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NORMAL SMASH PRODUCTS

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Abstract: Let H be a co-Frobenius Hopf algebra over a field k and A a right H-comodule algebra. It is shown that A is H-faithful and $A_N \# N^* \in \Phi$ iff $A \# H^{*\mathrm{rat}} \in \Phi$, where N is a subgroup of $G(H) = \{g \in H \mid \Delta(g) = g \otimes g\}$ and A_N is N-coinvariants, Φ denotes a normal class. It is also shown that if A/A_1 is right H-Galois and A_1 is central simple, then so is $A \# H^{*\mathrm{rat}}$. In particular, if A_1 is a divisible ring, then $A \# H^{*\mathrm{rat}}$ is a dense ring of linear transformations of the vector space A over A_1. Let H be a finite dimensional Hopf algebra over the field k and A an H-module algebra, K is a unimodular and normal subHopfalgebra and $\overline{H} = H/K^+H$, it is obtained that $A^K \# \overline{H} \in \Phi$ and A is H^*-faithful iff $A \# H \in \Phi$.

0 – Introduction

The notation of normal class was first defined by Nicholson and Watters. There are many normal classes of rings. For example, the class of prime rings, the class of (left) primitive rings, the class of primitive rings with non-zero socle, the class of prime subdirectly irreducible rings and the class of prime left non-singular rings, are all normal classes. Other examples and properties of normal classes can be found in [NW1] and [NW2]. They can be applied to deduce some interesting results of prime rings.

Throughout this paper, we work over a fixed field k. For example, H is finite dimensional means H is a finite dimensional Hopf algebra over the field k, etc. Let H be a finite dimensional Hopf algebra and A an H-module algebra, A^H the invariants of A under the H-action, and $A \# H$ the associated smash product.

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In [Y], the author discussed normal connections between A^H and $A \# H$. Let H be a co-Frobenius Hopf algebra and A a right H-comodule algebra. In this paper, we establish a generalized Morita context and apply it to give some normal connections of smash products for induced Hopf actions and coactions.

In the second section, it is shown that A is H-faithful and $A_N \# N^* \in \Phi$ iff $A \# H^{*\mathrm{rat}} \in \Phi$, where N is a subgroup of $G = \{g \in H \mid \Delta(g) = g \otimes g\}$ and A_N is N-coinvariants, Φ denotes a normal class. Suppose H is finite dimensional and A an H-module algebra, K is a unimodular and normal subHopfalgebra and $\overline{H} = H/K^+H$, it is also shown that $A^K \# \overline{H} \in \Phi$ and A is H^*-faithful iff $A \# H \in \Phi$.

The main results of [CC2, Section 3] are straightforword to check that $H \# H^{*\mathrm{rat}}$ is central simple and it is a dense ring of finite-rank linear transformations of H over k. Can we have the analogous statements about $A \# H^{*\mathrm{rat}}$? In the third section, we discuss this problem generally. As a corollary, a simple proof of [CC2, Theorems 3.7–3.9] is given.

1 – Preliminaries

It is always assumed that H is a co-Frobenius Hopf algebra with antipode S. From [D, Theorem 2], $\dim \int^l = 1$ ($\dim \int^r = 1$), where $\int^l$ ($\int^r$) is the left (right) integral space of H^*. It is well-known that S is bijective and $H^{*\mathrm{rat}}$ is a dense ideal of H^*. Sigma notations are used and [M] is our basic reference. Fix $0 \neq t \in \int^l$ and $G = \{g \in H \mid \Delta(g) = g \otimes g\}$. Notice that G is the group of group-likes of H.

