PORTUGALIAE MATHEMATICA
Vol. 56 Fasc. 2 – 1999

GEOMETRIC PROOFS OF COMPOSITION THEOREMS FOR GENERALIZED FOURIER INTEGRAL OPERATORS

M.S. Joshi

1 – Introduction

The calculus of Fourier integral operators introduced by Hörmander in [11] has found widespread use throughout the study of linear partial differential equations. However, it is limited by the fact that in general the calculus is not closed under composition. Also various natural distributions which arise, for example the forward fundamental solution of the wave operator, are not contained within the calculus. Generalizations of the class have been introduced by Boutet de Monvel, Guillemin, Melrose, Uhlmann and the author, see [4], [10], [13], [15], [17] and [19]. Our purpose here is to introduce a new approach to composing these classes which can handle all the different classes simultaneously and will always give information about the resultant class regardless of the degeneracy of the composition. This will yield the clean composition theorem for Fourier integral operators as a simple special case as well as generalizations involving paired Lagrangian distributions. We will see that a general composition can be reduced to understanding its local symplectic geometry.

The basic object of the theory of Fourier integral operators is the Lagrangian distribution which was originally defined in [11] using oscillatory integrals. It was later noticed by Beals, Bony, Melrose that these spaces could be defined in terms of stability of application of operators. For the case of conormal bundles, this became distributions whose orders were stable under repeated application of vector fields tangent to the submanifold. This was later generalized to all conic Lagrangian submanifolds by replacing the vector fields with first order pseudo-differential operators characteristic on the submanifold. This second definition had the advantages that the geometry appeared more naturally and that it was *a priori* coordinate invariant.

Received: March 4, 1997; *Revised*: October 31, 1997.

The essential idea in our approach is to regard the modules, under which stability is required, to be the prime objects of study. Proving composition theorems then becomes a question of how does the module change under certain operations. This means that one always obtains information about the composite regardless of how degenerate the composition. Melrose in [16] and [18] has already exploited this point of view in the special case where the modules are generated by vector fields, in which case one can work with Lie algebras of vector fields instead. Testing by Lie algebras of vector fields has also been extensively used in proving propagation theorems for semi-linear wave equations.

A composition is a sequence of three operations, exterior product, restriction to the diagonal and a push-forward along the diagonal so it is enough to understand the transformation of the defining modules in each of these three cases.

Definition 1.1. We shall say $\mathcal{M}$ is a defining class of symbols on a smooth manifold X, if $\mathcal{M}$ is submodule of the spaces of homogeneous smooth functions of order one (over the functions of order zero) which is closed under Poisson bracket.

Definition 1.2. If $\mathcal{M}$ is a defining class of symbols and $\mathcal{H}$ is a class of distributions then $I_l(\mathcal{M}, \mathcal{H})$ is the set of $u \in \mathcal{H}$ such that

$$P_1 \dots P_k u \in \mathcal{H}$$

for all $k \leq l \in \{\mathbb{N}\} \cup \{\infty\}$ and $P_j \in \mathcal{M}$. If $\mathcal{H} = H^s_{\text{loc}}$ we write the class as $I_l^{(s)}(\mathcal{M})$. We shall drop the subscript when $l = \infty$.

In this note, we shall work in the context of Sobolev spaces but it is important to realize that the only important properties we use are the L^2 continuity of zeroth order operators and their micro-localizability — one could equally well work with Besov spaces. Note one could also work with line bundles.

Suppose X, Y are smooth manifolds and π is the projection from $X \times Y$ to X. Upon pushing-forward a distribution from $X \times Y$ to X, wavefront which is not conormal to the fibres will be killed, i.e. only wavefront set contained in

$$N^*_X(Y) = \bigcup_{x \in X} N^*(\{x\} \times Y)$$

will show up below. Note that a function on $T^*(X)$ can be pulled-back to a function on $N^*_X(Y)$ by

$$\bar{\pi}^*(f)(x, \xi, y, 0) = f(x, \xi) \ .$$

We prove,

Theorem 1.1. *Let $\mathcal{M}$ be a defining class of symbols on $X \times Y$ where X, Y are smooth manifolds and Y has a smooth density. If $u \in I_l^{(s)}(\mathcal{M})$ and the support of u is proper with respect to projection onto Y then $\pi_* u \in I_l^{(s)}(\widetilde{\mathcal{M}})$ where $\widetilde{\mathcal{M}}$ is the class of symbols on X, which after being pulled back to $N_X^*(Y)$ are restrictions of elements of $\mathcal{M}$.*

We can view restrictions in a similar way. If Y is a submanifold of X then a function, $f(y, \eta)$, on the cotangent bundle of Y, T^*Y, can be pulled back to a function on $T_Y^*(X)$, the cotangent bundle of X restricted to Y, by the dual to the derivative of the natural inclusion

$$i \colon Y \to X \ .$$

Theorem 1.2. *Let $\mathcal{M}$ be a defining class of symbols on X. If $u \in I_l^{(s)}(\mathcal{M})$ and Y is a submanifold of X of codimension k such that*

$$\Big(\bigcap_{p \in \mathcal{M}} p^{-1}(0) \Big) \cap N^*(Y) = \emptyset \tag{1.1}$$

and $\mathrm{WF}(u) \cap N^*(Y) = \emptyset$ then $i_Y^* u \in I_l^{(s)}(\widetilde{\mathcal{M}})$ where $\widetilde{\mathcal{M}}$ is the class of symbols which after being pulled back to $T_Y^*(X)$ are restrictions of elements of $\mathcal{M}$ close to $\bigcap_{p \in \mathcal{M}} p^{-1}(0)$.

Note that the condition (1.1) ensures that $\mathrm{WF}(u) \cap N^*(Y) = \emptyset$ when $l = \infty$ and hence that the restriction exists.

The case of exterior product is much simpler.

Theorem 1.3. *Let $\mathcal{M}_i$ be a defining class of symbols on a smooth manifold X_i. Suppose $u_i \in I_l^{(s)}(X_i)$ then*

$$u_1 \otimes u_2 \ \in \ I_l^{(\min(s,t,s+t))}(\mathcal{M})(X_1, X_2)$$

where $\mathcal{M}$ is the set of first order homogeneous functions on $T^*(X_1 \times X_2) - \{0\}$ of the form

$$\sum a_j \, \bar{p}_{1,j} + \sum b_k \, \bar{p}_{2,k}$$

where a_j, b_k are zeroth order functions on $T^*(X_1 \times X_2) - \{0\}$ and with a_j zero near $N_{X_1}^*(X_2)$ and b_j zero near $N_{X_2}^*(X_1)$ and $p_{i,l} \in \mathcal{M}_i$ and $\bar{p}_{i,k}$ denotes the pull-back to product space.

With these three theorems proven, we show that proving the clean composition theorem for Fourier integral operators is then just the observation that if two submanifolds intersect cleanly and if a smooth function defined on one vanishes on the intersection then it can be extended smoothly to vanish on the second.

Paired Lagrangian distributions often arise as pseudo-differential operators with singular symbols, in which case one Lagrangian is the conormal bundle to the diagonal, and the questions then arises of when are these classes closed under composition. We study two cases. The first comes from when the second Lagrangian submanifold is obtained by flowing out a submanifold by Hamiltonians associated to Poisson commuting functions. This arises when studying operators of the form

$$P = \sum Q_j^2$$

or

$$P = Q_1 + i\,Q_2$$

with Q_j of real principal type (see for example [10]). We show that in this case the class is closed under composition.

The second case we examine is that of when the second Lagrangian is the graph of a symplectomorphism which arises in the study of singular Radon transforms. In this case, the class is not closed under composition but we identify a class in which R^tR, RR^t lies for R in this class.

I would like to thank Richard Melrose and Gunther Uhlmann for helpful conversations.

2 – Stability under testing

Our purpose in this section is to review how to define some of the basic objects in micro-local analysis in terms of stability under repeated application of elements of modules of pseudo-differential operators and to examine how these modules behave under exterior products, restrictions and push-forwards. As a composition can be decomposed into an exterior product, a restriction and a push-forward these theorems will be enough to deduce general composition theorems.

Throughout, M will be a smooth manifold with cotangent bundle T^*M. We will denote the "classical" (or polyhomogeneous with integer step) pseudo-differential operators of order m by $\Psi\mathrm{DO}^m_{phg}(M)$. We will work solely with pseudo-differential operators which are proper and classical. We will assume our manifolds are equipped with smooth densities and so smooth functions can be regarded as a subset of the space of distributions.

