) = -a(1-e^t) \;.$$

Let $b = c'(t) = a\,e^t$. That means $t = \log \frac{b}{a}$. We have

$$c^*(b) \;=\; t\,c'(t) - c(t) \;=\; \log \frac{b}{a} \cdot b + (a-b) \;. \;\square$$

Thus we get

Theorem 4. *Suppose X is a Poisson variable with parameter $a > 0$. Then there is an interval J_a such that for $b \in J_a$, almost surely*

$$\dim E_b \;=\; 1 - \frac{1}{\log 2} \left[(a-b) + b \log \frac{b}{a} \right] \;;$$

for $b \notin J_a$, $E_b = \emptyset$. The interval J_a consists of $b \ge 0$ such that $F(b) \le \log 2$ where $F(b) = a - b + b \log \frac{b}{a}$.

The interval J_a may be calculated explicitly. Notice that $x_{\min} = 0$, $x_{\max} = \infty$ and

$$c^*(0) = a\,, \qquad c^*(\infty) = \infty \;.$$

Let $B > a$ be the solution of $F(B) = \log 2$. If $a \le \log 2$, $J_a = [0, B]$. If $a > \log 2$, $J_a = [A, B]$ where $0 < A < a$ is the other solution of $F(A) = \log 2$.

Remark that if $a \le \log 2$, the above theorem implies that $\lim \frac{N_n(t)}{n}$ may be as small as possible. But if $a > \log 2$, it is uniformly (in t) bounded from below by $A > 0$.

Remark also that the variable X takes all positive integers as values. A priori, one might assume that $\lim \frac{N_n(t)}{n}$ may take large values. But by the above theorem, it is uniformly (in t) bounded by B.

Example 2. Suppose X is a Bernoulli variable with parameter $p > 0$ (i.e. $P(X = 1) = p = 1 - P(X = 0)$). Then

$$\varphi(t) = 1 - p + p\,e^t\,, \qquad c(t) = \log(1 - p + p\,e^t)\ .$$

Let $b = c'(t) = \frac{pe^t}{1-p+pe^t}$. That means $t = \log \frac{b(1-p)}{p(1-b)}$. We have

$$c^*(b) = t\,c'(t) - c(t) = b\,\log \frac{b}{p} + (1-b)\,\log \frac{1-b}{1-p}\ . \ \square$$

Thus we get

Theorem 5. *Suppose X is a Bernoulli variable with parameter $0 < p < 1$. Then there is an interval I_p such that for $b \in I_p$,*

$$\dim E_b = 1 - \frac{1}{\log 2}\left[b\,\log\frac{b}{p} + (1-b)\,\log\frac{1-b}{1-p}\right] \quad a.s.\ ;$$

for $b \notin I_p$, $E_b = \emptyset$. The interval I_p consists of $0 \le b \le 1$ such that $F(b) \le \log 2$ where $F(b) = b\,\log \frac{b}{p} + (1-b)\log\frac{1-b}{1-p}$.

The interval I_p may be calculated as follows. Notice first that $x_{\min} = 0$, $x_{\max} = 1$ and

$$c^*(0) = \log \frac{1}{1-p}\,, \qquad c^*(1) = \log \frac{1}{p}\ .$$

If $p = \frac{1}{2}$, $I_p = [0,1]$. If $p > \frac{1}{2}$, $I_p = [A, 1]$ where $0 < A < p$ is the solution of $F(A) = \log 2$; if $p < \frac{1}{2}$, $I_p = [0, B]$ where $p < B < 1$ is the solution of $F(B) = \log 2$.

Notice that if $p < \frac{1}{2}$, $\limsup \frac{N_n(t)}{n} \le B$ $(\forall\, t \in \mathbb{D})$ for some $B < 1$; if $p > \frac{1}{2}$, $\liminf \frac{N_n(t)}{n} \ge A$ $(\forall\, t \in \mathbb{D})$ for some $A > 0$.

Finally we remark that the condition $\sqrt{n \log\log n} = o(s_n)$ in Theorem 1 is not always necessary. Consider the Bernoulli covering with $0 < p < 1/2$. Let

$$\xi_n = \log\left(\frac{1-p}{p}\,(s_n - s_{n-1})\right)\ .$$

Notice that $\xi_n \to -\infty$ because of $s_n - s_{n-1} = o(1)$. Consider the random measure $Q\lambda$ defined by

$$\widetilde{W}_n = \frac{e^{\xi_k X}}{\varphi(\xi_k)} \ .$$

It may be checked that the Peyrière measure $\mathcal{Q}$ is well defined and $\dim Q\lambda = 1 - \log_2 \frac{1}{1-p}$ (by Lemma 2 and Lemma 3). Notice that $X_{t_1,\ldots,t_k} = 0$ or 1. We have

$$\mathbb{E}_{\mathcal{Q}} X_{t_1,\ldots,t_k} \;=\; \mathbb{E}_{\mathcal{Q}} X^2_{t_1,\ldots,t_k} \;=\; \mathbb{E}\, X_{t_1,\ldots,t_k}\, W_{t_1,\ldots,t_k} \;=\; c'(\xi_k) \ .$$

It follows that the variance

$$\mathrm{Var}_{\mathcal{Q}} X_{t_1,\ldots,t_k} \;=\; c'(\xi_k) - c'(\xi_k)^2 \ .$$