Lemma 1. *Let H be a co-Frobenius Hopf algebra, then there exists an $\alpha \in G$ such that $t\, f = \langle f, \alpha\rangle\, t$ for all $f \in H^*$.*

Proof: Choose $0 \neq t_r \in \int^r$. Since the map $H \to H^{*\mathrm{rat}}$, $x \to (t_r \leftharpoonup S(x))$ is bijective, there exists some $h \in H$ such that $t = t_r \leftharpoonup S(h)$. For all $g^* \in H^*$, $tg^* = n_g t$ for some $n_g \in k$. Since

$$\begin{aligned} tg^* = (t_r \leftharpoonup S(h))\, g^* &= \sum_{(h)} (t_r(g^* \leftharpoonup h_1)) \leftharpoonup S(h_2) \\ &= \sum_{(h)} \langle g^*, h_1\rangle\, t_r \leftharpoonup S(h_2) = t_r \leftharpoonup S(h \leftharpoonup g^*) \ , \end{aligned}$$

$h \leftharpoonup g^* = n_g h$ for all $g^* \in H^*$. Now $n_g\, \varepsilon(h) = \langle \varepsilon, h \leftharpoonup g^*\rangle = \langle g^*, h\rangle$ and $\langle g_0^*, h\rangle \neq 0$ for some $g_0^* \in H^*$, so $\varepsilon(h) \neq 0$. Set $\alpha = \varepsilon(h)^{-1} h$, then $\alpha \in H$ and $h = \varepsilon(h)\, \alpha$. One has $n_g = \langle g^*, \alpha\rangle$ and $\alpha \leftharpoonup g^* = \langle g^*, \alpha\rangle\, \alpha$, hence $\sum_{(\alpha)} \langle g^*, \alpha_1\rangle\, \alpha_2 = \langle g^*, \alpha\rangle\, \alpha$

for all $g^* \in H^*$. Thus $\Delta(\alpha) = \alpha \otimes \alpha$ and the result follows. ∎

Denote $f^g = g \rightharpoonup f$ for all $g \in G$ and $f \in H^*$.

Lemma 2. *With the notations as above, then $S(t) = t^\alpha$.*

Proof: For all $h \in H$ and $h^* \in H^*$, it is easy to see that

$$\begin{aligned}\langle t^\alpha h^*, h\rangle &= \sum_{(h)} \langle t^\alpha, h_1\rangle \, \langle h^*, h_2\rangle = \sum_{(h)} \langle t, h_1\alpha\rangle \, \langle h^{*\alpha^{-1}}, h_2\alpha\rangle \\ &= \langle t h^{*\alpha^{-1}}, h\alpha\rangle = \Big\langle \langle h^*, 1\rangle \, t^\alpha, h \Big\rangle\end{aligned}$$

and $t^\alpha h^* = \langle h^*, 1\rangle \, t^\alpha$. Thus $S^{-1}(t^\alpha) \in \int^l$. Applying $\sum_{(h)} \langle t', h_2\rangle \, h_1 = \langle t', h\rangle \, 1_H$ for $t' \in \int^l$, one has

$$\begin{aligned}(t \leftharpoonup h)\, t &= \sum_{(h)} (t \leftharpoonup h_1)\, (t \leftharpoonup S^{-1}(h_3)\, h_2) = \sum_{(h)} t\, (t \leftharpoonup S^{-1}(h_2)) \leftharpoonup h_1 \\ &= \sum_{(h)} \langle t \leftharpoonup S^{-1}(h_2), \alpha\rangle \, t \leftharpoonup h_1 = \sum_{(h)} \langle t, S^{-1}(h_2)\, \alpha\rangle \, t \leftharpoonup h_1 \\ &= \sum_{(h)} \langle S^{-1}(t^\alpha), h_2\rangle \, t \leftharpoonup h_1 = \sum_{(h)} t \leftharpoonup \langle S^{-1}(t^\alpha), h_2\rangle \, h_1 \\ &= \langle S^{-1}(t^\alpha), h\rangle \, t \ .\end{aligned}$$

On the other hand, $(t \leftharpoonup h)\, t = \langle t, h\rangle \, t$ hence $\langle t, h\rangle = \langle S^{-1}(t^\alpha), h\rangle$, it follows that $S(t) = t^\alpha$. ∎

Suppose that A is a right H-comodule algebra with the structure map ρ. Define

$$A_g = \Big\{ a \in A \mid \rho(a) = a \otimes g \Big\} \qquad \text{for all} \ \ g \in G \ .$$