Example 2.1. Let Λ be a conic Lagrangian submanifold of $T^*M - \{0\}$ and let $\mathcal{M}$ be the set of first order functions which vanish on Λ. Then

$$I(\mathcal{M}) = \bigcup_s I(\mathcal{M}, H^s_{\mathrm{loc}})$$

is the set of Lagrangian distributions associated with Λ. These are the Schwartz kernels of Fourier integral operators.

Example 2.2. Let Σ be an isotropic submanifold of $T^*M - \{0\}$ and then let $\mathcal{M}$ be the first order functions which are zero on Σ and whose Hamiltonians are tangent to Σ. Then

$$I(\mathcal{M}) = \bigcup_s I(\mathcal{M}, H^s_{\mathrm{loc}})$$

is the set of isotropic distributions with respect to Σ.

Example 2.3. Let Λ_1, Λ_2 be cleanly intersecting conic Lagrangian submanifold of $T^*M - 0$ and let $\mathcal{M}$ be the first order symbols which are characteristic on $\Lambda_1 \cup \Lambda_2$. Then

$$I(\mathcal{M}) = \bigcup_s I(\mathcal{M}, H^s_{\mathrm{loc}})$$

is the set, $I(\Lambda_1, \Lambda_2)$, of paired Lagrangian distributions with respect to (Λ_1, Λ_2). Note that this is a very weak definition of a paired Lagrangian distribution — in particular it includes the set of isotropic distributions with respect to the intersection.

Example 2.4. Let Λ be a conic Lagrangian submanifold of $T^*M - \{0\}$ and suppose $\Sigma_1, \dots, \Sigma_k$ are nested submanifolds of Λ, then let $\mathcal{M}$ be the first order symbols which are characteristic on Λ and whose Hamiltonians are tangent to each of the Σ_j. Then

$$I(\mathcal{M}) = \bigcup_s I(\mathcal{M}, H^s_{\mathrm{loc}})$$

is the set of marked Lagrangian distributions, I_{ma}, with respect to $(\Lambda, \Sigma_1, \dots, \Sigma_k)$ (see [17]).

Note that if a class of symbols, $\mathcal{M}$, fails to be a defining class because of non-closure under Poissson bracket then it will still define a class of distributions but in the infinite regularity case, the class will be equal to that obtained by using the smallest class of operators containing $\mathcal{M}$ which is closed under Poisson bracket. This is because if

$$P_1 P_2 u,\ P_2 P_1 u \in H^s_{\mathrm{loc}}$$

then so is

$$[P_1, P_2]\, u \ .$$

This is one reason why Lagrangian submanifolds are natural: if two functions vanish on a Lagrangian submanifold then so does their Poisson bracket and Lagrangian submanifolds are of minimal dimension amongst submanifolds having this property.

If there is an element P of $\mathcal{M}$ which is not characteristic at a point (x, ξ) then it is immediate from micro-ellipticity that $(x, \xi) \notin \mathrm{WF}(u)$ for $u \in I(\mathcal{M}, H^s_{\mathrm{loc}})$ so we have

Proposition 2.1. *If* $u \in I(\mathcal{M}, H^s_{\mathrm{loc}})$ *then*

$$\mathrm{WF}(u) \subset \bigcap_{p \in \mathcal{M}} p^{-1}(0) \ .$$

With our definitions now made, we consider exterior products, push-forwards and restrictions.

If π_y is the projection of $X \times Y$ onto X then it defines a push-forward map of distributions whose supports are proper for the projection by

$$\langle (\pi_y)_* u, \phi \rangle = \langle u, (\pi_y)^* \phi \rangle \tag{2.1}$$

for $u \in \mathcal{D}'(X \times Y)$ and $\phi \in C^\infty_c(X)$. This defines a continuous map from $H^s_c(X \times Y)$ to $H^s_c(X)$ (see Section 3). A simple upper bound on the wavefront set holds expressing the fact that only singularities conormal to the push-forward are retained, letting

$$N^*_X(Y) = \bigcup_{x \in X} N^*(\{x\} \times Y) \ ,$$

we have

$$\mathrm{WF}((\pi_y)_* u) \subset \Big\{ (x, \xi) \in T^*(X) - \{0\} \colon \exists y \ (x, \xi, y, 0) \in \mathrm{WF}(u) \Big\} \tag{2.2}$$

$$= (\pi_y)_* \Big(N^*_X(Y) \cap \mathrm{WF}(u) \Big) \ . \tag{2.3}$$

For a proof of this see [11]. There is a natural projection of $N^*_X(Y)$ onto $T^*(X)$

$$\bar{\pi}(x, \xi, y, 0) = (x, \xi) \ .$$

Thus functions on $T^*(X) - \{0\}$ can be pulled back using $\bar{\pi}$ to functions on $N^*_X(Y)$ by

$$\bar{\pi}^* q(x, \xi, y, 0) = q(x, \xi) \ .$$

Since we know that only wavefront set meeting $N^*_X(Y)$ shows up after the push-forward, it is not surprising that it is the behaviour of the defining classes at $N^*_X(Y)$ which affects the regularity properties below. We now proceed to the proof of Theorem 1.1 which expresses this relationship.

Proof of Theorem 1.1: See Section 3 for a proof that π_* maps elements of $H^s_{\mathrm{loc}}(X \times Y)$ which are proper to the projection into $H^s_{\mathrm{loc}}(X)$. We need to show stability for $(\pi_y)_* u$ under application of operators with principal symbols in $\widetilde{\mathcal{M}}$. We do this by lifting the pseudo-differential operators to act on $X \times Y$ and then add operators which are killed by the push-forward to obtain operators with symbols in $\mathcal{M}$.

Taking local coordinates and a partition of unity, it is enough to take Y to be $\mathbb{R}^n$. When pushing-forward with respect to Lebesgue measure on $\mathbb{R}^n$ derivatives will vanish. So with a general density and Q a zeroth order pseudo-differential operator, for any j we have

$$(\pi_y)_*(D_{y_j} Q u) = (\pi_y)_*(Q' u)$$

with Q' a zeroth order pseudo-differential operator. So

$$P(\pi_y)_*(u) = (\pi_y)_*\Big(\big(\bar{P} + \sum D_{y_j} Q_j\big)u + Ru\Big) \,, \tag{2.4}$$

where $\bar{P}$ is P lifted and Q_j is an arbitrary zeroth order pseudo-differential operator and R is zeroth order.

Note that if P has Schwartz kernel $K(x, x')$ then $\bar{P}$ will have Schwartz kernel $K(x, x')\,\delta_y(y')$ and so $\bar{P}$ is not quite a pseudo-differential operator as its wavefront set contains $N^*(y = y')$ — it is in fact a paired Lagrangian distribution. However the additional wavefront set will be killed by the push-forward and is therefore irrelevant. So provided there exists Q_j such that the principal symbol of $\bar{P} + \sum D_{y_j} Q_j$ is in $\mathcal{M}'$ near $N^*_X(Y)$ then $(\pi_y)_*(u)$ will retain membership of H^s_{loc} under application of P as then $(\bar{P} + \sum D_{y_j} Q_j)u$ will remain in $I(\mathcal{M}, H^s_{\mathrm{loc}})$. Inductively, this remains true for any application $P_1 P_2 \dots P_k$ for P_j with this property. But the condition on P_j is equivalent to the hypothesis of our theorem and so the result follows. ∎

Corollary 2.1. *If the conic Lagrangian submanifold Λ intersects $N^*_X(Y)$ cleanly then for properly supported elements*

$$(\pi_y)_* \colon I^{(s)}_l(\Lambda) \to I^{(s)}_l(\Lambda') \,,$$

where $\Lambda' = \{(x, \xi) \colon \exists y\ (x, \xi, y, 0) \in \Lambda\}$.

Proof: If a function defined on $N_X^*(Y)$ vanishes on the intersection with Λ then it can immediately be extended to vanish on Λ (see Corollary 4.1) and the result is immediate.

NB it follows from the discussion in [12] Chap. 21 that Λ' is at least locally a Lagrangian submanifold. ∎

We remark that this argument works equally well if the testing functions are required to vanish to fixed finite order k, (or even infinite order) on Λ — such classes could be interpreted as having symbols of type $1/k$.