Since

$$c'(t) \;=\; \frac{pe^t}{1-p+p\,e^t} \;=\; \frac{p}{1-p}\, e^t + O(e^{2t}) \qquad (t \to -\infty) \ ,$$

we have

$$\mathbb{E}_{\mathcal{Q}} X_{t_1,\ldots,t_k} \;=\; \mathrm{Var}_{\mathcal{Q}} X_{t_1,\ldots,t_k} \;=\; s_k - s_{k-1} + O((s_k - s_{k-1})^2) \ .$$

Since $s_n - s_{n-1} = o(1)$, we have

$$\sum_{k=1}^{n} (s_k - s_{k-1})^2 \;=\; o(s_n) \ .$$

By using the law of the iterated logarithm, $\mathcal{Q}$-almost everywhere we have

$$N_n(t) \;\sim\; \frac{p}{1-p} \sum_{k=1}^{n} e^{\xi_k} \;=\; \sum_{k=1}^{n} (s_k - s_{k-1}) \;=\; s_n \ .$$

Thus for the Bernoulli covering with $0 < p < 1/2$, for any sequence such that $s_n - s_{n-1} = o(1)$ we have a.s.

$$\dim E_{0,s} \;=\; 1 - \log_2 \frac{1}{1-p} \;=\; 1 - \frac{c^*(0)}{\log 2} \ .$$

ACKNOWLEDGEMENT – The authors thank Professor A.H. Dooley for his careful reading and linguistic suggestions.

REFERENCES

[B] Biggins, J.D. – Chernoff's theorem in the branching random walk, *J. Appl. Prob.,* 14 (1977), 630–636.

[BP] Benjamini, I. and Peres, Y. – Tree-indexed random walks and first passage percolation, *Probab. Th. Rel. Fields,* 98 (1994), 91–112.

[D] Dvoretzky, A. – On covering a circle by randomly placed arcs, *Pro. Nat. Acad. Sci. USA,* 42 (1956), 199–203.

[E] Ellis, R.S. – *Entropy, Large Deviation and Statistical Mechanics*, Springer-Verlag, 1985.

[F1] Fan, A.H. – Sur certains processus de naissance et de mort, *C.R. Acad. Sci. Paris, Sér. I,* 310 (1990), 441–444.

[F2] Fan, A.H. – Sur les dimensions de mesures, *Studia Math.,* 111 (1994), 1–17.

[F3] Fan, A.H. – How many intervals cover a point in Dvoretzky covering?, submitted to *Israel J. Math.*.

[FK] Fan, A.H. and Kahane, J.P. – Rareté des intervalles recouvrant un point dans un recouvrement aléatoire, *Ann. Inst. Henri Poincaré,* 29(3) (1993), 453–466.

[K1] Kahane, J.P. – Sur le modèle de turbulence de Benoît Mandelbrot, *C. R. Acad. Sci. Paris,* 278 (1974), 621–623.

[K2] Kahane, J.P. – *Some Random Series of Functions*, Cambridge University Press, 1985.

[K3] Kahane, J.P. – Positive martingales and random measures, *Chinese Ann. Math.,* 8B1 (1987), 1–12.

[K4] Kahane, J.P. – Multiplications aléatoires et dimensions de Hausdorff, *Ann. Inst. H. Poincaré,* 23 (1987), 289–296.

[K5] Kahane, J.P. – *Produits de poids aléatoires indépendants et applications*, in "Fractal Geometry and Analysis" (J. Bélair and S. Duduc, Eds.), Kluwer Acad. Publ., 1991, 277–324.

[K6] Kahane, J.P. – *Random coverings and multiplicative processes*, in "Fractal Geometry and Stochastics II" (Ch. Bandt, S. Graf and M. Zähle, Eds.), Progress in Probability, vol. 46, Birkhäuser, 2000.

[KP] Kahane, J.P. and Peyrière, J. – Sur certaines martingales de Benoît Mandelbrot, *Advances in Math.,* 22 (1976), 131–145.

[L] Lyons, R. – Random walks and percolation on trees, *Ann. Probab.,* 18 (1990), 931–958.

[LP] Lyons, R. and Pemantle, R. – Random walks in a random environment and first passage percolation on trees, *Ann. Probab.,* 20 (1992), 125–136.

[M1] Mandelbrot, B.B. – Multiplications aléatoires itérées et distributions invariantes par moyenne pondérée aléatoire*, C. R. Acad. Sci. Paris, Sér. I,* 278 (1974), 289–292 and 355–358.

[M2] Mattila, P. – *Geometry of Sets and Measures in Euclidean Spaces, Fractals and Rectifiability*, Cambridge University Press, 1995.

[PP] Pemantle, R. and Peres, Y. – Critical random walk in random environment on trees, *Ann. Probab.,* 23 (1995), 105–140.

[S] Shepp, L. – Covering the circle with random arc, *Israel J. Math.,* 11 (199-72), 163–170.

Ai Hua Fan
Département de Mathématiques et Informatique, Université de Picardie Jules Verne
33, Rue Saint Leu, 80039 Amiens Cedex 1 – FRANCE
E-mail: ai-hua.fan@u-picardie.fr

and

Jean Pierre Kahane
Analyse harmonique, Université de Paris Sud
91405 Orsay Cedex – FRANCE
E-mail: jean-pierre.kahane@math.u-psud.fr