If N is a subgroup of G, $A_N = \sum_{g \in N} A_g$ which is called the N-coinvariants of A, is an N-graded algebra. A_G is the semicoinvariants and $A_1 \subseteq A_N$ is the H-coinvariants of A. $A \# H^{*\mathrm{rat}}$ is a ring, which is $A \otimes_k H^{*\mathrm{rat}}$ as a vector space, the multiplication is

$$(a \# h^*)\, (b \# g^*) = \sum_{(b)} a\, b_0 \# (h^* \leftharpoonup b_1)\, g^*, \qquad \text{for} \ \ a, b \in A, \quad h^*, g^* \in H^{*\mathrm{rat}} \ .$$

A is a left $A \# H^{*\mathrm{rat}}$-module via $(a \# h^*) \cdot b = a(h^* \cdot b)$ and a right $A \# H^{*\mathrm{rat}}$-module via $b \cdot (a \# h^*) = \sum_{(b),(a)} \langle h^*, S^{-1}(b_1 a_1)\, \alpha\rangle \, b_0\, a_0$ for $h^* \in H^{*\mathrm{rat}}$, $a, b \in A$.

Proposition 1. *$\mu_1 = (A_1,\ A,\ A,\ A \# H^{*\mathrm{rat}})$ is a Morita context.*

Proof: Obviously, A is a left and right A_1-module via ring multiplications. One sees that A is an $(A\#H^{*\mathrm{rat}}, A_1)$- and an $(A_1, A\#H^{*\mathrm{rat}})$-bimodule. Define

$$[\ ,\ ]\colon\quad A\otimes_{A_1} A \longrightarrow A\#H^{*\mathrm{rat}},\quad [a,b]=\sum_{(b)}(a\,b_0)\#(t\leftharpoondown b_1)\ ,$$

$$(\ ,\ )\colon\quad A\otimes_{A\#H^{*\mathrm{rat}}} A \longrightarrow A_1,\quad (a,b)=t\cdot(a\,b)\ .$$

Applying Lemma 1 and Lemma 2, the result follows in the similar way as [CC1, Theorem 2.4]. ∎

Remark. If $\int^l=\int^r\neq 0$, μ_1 coincides with [CC1, Theorem 2.4]; if A is a G-graded algebra for any group G, μ_1 coincides with [B, Theorem 1.2]; if H is a finite dimensional Hopf algebra and A an H-module algebra, μ_1 also coincides with [CFM, Theorem 2.10].

In the sequel we apply the Morita context μ_1 to study normal smash products. All notations as noted above unless otherwise specified.

2 – Normal connections

A Morita context $u=(R,V,W,T)$ is called T-faithful, if $(V,t\cdot W)=(V\cdot t,W)=0$ with $t\in T\neq 0$, then $t=0$. Similarly, one can define R-faithful. A class Φ of prime rings is said to be normal if u is a T-faithful context with $R\in\Phi$, then necessarily $T\in\Phi$. Note that the Morita context $\mu_1=(A_1,A,A,A\#H^{*\mathrm{rat}})$ is always A_1-faithful. Let A be a right H-comodule algebra, A is called H-faithful if A is $A\#H^{*\mathrm{rat}}$-faithful as both a left and a right $A\#H^{*\mathrm{rat}}$-module.

Lemma 3. *A is H-faithful iff μ_1 is $A\#H^{*\mathrm{rat}}$-faithful.*

Proof: Suppose A is H-faithful, $r\in A\#H^{*\mathrm{rat}}$ and

$$(A,r\cdot A)=(A\cdot r,A)=0\ ,$$

then

$$[A,A\cdot r]\cdot A=A\,(A\cdot r,A)=0\ ,$$

which implies that $[A,A\cdot r]=0$ and $A\cdot r=0$, hence $r=0$. So μ_1 is $A\#H^{*\mathrm{rat}}$-faithful. It is easy to check the converse. ∎

Let $N\le G$, we may form $A_N\#N^*$(usually unrelated to $A\#H^{*\mathrm{rat}}$) and have

$$\mu_2=\Big(A_1,\ {}_{A_1}A_{N\,A_N\#N^*},\ {}_{A_N\#N^*}A_{N\,A_1},\ A_N\#N^*\Big)\ .$$

For the definition of graded nondegenerate (resp. graded faithfully), the reader is referred to [CM].