The arguments for restrictions are very similar. An embedded submanifold X of M has a natural inclusion

$$i\colon\ X \to M$$

and the derivative di provides a natural inclusion of tangent spaces. There is therefore a natural projection of the bundle T^*M restricted to X onto T^*X dual to this inclusion. We use this projection to pull-back functions on T^*X to functions on $T^*M_{|X}$. In local coordinates such that $X = \{y = 0\}$, this becomes, for q a function on T^*X,

$$i_*\, q(x, \xi, 0, \eta) = q(x, \xi)\ .$$

Now a distribution, u, on a manifold, M, can be restricted to a submanifold, X, (or pulled back by the inclusion map i) if its wavefront set is disjoint from the conormal bundle of X and so we work with defining classes whose wavefront sets are disjoint from the conormal bundle.

Proof of Theorem 1.2: The mapping property on Sobolev spaces is proven in Section 3. Our approach to this proof is very similar to that of Theorem 1.1 and in some sense our argument is dual to the one there. Since X is an embedded submanifold, taking a partition of unity we can work in local coordinates (x, y) such that X is given by $y = 0$. For any $u \in I(\mathcal{M}, H^s_{\text{loc}})$, $\text{WF}(u) \cap N^*(X)$ is empty so $i_X^*(u)$ can be defined by

$$\Big\langle i_X^*(u), \phi(x) \Big\rangle = \Big\langle u, \delta(y)\phi(x) \Big\rangle \tag{2.5}$$

(see for example [11]). This means that if a smooth function f vanishes on X that $i_X^*(fu)$ is equal to zero. At the pseudo-differential operator level this becomes

$$P i_X^*(u) = i_X^*\Big((\bar{P} + Q)\, u \Big)\ ,$$

where $\bar{P}$ is the lifting of P to act in extra coordinates and Q is a first order pseudo-differential operator of which the total symbol vanishes on X. The extra wavefront set in $\bar{P}$ will be irrelevant as the condition on $\mathrm{WF}(u)$ ensures that it will not interact, so if there exists Q such that the principal symbol of $\bar{P}+Q$ is in $\mathcal{M}$ near $\mathrm{WF}(\mathcal{M})$ then $(\bar{P}+Q)u$ will remain in $I(\mathcal{M}, H^s_{\mathrm{loc}})$ and so stability under application of P follows. This condition on the symbol is equivalent to our hypothesis and the result is now immediate. ∎

Corollary 2.2. *If Λ is an embedded Lagrangian submanifold of $T^*M - \{0\}$ disjoint from $N^*(X)$ and Λ intersects $T^*M_{|X}$ cleanly, then*

$$i_X^*\colon\, I_l^{(s)}(\Lambda) \to I_l^{(s+k/2)}(\Lambda')$$

where Λ' is the projection of $\Lambda \cap T^*M_{|X}$ onto $T^*(X)$. If $l < \infty$, we also require $\mathrm{WF}(u) \cap N^*(X) = \emptyset$.

Proof: This is the same as the proof of Corollary 2.1. ∎

Exterior product is the simplest of the three operations — the distributions are simply adjoined by making them act in different variables. So if we have that $\mathcal{M}_i$ is a defining class on X_i and that $u_i \in I(\mathcal{M}_i, \mathcal{H}_i)$ then it is immediate that $u_1 \otimes u_2$ will be stable under applications of $\bar{P}_i$ — the lifting of an operator with symbol in $\mathcal{M}_i$ to the product space:

$$\bar{P}_1(u_1 \otimes u_2) = (P_1 u_1 \otimes u_2)\,, \qquad \bar{P}_2(u_1 \otimes u_2) = (u_1 \otimes P_2 u_2)\ .$$

Theorem 1.3 is now obvious.

We now compute some examples to illustrate the use of the push-forward theorem when the usual composition calculus fails.

Example 2.5. Let $X = \mathbb{R}^l_s \times \mathbb{R}_x$ and let $Y = \mathbb{R}_y$ with dual coordinates (μ, ξ, η). Then the Lagrangian submanifold in $T^*(X \times Y)$, $\Lambda = N^*(x = y^{k+1})$ is tangent to $N^*_X(Y) = \{\eta = 0\}$:

$$T_p\Big(\Lambda \cap \{\eta = 0\}\Big) \ \subsetneq\ T_p\Lambda + T_p(\{\eta = 0\})$$

so the composition with push-forward in y is not clean. Using Taylor's theorem, we have that $p \in \mathcal{M}(\Lambda)$ if and only if

$$p(s, \mu, x, \xi, y, \eta) = (x - y^{k+1})\, p_1 + \Big(\eta + (k+1)\, y^k\, \xi\Big)\, p_2 + \mu\, p_3\ .$$

So $q(s, \mu, x, \xi) \in \widetilde{\mathcal{M}}$ if and only if there exists smooth r such that

$$q + \eta\, r = (x - y^{k+1})\, p_1 + \Big(\eta + (k+1)\, y^k\, \xi\Big)\, p_2 + \mu\, p_3$$

for some smooth p_j. Evaluating at $\eta = 0$, $\mu = 0$, we obtain,

$$q(s,0,x,\xi) = (x - y^{k+1})\,p_1 + (k+1)\,y^k\,\xi\,p_2\ .$$

Putting $y = 0$ and observing that the left side is independent of y we deduce that $q(s,0,x,\xi)$ must vanish on $x = 0$ and so

$$q = x\,q_1 + \mu\,q_2\ .$$

So we have that $q \in \mathcal{M}(N^*(x=0))$. Any q of this form will be extendible to an element of $\mathcal{M}(\Lambda)$ as we can take

$$p = (x - y^{k+1})\,q_1 + \mu\,q_2 + \Big(\eta + (k+1)\,y^k\,\xi\Big)\,\frac{y}{(k+1)\xi}\,q_1\ .$$

So we conclude that

$$(\pi_y)_*\colon\ I\Big(N^*(x = y^{k+1}),\, H^s_{\mathrm{loc}}\Big) \to I\Big(N^*(x=0),\, H^s_{\mathrm{loc}}\Big)\ .$$

We now consider an example where the nature of the intersection varies — we do not obtain a Lagrangian distribution but instead a paired Lagrangian distribution.

Example 2.6. Let (s,x,y) be as in Example 2.5 but now let

$$\Lambda = N^*(x = y^{k+1},\ s = 0)\ .$$

Away from $(x,y) = 0$ the push-forward in y is transversal and there are no problems but at the origin there is tangency and a second Lagrangian appears. Similarly to above, the elements of $\mathcal{M}(\Lambda)$ are of the form

$$p = (x - y^{k+1})\,p_1 + s\,p_2 + \Big(\eta + (k+1)\,y^k\,\xi\Big)\,p_3\ .$$

The same arguments as above show that the elements of $\widetilde{\mathcal{M}}$ are of the form

$$x\,q_1 + s\,q_2\ .$$

However, general q_1 are not possible, putting $s = 0$, $\xi = 0$ and $\eta = 0$ we have that

$$x\,q_1(0,\mu,x,0) = (x - y^{k+1})\,p_1(0,\mu,x,0,y,0)\ .$$

The left-hand side is independent of y and the right-hand side vanishes when $x = y^{k+1}$. So if k is even then

$$q = x\,\xi\,q_1^{'} + s\,q_2$$

and when q is odd we have

$$q = x\,\xi\,q_1' + s\,q_2 + f$$

where f is identically zero on $x \geq 0$. Thus, as above, when k is even we obtain

$$(\pi_y)_*\colon\ I\Big(N^*(x = y^{k+1}),\, H^s_{\text{loc}}\Big) \to I\Big(N^*(s = 0),\, N^*(x = 0, s = 0),\, H^s_{\text{loc}}\Big)$$

and when k is odd we obtain

$$(\pi_y)_*\colon\ I\Big(N^*(x = y^{k+1}),\, H^s_{\text{loc}}\Big) \to I\Big(N^*(s = 0, x \geq 0),\, N^*(x = 0, s = 0),\, H^s_{\text{loc}}\Big)\ .$$

We give an example where the degeneracy of a restriction yields a paired Lagrangian distribution from a Lagrangian distribution and a slight modification of this example gives an example of paired Lagrangians going to paired Lagrangians.

Example 2.7. Let Λ be the closure of the conormal bundle of the smooth part of the cone $t^2 = x^2$ in the space $\mathbb{R}_t \times \mathbb{R}^n_x$. If we let (ξ, τ) be the dual coordinates, the smooth Lagrangian submanifold Λ can be defined by

$$\begin{aligned} x_j\,\tau + t\,\xi_j &= 0\,, \qquad j = 1, ..., n\ , \\ \tau^2 - \xi^2 &= 0\ . \end{aligned}$$

So the symbols in the defining class are of the form

$$\sum_{j=1}^{n} (x_j\,\tau + t\,\xi_j)\,q_j + (\tau^2 - \xi^2)\,r\ .$$

We consider the restriction to $x_1 = 0$ — this exists as

$$\Lambda \subset \{\tau^2 - \xi^2 = 0\}$$

and the conormal bundle to $x_1 = 0$ is contained in $\{\tau = 0\}$. We claim that symbols of the form

$$p(x'', \xi'', t, \tau) = \sum_{j=2}^{n} (x_j\,\tau + t\,\xi_j)\,q_j + t(\tau^2 - (\xi'')^2)\,r + f\ ,$$

where f is identically zero on $(\tau^2 - (\xi'')^2) \geq 0$, are in the restricted class $\mathcal{M}'$. The extension of elements of the form $(x_j\,\tau + t\,\xi_j)\,q_j$ is obvious and if $f(x'', \xi'', t, \tau)$ is zero on $(\tau^2 - (\xi'')^2) \geq 0$ then it will be zero on $\tau^2 = \xi^2$ when lifted to the extra variable and so is extendible. This leaves $t(\tau^2 - (\xi'')^2)\,r$ but

$$t(\tau^2 - (\xi'')^2)\,r + \tau\,x_1\,\xi_1\,r = t(\tau^2 - (\xi)^2)\,r + (t\,\xi_1 + \tau\,x_1)\,\xi_1\,r$$

is a valid extension and so the claim is proven.

These symbols are precisely the ones which are zero on

$$N^*((x'')^2 = t^2)$$

and on

$$\left\{x'' = 0,\ t = 0,\ (\xi'')^2 \le \tau^2\right\} .$$

We conclude that the restriction is paired Lagrangian. In fact, the same arguments show that if one takes a paired Lagrangian with respect to Λ and $N^*(x=t=0)$ cut off away from $\xi_1=0$ then the restriction is in $I(N^*((x'')^2=t^2)$, $N^*(x''=0,\ t=0))$.

3 – Operations on Sobolev spaces

In this section, we examine how Sobolev spaces behave under three basic operations thus ensuring that our theorems are not vacuous.

Theorem 3.1. *If X, Y are smooth manifolds and Y is equipped with a smooth density then push-forward along Y induces a map on Sobolev spaces*

$$(\pi_y)_*\colon\ H^s_c(X \times Y) \to H^s_c(X) .$$

Proof: Taking a partition of unity it is enough to consider the case when X, Y are open subsets of $\mathbb{R}^n$ with compact closure. As the smooth density on Y will be equal to a smooth function times the Euclidean density and the function will be bounded above it is enough to prove the result for the Euclidean density.

First of all suppose $s = 0$, then we need to show the continuity of push-forward on L^2_c. Assume that $f \in L^2$ has support contained in some box $\mathbb{R}^k_x \times [-C, C]^l_y$ then using Cauchy–Schwartz, we have that

$$\int \left|\int f(x,y)\,dy\right|^2 dx \;\le\; \iint |f(x,y)|^2\,dx\,dy\,(2C)^l \tag{3.1}$$

and so in terms of norms we have $\|(\pi_y)_* f\|_0 \le (2\,C)^{l/2}\,\|f\|_0$.

The case for positive integers follows come from considering estimates on derivatives and the negative integers follow by duality. Once one has all the integral cases one can interpolate to prove the estimate for all s. ∎

Theorem 3.2. *Exterior product induces a continuous map*

$$H^s_{\mathrm{loc}}(X) \times H^t_{\mathrm{loc}}(Y) \to H^{\min(s,t,s+t)}_{\mathrm{loc}}(X, Y) .$$

Proof: It is enough to show that for $s, t \leq 0$ there is a continuous map into H^{s+t} and for $0 \leq s \leq t$ there is a continuous map into H^s.

For $s, t \leq 0$, we have

$$\begin{aligned}\iint |\hat{f}(\xi)|^2\, |\hat{g}(\eta)|^2\, (1+|\xi|^2)^s\, (1+|\eta|^2)^t\, d\xi\, d\eta \geq \\ \geq \iint |\hat{f}(\xi)|^2\, |\hat{g}(\eta)|^2\, (1+|\xi|^2+|\eta|^2)^{s+t}\, d\xi\, d\eta\ ,\end{aligned} \tag{3.2}$$

i.e. $\|f(x)\, g(y)\|_{s+t} \leq \|f(x)\|_s\, \|g(y)\|_t$ and the estimate follows in the Euclidean case. Taking local coordinates the result then follows in general.

For $0 \leq s \leq t$, we have that

$$\begin{aligned}\iint |\hat{f}(\xi)|^2\, |\hat{g}(\eta)|^2\, (1+|\xi|^2+|\eta|^2)^s\, d\xi\, d\eta \leq \\ \leq \int |\hat{f}(\xi)|^2\, (1+|\xi|^2)^s\, d\xi \int |\hat{g}(\eta)|^2\, (1+|\eta|^2)^t\, d\eta\end{aligned} \tag{3.3}$$

and so that case follows also. ∎

Theorem 3.3. *If X is a smooth embedded submanifold of codimension k in a manifold M and $u \in H^s_{\mathrm{loc}}(M)$ is such that $\mathrm{WF}(u) \cap N^*(X) = \emptyset$ then $i^*_X u$ is well-defined and is in $H^{s-k/2}_{\mathrm{loc}}(X)$.*

Proof: That the restrictions is well-defined follows from the calculus of wavefront sets (see for example [11]).

It is sufficient to work locally so suppose (x', x'') is a splitting of the coordinates in $\mathbb{R}^n$ such that $X = \{x' = 0\}$ then if $p \in X$ we can put $u = u_1 + u_2 + u_3$ with u_1 smooth, u_2 identically zero near p and the Fourier transform of u_3 is supported in a cone which does not meet $\xi'' = 0$. The regularity properties of $i^*_X u$ near p will be determined purely by u_3. Thus we need to estimate the H^s norm of $i^*_X(u_3)$.

Now there exists C such that $\mathrm{supp}(\hat{u}_3) \subset \{|\xi'| \leq C|\xi''|\}$ and so we have

$$\|\widehat{i^*_X u_3}\|^2_{s'} = \int \left| \int \hat{u}(\xi)\, d\xi' \right|^2 (1+|\xi''|^2)^{s'}\, d\xi'' \tag{3.4}$$

$$\leq \int \int_{|\xi'| \leq C|\xi''|} |\hat{u}(\xi)|^2\, (1+|\xi''|^2)^{s'+\frac{k}{2}}\, d\xi \tag{3.5}$$

$$\leq C' \iint |\hat{u}(\xi)|\, (1+|\xi|^2)^{s'+\frac{k}{2}}\, d\xi\ . \tag{3.6}$$

Thus the result follows. ∎

In this note, we do not use the regularity of the distributions under testing by operators to improve these estimates, however work of Gerard and Golse has shown this is possible for push-forwards and probably in general, [8].

4 – Clean intersections

Much of this paper involves reducing analysis to questions of differential and symplectic geometry. We assume that the reader is familiar with the contents of Chap. 21 of [12] which is a good reference for the symplectic geometry needed in micro-local analysis. It is customary in micro-local analysis to work with twisted symplectic forms on products of symplectic manifolds i.e. on the product $S_1 \times S_2$ to take the difference of the pull-back of the symplectic forms rather than the sum. However, here we shall work with the sum instead as it is more natural when comparing cotangent bundles of products with products of cotangent bundles. If Λ_j is a Lagrangian submanifold of $T^*X_j \times T^*X_{j+1}$ and we regard it as a relation, we have the twisted composition:

$$\Lambda_1 \circ \Lambda_2 = \Big\{(x,\xi,z,\zeta)\colon\ (x,\xi,y,\eta,y,-\eta,z,\zeta) \in \Lambda_1 \times \Lambda_2\Big\} .$$

One situation we will need to consider often is that of submanifolds intersecting; a theorem due to Bott ([3] or [12]) gives us a necessary and sufficient condition for two intersecting embedded submanifolds to be modelable by the vanishing of coordinates.