Suppose H is a finite dimensional Hopf algebra and A an H-module algebra. From [CFM, Theorem 2.14], one knows that A^H is prime and A is H^*-faithful iff $A \# H$ is prime. Applying [Y, Theorem 3], one can deduce $A^H \in \Phi$ and A is H^*-faithful iff $A \# H \in \Phi$. In fact, we have the following result.

Theorem 1. *Let H be a co-Frobenius Hopf algebra and A an H-comodule algebra. Then A is H-faithful and $A_N \# N^* \in \Phi$ iff $A \# H^{*\mathrm{rat}} \in \Phi$.*

Proof: μ_1 is always A_1-faithful, so if $A \# H^{*\mathrm{rat}} \in \Phi$, then $A_1 \in \Phi$.

Suppose $A \# H^{*\mathrm{rat}} \in \Phi$. It is easy to check that μ_1 is $A \# H^{*\mathrm{rat}}$-faithful and A is H-faithful by Lemma 3. First, $A_g \neq 0$ for any $g \in N$. Indeed, if $0 \neq a \in A$, then $(a \# t^{g^{-1}}) \cdot A \neq 0$, that is $a(t^{g^{-1}} \cdot A) \neq 0$, so $t^{g^{-1}} \cdot A \neq 0$. Hence $A_g \neq 0$ for all $g \in N$ since $t^{g^{-1}} \cdot A \subseteq A_g$. Next, A_N is graded nondegenerate. Let $a_g \in A_g$. If $a_g A_\tau = 0$, then $a_g(t^{\tau^{-1}} \cdot A) = 0$. So $(a_g \# t^{\tau^{-1}}) \cdot A = 0$, $a_g \# t^{\tau^{-1}} = 0$ and $a_g = 0$. Hence A_N is a faithful left $A_N \# N^*$-module [cf. CM, Lemma 2.5] and $a_g A_{g^{-1}} \neq 0$ for $0 \neq a_g \in A_g$. Note that $y \in A_g$, Ay is an H-subcomodule left ideal of A, if $0 \neq a_g \in A_g$, $A_{g^{-1}} a_g \supseteq (t^g \cdot A)\, a_g = t \cdot (A a_g) = A \cdot (a_g \# t) \neq 0$ and A_N is graded nondegenerate. In fact, A_N is N-graded faithfully. Let $r \in A_N \# N^*$ and $A_N \cdot r = 0$, then $(A_N, r \cdot A_N) = (A_N \cdot r, A_N) = 0$ and $r \cdot A_N = 0$, so $r = 0$. Hence A_N is right $A_N \# N^*$-faithful. It follows that μ_2 is A_1- and $A_N \# N^*$-faithful.

If $A \# H^{*\mathrm{rat}} \in \Phi$, then $A_1 \in \Phi$ and μ_2 is $A_N \# N^*$-faithful, hence $A_N \# N^* \in \Phi$. Conversely, if $A_N \# N^* \in \Phi$ and A is H-faithful, then $A_1 \in \Phi$ and μ_1 is $A \# H^{*\mathrm{rat}}$-faithful by Lemma 3, $A \# H^{*\mathrm{rat}} \in \Phi$. ∎

Example 1. Group-graded rings: Let $A = \bigoplus_{g \in G} A_g$ be a G-graded algebra, $N \leq G$, then $A_N = \sum_{n \in N} A_n$ is N-graded algebra. By Theorem 1, $A \# G^* \in \Phi$ iff A is G-graded faithfully and $A_N \# N^* \in \Phi$.