Definition 4.1. Embedded submanifolds Y, Z of a smooth manifold X are said to intersect cleanly if $Y \cap Z$ is a smooth submanifold such that

$$T_p(Y \cap Z) = T_p(Y) \cap T_p(Z)\,, \qquad p \in Y \cap Z\ .$$

Theorem 4.1. *If Y and Z intersect cleanly then there exist local coordinates, x, such that*

$$Y = \Big\{x_1 = \ldots = x_e = 0,\ x_{e+1} = \ldots = x_k = 0\Big\}\ ,$$
$$Z = \Big\{x_1 = \ldots = x_e = 0,\ x_{k+1} = \ldots = x_l = 0\Big\}\ .$$

Note that e is the codimension of $T_p(Y) + T_p(Z)$ and is often called the *excess* of the intersection — it expresses how much the intersection has failed to be transversal. One way of viewing the theorem is that locally there exists a submanifold containing Y, Z in which they are transversal — it is easy to see that any transversal intersection must be clean. The important fact about clean intersections for us is,

Corollary 4.1. *If submanifolds Y, Z of X intersect cleanly and the smooth function f on Y vanishes on $Y \cap Z$ then f is the restriction of a smooth function on X which vanishes on Z.*

Note that this is not true for general intersections. We recall from [12] Chap. 21 that for clean intersection of Lagrangian submanifolds, a model holds within the homogeneous symplectic category.

Theorem 4.2. *If Λ_1 and Λ_2 are cleanly intersecting conic Lagrangian submanifolds and $p \in \Lambda_1 \cap \Lambda_2$ then there are local homogeneous symplectic coordinates (x, ξ) such that*

$$\Lambda_1 = \Big\{ x = 0 \Big\}, \quad \Lambda_2 = \Big\{ x_1 = \ldots = x_k = 0, \ \xi_{k+1} = \ldots = \xi_n = 0 \Big\} .$$

When studying the compositions of paired Lagrangian distributions, we shall need models for triple intersections.

Definition 4.2. We shall say the submanifolds X, Y, Z form a cleanly intersecting triple if all the pairwise intersections are clean, the pairwise intersections of the pairwise intersections and the original submanifolds are clean and for all $p \in X \cap Y \cap Z$

$$\Big(T_p(X) + T_p(Y) \Big) \cap T_p(Z) = \Big(T_p(X) \cap T_p(Z) \Big) + \Big(T_p(Y) \cap T_p(Z) \Big) . \tag{4.1}$$

The condition (4.1) can be rephrased in a symmetric way:

Proposition 4.1. *If A, B, C are subspaces of a vector space V then following properties are equivalent*

(**1**) $A \cap (B + C) = (A \cap B) + (A \cap C)$.

(**2**) *There is a basis for V such that each of A, B and C is a span of elements of the basis.*

(**3**) $\dim(A) + \dim(B) + \dim(C) - \dim(A \cap B) - \dim(A \cap C) - \dim(B \cap C) = \dim(A + B + C)$.

Example 4.1. In $\mathbb{R}^3$, let

$$X = \Big\{ x_1, x_3 = 0 \Big\} , \tag{4.2}$$

$$Y = \Big\{ x_2, x_3 = 0 \Big\} , \tag{4.3}$$

$$Z = \Big\{ x_1 = x_2, \ x_3 = x_1^k \Big\}, \quad k \geq 2 , \tag{4.4}$$

then X, Y, Z do not form an intersecting triple as (4.1) is violated.

Theorem 4.3. *If the submanifolds X, Y, Z form an intersecting triple then near any point there exist local coordinates such that each is given by a subset of the coordinates vanishing.*

Clearly, a homogeneous version of the theorem also holds. Note that the converse to the theorem is trivial. Note also that the conclusion of our theorem is stronger than requiring the existence of a linear model and is not equivalent. For example, three collinear lines in a plane would not form an intersecting triple.

Proof: We only need consider points near the triple intersection as otherwise Bott's theorem is sufficient. Since we have that the triple intersection is a submanifold, we can certainly pick coordinates on it and so the behaviour along $X \cap Y \cap Z$ is parametrised by them. This means that it is enough to consider the case where the triple intersection is a single point.

Now we know by Bott's result that there exist coordinates, f, on the ambient manifold such that locally

$$Y = \Big\{f_1, \dots, f_l = 0 = f_{l+1} = \dots = f_m\Big\} , \tag{4.5}$$

$$Z = \Big\{f_1, \dots, f_l = 0 = f_{m+1} = \dots = f_r\Big\} . \tag{4.6}$$

Bott's result also allows us to pick coordinates on X, x, such that

$$X \cap Y = \Big\{x_1 = \dots = x_\alpha = x_{\alpha+1} = \dots = x_{\alpha+\beta} = 0\Big\} , \tag{4.7}$$

$$X \cap Z = \Big\{x_1 = \dots = x_\alpha = x_{\alpha+\beta+1} = \dots = x_{\alpha+\beta+\gamma} = 0\Big\} . \tag{4.8}$$

Our procedure will be to perform non-singular operations on the f coordinates in such a way that they will agree with the x coordinates on X and then the result will follow. Let g_j denote f_j restricted to X.

We know from our hypothesis that

$$T_pX \cap (T_pY + T_pZ) = (T_pX + T_pZ) \cap (T_pY + T_pZ) .$$

This means precisely that $\langle\{dg_1, \dots, dg_l\}\rangle = \langle\{dx_1, \dots, dx_\alpha\}\rangle$. So by reordering the f_i and subtracting multiples we can assume that the set $\{dg_1, \dots, dg_l\}$ is linearly independent and that $dg_{\alpha+1}, \dots, dg_l$ are equal to zero.

Since we know that g_i vanish on $X \cap Y$ and $X \cap Z$ we have that

$$g_i = \sum_{j=1}^{\alpha} a_{ij}\, x_j + \sum_{j=\alpha+1}^{\alpha+\beta} \sum_{j=\alpha+\beta+1}^{\alpha+\beta+\gamma} h_{ijk}\, x_j\, x_k ,$$

for some smooth functions a_{ij}, h_{ijk}.

Now consider $g_{l+1}, \ldots, g_m$:

$$X \cap Y = \bigcap_{i=1}^{m} g_i^{-1}(0)$$

and

$$dg_i \, \frac{\partial}{\partial x_j} = 0\,, \qquad j = \alpha+1, \ldots, \alpha+\beta, \quad i = 1, \ldots, l\ ,$$

since $\frac{\partial}{\partial x_j}$, $j = \alpha{+}1, \ldots, \alpha{+}\beta$, are not tangent to $X \cap Y$ and are tangent to X, they are not tangent to Y. So $df_{l+1}, \ldots, df_m$ are full rank on $\frac{\partial}{\partial x_j}$, $j = \alpha+1, \ldots, \alpha+\beta$, and thus we have that $dg_{l+1}, \ldots, dg_m$ is full rank on $\frac{\partial}{\partial x_j}$, $j = \alpha+1, \ldots, \alpha+\beta$.

So renumbering again we can take $\{dg_{l+1}, \ldots, dg_{l+\beta}\}$ to be linearly independent. As g_i vanishes on $X \cap Y$, we have

$$g_i = \sum_{j=1}^{\alpha} a_{ij}\, x_j + \sum_{k=\alpha+1}^{\beta} b_{ik}\, x_k\,, \qquad i = l+1, \ldots, l+\beta\ ,$$

for some smooth a_{ij}, b_{ik}. We also have

$$dg_i \, \frac{\partial}{\partial x_k} = b_{ik}\,, \qquad i = l+1, \ldots, l+\beta, \quad k = \alpha+1, \ldots, \beta\ .$$

This means that $b_{ik}(x)$ is non-singular and so has a smooth inverse $c_{ik}(x)$ which we extend smoothly to a small neighbourhood in the whole manifold.

We now define

$$\bar{f}_i = \sum_{k=\alpha+1}^{\alpha+\beta} c_{ik}\, f_k\,, \qquad i = l+1, \ldots, l+\beta\ ,$$

and leave the other f_i invariant for now.

Putting $\bar{g}_i$ equal to f_i restricted to X, we have

$$\bar{g}_i = x_{\alpha+i-l} + \sum_{j=1}^{\alpha} a'_{ij}\, x_j\,, \qquad i = l+1, \ldots, l+\beta\ .$$

We can subtract multiples of $\bar{f}_i, \ldots$ $i = l+1, \ldots, l+\beta$ from f_i, $i = l+\beta+1$, $\ldots, m$ to obtain $\bar{f}_i$, $i = l+\beta+1, \ldots, m$, such that

$$\bar{g}_i = (\bar{f}_i)_{|X} = \sum_{j=1}^{\alpha} a'_{ij}\, x_j\,, \qquad i = l+\beta+1, \ldots, m\ .$$

We can also apply the same process to $X \cap Z$ and obtain

$$\bar{g}_i = x_{\alpha+\beta+i-m} + \sum a_{ij}\, x_j\,, \qquad i = m+1, \dots, m+\gamma\ ,$$
$$\bar{g}_i = \sum a_{ij}\, x_j\,, \qquad i = m+\gamma+1, \dots, r\ .$$

Now let

$$\bar{f}_i = f_i - \sum_{j=\alpha+1}^{\alpha+\beta} \sum_{k=\alpha+\beta+1}^{\alpha+\beta+\gamma} h_{ijk}\, \bar{f}_{j+l-\alpha}\, \bar{f}_{i+m-\alpha-\beta}\,, \qquad i = 1, \dots, l\ ,$$

then

$$\bar{g}_i = \sum_{j=1}^{\alpha} a_{ij}\, x_j\,, \qquad i = 1, \dots, \alpha\ .$$

All the operations we have carried out on the f_i to obtain $\bar{f}_i$ have been non-singular so the $\bar{f}_i$ retain the properties (4.5), (4.6). We also still have that $\{d\bar{g}_1, \dots, d\bar{g}_\alpha\}$ is linearly independent and that $d\bar{g}_j$, $j = \alpha+1, \dots, j = l$, are zero at the intersection. So we can do a matrix inversion at the intersection to reduce to the case where

$$g_i = \begin{cases} x_i\,, & i = 1, \dots, \alpha\ , \\ 0\,, & i = \alpha+1, \dots, l\ . \end{cases}$$

Having done this, we can subtract multiples of the first l f_i from the higher ones to yield that

$$g_i = \begin{cases} x_i\,, & i = 1, \dots, \alpha\ , \\ 0\,, & i = i = \alpha, \dots, l, l+\beta+1, \dots, m, m+\gamma+1, \dots r\ , \\ x_{i+\alpha-l}\,, & i = l+1, \dots, l+\beta\ , \\ x_{i+\alpha+\beta-m}\,, & i = m+1, \dots, m+\gamma\ . \end{cases}$$

These new f_i now have the requisite properties and the theorem follows. ∎

When one takes a clean composition of two Lagrangian submanifolds, one obtains a Lagrangian submanifold, at least locally. (Theorem 21.2.4 in [12].) (Globally, one could obtain self intersections.) So if we compose a pair of Lagrangian submanifolds with a third one then we will obtain a pair of Lagrangian submanifolds. However, it is not so trivial that if we start with a cleanly intersecting pair the output is also cleanly intersecting.

Theorem 4.4. *Let X_i, $i = 1, \dots, 3$ be smooth manifolds and let (Λ_1, Λ_2) be a cleanly intersecting pair of Lagrangian submanifolds in $T^*X_1 \times T^*X_2$ and suppose that Λ_3 is a Lagrangian submanifold in $T^*X_2 \times T^*X_3$ such that*

$$\Big(\Lambda_1 \times \Lambda_3,\ \Lambda_2 \times \Lambda_3,\ T^*X_1 \times N^*\Delta_{X_2} \times T^*X_3\Big)$$

is an intersecting triple than locally $(\Lambda_1 \circ \Lambda_3,\ \Lambda_2 \circ \Lambda_3)$ *are cleanly intersecting Lagrangian submanifolds.*

Proof: We need to check that the intersection is a submanifold and that its tangent space is equal to the tangent space of the intersection. The fact that the push-forward of the intersection is a submanifold comes from simply checking the proof of Theorem 21.2.4 in [12] works equally well for an isotropic submanifold. The result then follows from checking its tangent space is equal to the intersection of the tangent spaces of the two pushed-forward Lagrangian submanifolds.

Our proof of the second property relies heavily on the fact that we have an intersecting triple. For brevity write $E = T_p(T^*X_1 \times N^*\Delta_{X_2} \times T^*X_3)$ and $L_j = T_p(\Lambda_j \times \Lambda_3)$ then we have that

$$(L_1 + L_2) \cap E = (L_1 \cap E) + (L_2 \cap E)$$

and so taking orthogonal complements we obtain

$$(L_1 \cap L_2) + E^\perp = (L_1 + E^\perp) \cap (L_2 + E^\perp)\ .$$

But this means that

$$\frac{L_1 \cap L_2}{E^\perp} = \frac{L_1}{E^\perp} \cap \frac{L_2}{E^\perp}\ .$$

The result is now immediate as the differential of the projection onto $T^*X_1 \times T^*X_3$ yields an isomorphism of the left-hand side with $T_p((\Lambda_1 \circ \Lambda_3) \cap (\Lambda_2 \circ \Lambda_3))$ and of the right-hand side with $(T_p(\Lambda_1 \circ \Lambda_3)) \cap (T_p(\Lambda_2 \circ \Lambda_3))$. ∎

5 – The clean composition theorem for Lagrangians and paired Lagrangians

Our purpose in this section is to show how the clean composition theorem for Fourier integral operators follows simply from Theorems 1.1 and 1.2.

Theorem 5.1. *Suppose* $\Lambda_1 \subset (T^*X - \{0\}) \times (T^*Y - \{0\})$ *and* $\Lambda_2 \subset (T^*Y - \{0\}) \times (T^*Z - \{0\})$ *are conic, Lagrangian submanifolds such that*

$$(\Lambda_1 \times \Lambda_2) \cap \Big((T^*X - \{0\}) \times N^*(\Delta_Y) \times (T^*Y - \{0\})\Big)$$

is clean where Δ_Y *is the diagonal in* $Y \times Y$ *then for compositions which are proper with respect to supports and wavefronts, we have*

$$\circ\colon\ I_l(\Lambda_0, H^s_{\mathrm{loc}}) \times I_l(\Lambda_1, H^t_{\mathrm{loc}}) \ \to\ Il(\Lambda_1 \circ \Lambda_2, H^{\min(s,t,s+t) - n_y/2})\ ,$$

where n_Y *is the dimension of* Y.

Note that this is equivalent to the usual hypothesis for the clean intersection theorem — we have simply untwisted the Lagrangian submanifolds.

Proof: We can decompose $\circ$ into three operations: exterior product, restriction to $X \times \Delta_Y \times Z$ and the push-forward from $X \times \Delta_Y \times Z$ to $X \times Z$.

If $K \in I(\Lambda_1)$ and $L \in I(\Lambda_2)$ then $K \otimes L \in I(\Lambda_1 \times \Lambda_2)$ except for components near $(T^*X - \{0\}) \times (T^*Y - \{0\}) \times Y_0 \times Z_0$ and $X_0 \times Y_0 \times (T^*Y - \{0\}) \times (T^*Z - \{0\})$. These additional components are killed by the push-forward part of the composition and so our theorem reduces to proving that if Λ is a Lagrangian submanifold of

$$(T^*X - \{0\}) \times (T^*(Y) - \{0\}) \times (T^*Y - \{0\}) \times (T^*Z - \{0\})$$

which meets

$$(T^*X - \{0\}) \times (N^*(\Delta_Y)) \times (T^*Z - \{0\})$$

cleanly then restriction followed by push-forward yields an element of $I(\Lambda')$ with

$$\Lambda' = \Big\{(x,\xi,z,\mu)\colon \exists\,(y,\eta)\ \ (x,\xi,y,-\eta,y,\eta,z,\mu) \in \Lambda\Big\}\ .$$

Our composition is properly supported so we can reduce to a locally finite sum of pieces supported in coordinate patches and thus we are reduced to considering the case where Y is a subset of $\mathbb{R}^n$. We enact a change of coordinates on $Y_y \times Y_{y'}$,

$$w = y + y'\ , \tag{5.1}$$

$$t = y - y\ , \tag{5.2}$$

and then we have to consider restriction to $w = 0$, followed by integration in t. Letting w^*, t^* be the dual variables, our hypothesis is now that Λ meets $w = 0$, $t^* = 0$ cleanly. The restriction and the push-forward commute so when combining the two theorems we have that the composite is stable under operators with principal symbols which are restrictions of symbols vanishing on Λ. The result is now immediate — from the restriction and push-forward theorems, the composite is stable under first order symbols vanishing on $\Lambda \cap \{w = 0,\ t^* = 0\}$ which are restrictions of symbols vanishing on Λ. But the cleanness means that all such symbols are restrictions and the result follows. The orders follow from the results of Section 3. ∎

The same techniques establish the analogous result for paired Lagrangian distributions.

Theorem 5.2. *Suppose (Λ_1, Λ_2) are cleanly intersecting conic Lagrangian submanifolds in $(T^*X - \{0\}) \times (T^*Y - \{0\})$ and Λ is a conic Lagrangian submanifold of $(T^*Y - \{0\}) \times (T^*Z - \{0\})$ such that*

$$\Lambda_1 \times \Lambda,\ \ \Lambda_2 \times \Lambda,\ \ (T^*X - \{0\}) \times N^*(\Delta_Y) \times (T^*Y - \{0\})$$

is an intersecting triple then if for compositions which are proper with respect to supports and wavefronts we have

$$\circ\colon\ I_l(\Lambda_0,\Lambda_1,H^s_{\mathrm{loc}})\times I_l(\Lambda,H^t_{\mathrm{loc}})\ \to\ I_l(\Lambda_0\circ\Lambda,\ \Lambda_1\circ\Lambda,\ H^{\min(s,t,s+t)-n_Y/2})\ ,$$

where n_Y is the dimension of Y.

Proof: This is the same as for the Clean Composition Theorem except that we invoke the triple intersection Theorem 4.3 to obtain the existence of a linear model and hence the extendibility follows. ∎

6 – Pseudo-differential operators with singular symbols

Our purpose in this section is to study the composition of paired Lagrangian distributions in a special case where the first Lagrangian submanifold is the conormal bundle to the diagonal and the second one, Λ, is such that $\Lambda\circ\Lambda=\Lambda$. It is then immediate from the calculus of wavefront sets that no new Lagrangian submanifolds are generated by the composition and so one would expect the class of paired Lagrangian distributions to be closed under composition. These operators can be thought of as pseudo-differential operators with singular symbols, as on the diagonal they are pseudo-differential operators with a symbol which will typically blow up on approach to the intersection. Composition calculi have previously been established in the case where the intersection is of co-dimension one and one-sided ([1], [13], [15]). We shall study the case where Λ is given by the flow out of a characteristic variety by commuting principal symbols (cf. [10]).

Definition 6.1. Let p_j, $j=1,\ldots,k$, be real first order symbols such that $\{dp_j\}$ is linearly independent on their common zero set and such that

$$\{p_i,p_j\}=0\,,\qquad \forall\, i,j\ ,$$

then the set of points of the form

$$\exp(t_1p_1)\,\exp(t_2p_2)\,\cdots\,\exp(t_kp_k)(x_0,\xi_0)$$

shall be called the bicharacteristic leaf through (x_0,ξ_0) for $(x_0,\xi_0)\in\bigcap_j p_j^{-1}(0)$.

It is immediate that the bicharacteristic leaves form a foliation of the common zero set — this is the Hamiltonian foliation.

Definition 6.2. We shall say $p_1, \dots, p_k$ is pseudo-convex if for any (x_0, ξ_0) the map

$$\theta_{x_0,\xi_0} \colon \mathbb{R}^k \to T^*(M) \tag{6.1}$$

$$(t_1, t_2, \dots, t_k) \mapsto \exp(t_1 H_{p_1}) \exp(t_2 H_{p_2}) \cdots \exp(t_k H_{p_k})(x_0, \xi_0) \tag{6.2}$$

is injective and the composite map with projection to M is proper.

Proposition 6.1. *If the system $p_1, \dots, p_k$ is pseudo-convex and the set Λ in $T^*(M) \times T^*(M)$ is defined by $(x, \xi, y, -\eta) \in \Lambda$ if and only if (x, ξ) and (y, η) are in the same leaf of the bicharacteristic foliation then Λ is a conic Lagrangian submanifold.*

Proof: The pseudo-convexity condition ensures that Λ is a submanifold and the homogeneity of the p_j ensures that it is conic. To see that Λ is Lagrangian, note that it is sufficient to prove this at the diagonal as the symplectomorphisms

$$\exp(t_1 H_{p_1}) \exp(t_2 H_{p_2}) \cdots \exp(t_k H_{p_k})$$

can be used to take any other point to the diagonal. At the diagonal the tangent space is

$$T(N^*(\Delta) \cap \Lambda) + \langle \{H_{p_j}\}_{j=1,\dots,k} \rangle \ .$$

That $T(N^*(\Delta) \cap \Lambda)$ is isotropic is immediate as $N^*(\Delta)$ is Lagrangian and H_{p_j} are mutually symplectically orthogonal by hypothesis. As $D_v p_j = 0$ for any

$$v \in T(N^*(\Delta) \cap \Lambda)$$

we have also that v, H_{p_j} are orthogonal and the result follows. ∎

Note that any flow out Lagrangian will satisfy the relation $\Lambda \circ \Lambda = \Lambda$ as membership of a leaf is an equivalence relation.

Theorem 6.1. *Suppose Λ is a flow out Lagrangian submanifold then for compositions which are proper with respect to supports and wavefronts, we have*

$$\circ \colon \ I_l(N^*(\Delta), \Lambda, H^s_{\mathrm{loc}}) \times I_l(N^*(\Delta), \Lambda, H^t_{\mathrm{loc}}) \to I_l(N^*(\Delta), \Lambda, H^{\min(s,t,s+t)-n/2}) \ ,$$

where n is the dimension of the manifold.

Proof: Similarly to the proof of the Clean Composition Theorem 5.1, the problem is to prove that if a first order symbol q defined on $E = T^*X \times N^*(\Delta) \times T^*X$ vanishes on its intersection with $N^*(\Delta) \times N^*(\Delta)$, $N^*(\Delta) \times \Lambda$, $\Lambda \times N^*(\Delta)$ and $\Lambda \times \Lambda$, then it can be extended to a first order symbol nearby which still vanishes on them.

Now near any point in $T^*(X)$, there is a locally defined homogeneous symplectomorphism, f, defined on an open, conic subset U of $T^*(\mathbb{R}^n) - \{0\}$ such that $f^*(p_j) = \xi_j$ (see for example [12], Chap. 21). These can be extended in a "cylindrical neighbourhood" — $\{(x,\xi)\colon \exists\, y' \ (y', x'', \xi) \in U\}$ — by putting

$$
\begin{aligned}
f\Big(x_1+t_1, x_2+t_2, \ldots, x_k+t_k, x_{k+1}, \ldots, x_n, \xi_1, \xi_2, \ldots, \xi_n\Big) &= \\
= \exp(t_1 H_{p_1}) \, \exp(t_2 H_{p_2}) \cdots \exp(t_n H_{p_n}) \, f(x,\xi) \ . &
\end{aligned} \tag{6.3}
$$

This extension will be well-defined as $H_{\xi_j} = \frac{\partial}{\partial x_j}$ and will be a symplectomorphism as a Hamiltonian flow is a symplectomorphism. Thus we have symplectic coordinates in a neighbourhood of any leaf such that $\Lambda = N^*\,(x'' = y'')$. So if we put these coordinates on all four copies of $T^*(X)$, we have a linear model which is clearly equivalent to one in which all the submanifolds are given by the subsets of the coordinates vanishing. ∎

7 – Singular Radon transforms

In the study of Radon transforms, R, with singular densities under certain conditions one obtains an operator whose kernel is paired Lagrangian with respect to the diagonal and to the canonical graph of a symplectomorphism which cleanly intersect (see [9]). Our purpose in this section is to find a class of distributions within which R^tR, RR^t lie. From a straightforward composition of wavefront sets argument, one obtains that their wavefront sets lie within $N^*(\Delta) \cup \Gamma'_f \cup \Gamma'_{f^{-1}}$. We show that given a non-degeneracy assumption on the fixed points of f that R^tR, RR^t lie in $I(\mathcal{M})$ where $\mathcal{M}$ is the set of first order symbols which are zero on all three.

Definition 7.1. If f is a diffeomorphism of a manifold M then we shall say it has clean fixed point set if the set of fixed points form a submanifold X and for each $p \in X$

$$\overline{df}_p\colon\ T_p(M)/T_p(X) \to T_p(M)/T_p(X)$$

has no unit eigenvalues.

So in local coordinates, $x = (x', x'')$, with $X = \{x' = 0\}$, we have

$$f(x', x'') = \Big(A(x'')\, x',\ x'' + B(x'')\, x'\Big) + O\Big((x')^2\Big)$$

with $A(x'')$ a square matrix such that $I - A(x'')$ is invertible. Geometrically, an eigenvector of eigenvalue one would express a direction in which the fixed point remained fixed to some order.

Theorem 7.1. *Suppose f is a symplectomorphism of $T^*M - \{0\}$ with clean fixed point set then for compositions, we have*

$$\circ : \ I_l(N^*(\Delta), \Gamma'_f, H^s_{\text{loc}}) \times I_l(N^*(\Delta), \Gamma'_{f^{-1}}, H^t_{\text{loc}}) \ \to \ I_l(\mathcal{M}, H^{\min(s,t,s+t)-n/2})$$

where $\mathcal{M}$ is the set of first order symbols which vanish on $N^*(\Delta) \cup \Gamma'_f \cup \Gamma'_{f^{-1}}$.

Proof: As usual exterior product presents no problems and so our theorem reduces to showing that if a first order symbol vanishes on $N^*(\Delta) \cup \Gamma'_f \cup \Gamma'_{f^{-1}}$ and is pulled back to a function on $T^*(X) \times N^*(\Delta) \times T^*(X)$ then it can be extended to a first order symbol nearby which vanishes on each of

$$N^*(\Delta) \times N^*(\Delta), \quad N^*(\Delta) \times \Gamma'_{f^{-1}}, \quad \Gamma'_f \times N^*(\Delta), \quad \Gamma'_f \times \Gamma'_{f^{-1}}$$

(cf. proof of Theorem 5.1). We pick local homogeneous coordinates on $T^*X \times T^*X$ which reduce $N^*(\Delta)$ and Γ'_f to linear subspaces; the transpose coordinates do the same for $N^*(\Delta)$ and $\Gamma'_{f^{-1}}$. Thus we reduce all the subspaces to a linear model and the extendibility is then clear — as we are showing vanishing we do not need symplectic coordinates.

We work with twisted manifolds — i.e. the untwisted maps. We can reduce by the radial direction as all our manifolds are conic and maps homogeneous and then extend homogeneously at the end. So suppose g is a map of a manifold Y to itself with clean fixed point set, we show that there are coordinates on $Y \times Y$ which preserve the diagonal in which $\Gamma_g = (y, 0, y'')$ for some splitting of the coordinates (y', y''). Pick local coordinates on Y such that the fixed point set is given by $y' = 0$ and take these coordinates on each copy of Y. We now take coordinates, $\tilde{z}$ on the second copy indexed by the first copy to reduce Γ_g to the requisite form. Clearly, to preserve the diagonal we will need

$$\tilde{z}(y, y) = y$$

and so in general

$$\tilde{z}(y, z) = z + H(y, z)\,(y - z) \ ,$$

with H a square matrix. Thus we have

$$\tag{7.1} \begin{aligned} \tilde{z}(y, f(y)) = (0, y'') + \Big(G_1(y)\,y', \, G_2(y)\,y'\Big) \\ + H(y, f(y)) \Big((0, y'') + \Big(G_1(y)\,y', \, G_2(y)\,y'\Big) - (y', y'') \Big) \ , \end{aligned}$$

$$\tag{7.2} \begin{aligned} \tilde{z}(y, f(y)) = (0, y'') + \Big(G_1(y)\,y', \, G_2(y)\,y'\Big) \\ + H(y, f(y)) \Big((G_1(y) - I)\,y', \, G_2(y)\,y'\Big) \ . \end{aligned}$$

So now picking H to act only the first coordinates, this becomes, with H_1, H_2 to be chosen,

$$\tilde{z}(y, f(y)) =$$

$$= \Big(\Big(G_1(y) + H_1(y, f(y))\,(G_1(y) - I)\Big)\,y',\; y'' + \Big(G_2(y) + H_2 G_1(y)\Big)\,y'\Big) \tag{7.3}$$

but then defining

$$H_1(y) = -G_1(y)\,(G_1(y) - I)^{-1}\;, \tag{7.4}$$

$$H_2(y) = -G_2(y)\,(G_1(y) - I)^{-1}\;, \tag{7.5}$$

our result follows. Note that our hypothesis implies that $(G_1(y) - I)^{-1}$ exists near $y' = 0$ and that this also guarantees that

$$(y, z) \mapsto (y, \tilde{z}(y, z))$$

is indeed a diffeomorphism. ∎

REFERENCES

[1] Antoniano, J. and Uhlmann, G. – A functional calculus for a class of pseudo-differential operators with singular symbols, *Proceedings of Symposia in Pure Mathematics,* 43 (1985), 5–16.

[2] Bony, J.M. – *Propagation des Singularites pour les Equations aux Derivées Partielles Non-Lineaires,* Séminaire Goulaouic–Schwartz, 1979–1980.

[3] Bott, R. – Non-degenerate critical manifolds, *Ann. Math.,* 60(2), 248–261.

[4] Boutet de Monvel, L. and Guillemin, V. – The spectral theory of Toeplitz operators, *Annals of Mathematics Studies,* 99, Princeton University Press, 1981.

[5] Duistermaat, J.J. – Oscillatory integrals, Lagrange immersions and unfolding of singularities, *Communications on Pure and Applied Mathematics,* XXVII, (1974), 207–281.

[6] Duistermaat, J.J. and Hörmander, L. – Fourier integral operators II, *Acta Mathematicae,* 128(3–4) (1972), 183–269.

[7] Duistermaat, J.J. and Guillemin, V.W. – The spectrum of positive elliptic operators with periodic bicharacteristics, *Inventiones Math.,* 29 (1975).

[8] Gerard, P. and Golse, F. – Averaging regularity results for PDEs under transversality assumptions, *Comm. Pure Appl. Math.,* 45(1) (1992), 1–26.

[9] Greenleaf, A. and Uhlmann, G. – Estimates for singular Radon transforms and pseudo-differential operators with singular symbols, *Journal of Functional Analysis,* 89 (1990), 205–240.

[10] Guillemin, V.W. and Uhlmann, G. – Oscillatory integrals with singular symbols, *Duke Mathematical Journal,* 48(1) (1981), 251–267.

[11] Hörmander, L. – Fourier integral operators I, *Acta Mathematicae,* 127(1–2) (1971), 79–183.

[**12**] Hörmander, L. – *Analysis of Linear Partial Differential Operators*, Vol. 1 to 4, Springer-Verlag, Berlin 1985.
[**13**] Joshi, M.S. – *A Precise Calculus of Paired Lagrangian Distributions*, M.I.T. thesis, 1994.
[**14**] Joshi, M.S. – An intrinsic characterisation of polyhomogeneous Lagrangian distributions, *Proceedings of the A.M.S.,* 125(5) (1997), 1537–1543.
[**15**] Joshi, M.S. – *A Symbolic Construction of the Forward Fundamental Solution of the Wave Operator*, forthcoming.
[**16**] Melrose, R.B. – *Differential Analysis on Manifolds with Corners*, forthcoming.
[**17**] Melrose, R.B. – *Marked Lagrangian Distributions*, forthcoming.
[**18**] Melrose, R.B. – *The Atiyah–Patodi–Singer Theorem*, A.K. Peters, 1994.
[**19**] Melrose, R.B. and Uhlmann, G. – Intersecting lagrangians and the Cauchy problem, *Communications in Pure and Applied Mathematics,* 32(4) (1979), 483–519.

M.S. Joshi,
Department of Pure Mathematics and Mathematical Statistics, University of Cambridge,
16 Mill Lane, Cambridge CB2 1SB – England, U.K.
E-mail: `joshi@pmms.cam.ac.uk`