Corollary 1. *Suppose that $A \# H^{*\mathrm{rat}}$ is semiprime. Then A is a faithful left $A \# H^{*\mathrm{rat}}$-module and $A_N \# N^* \in \Phi$ iff $A \# H^{*\mathrm{rat}} \in \Phi$.*

Proof: It suffices to show that if $A \# H^{*\mathrm{rat}}$ is semiprime and A is a faithful left $A \# H^{*\mathrm{rat}}$-module, then A is H-faithful. If $x \in A \# H^{*\mathrm{rat}}$ and $A \cdot x = 0$, then $[A, A]\, x[A, A] = 0$, $x[A, A] = 0$ since $A \# H^{*\mathrm{rat}}$ is semiprime. So $[x \cdot A, A] = 0$ and $x \cdot A = 0$, that is $x = 0$. Hence A is H-faithful. ∎

Now, we focus on the factor Hopf actions. Let H be a finite dimensional Hopf algebra and A an H-module algebra. K is a normal subHopfalgebra of H.

It is well-known that $\overline{H} = H/K^+H$ is a Hopf algebra, where $K^+ = \operatorname{Ker}\varepsilon \cap K$. Ht_K ($t_K \in \int_K^l$) and A^K are $\overline{H}$-modules, $t_K H$ is a right $\overline{H}$-module and $(A^K)^{\overline{H}} = A^H$ [YC]. The normal connection between $A^K \# \overline{H}$ and $A \# H$ is stated as following.

Theorem 2. *Let H be a finite dimensional Hopf algebra, K a unimodular and normal subHopfalgebra of H and A an H-module algebra. Then A is H^*-faithful and $A^K \# \overline{H} \in \Phi$ iff $A \# H \in \Phi$.*

Proof: By Remark, Lemma 3 and Theorem 1, it suffices to check that if A is H^*-faithful, then A^K is $\overline{H}^*$-faithful.

Let $0 \neq t_K$ be a left integral element of K. It is easy to see that $\overline{h} t_K \neq 0$ if $\overline{h} \neq 0$. Similarly, if $\overline{h} \neq 0$ and K is unimodular, $t_K \overline{h} \neq 0$.

Let $0 \neq \sum a_h \# \overline{h} \in A^K \# \overline{H}$, then $0 \neq \sum a_h \# \overline{h} t_K = \sum a_h \# h t_K \in A \# H$. Since A is H^*-faithful, there exists $0 \neq x \in A$ such that

$$\Big(\sum a_h \# \overline{h}\Big) \cdot (t_K \cdot x) = \Big(\sum a_h \# h t_K\Big) \cdot x \neq 0 \ .$$

So A is a faithful left $A^K \# \overline{H}$-module. $0 \neq t_K (\sum a_h \# \overline{h}) = \sum a_h \# t_K \overline{h} = \sum a_h \# t_K h \in A \# H$ since K is unimodular. One can choose $x \in A$ such that

$$(t_K \cdot x) \cdot \Big(\sum a_h \# \overline{h}\Big) = x \cdot \Big(\sum a_h \# t_K h\Big) \neq 0 \ .$$

Hence A^K is a faithful right $A^K \# \overline{H}$-module. The result follows. ∎

Example 2. Skew group rings: Let G be a finite group of automorphisms of A. N a normal subgroup of G and $\overline{G} = G/N$, $A^N = \{a \in A \mid n \cdot a = a$ for all $n \in N\}$. Note that A is a faithful right $A * G$-module then A is a faithful left $A * G$-module [CFM]. Thus, if A is a faithful right $A * G$-module, the following statements are equivalent:

1) $A^N * \overline{G} \in \Phi$.

2) $A^G \in \Phi$.

3) $A * G \in \Phi$.

Noting that it is independent on the field k.

3 – Central simplicity

In this section, we give an application of normal classes. The readers can see the paper [NW3] for the definition of the extended centroid, we make a sketch of it. Suppose that R is any ring, $F(R) = \{I \mid 0 \neq I$ is an ideal of R and $Ir = 0$ implies $r = 0\} \neq \emptyset$. Let $I \xrightarrow{\alpha} R$ and $J \xrightarrow{\beta} R$ be two homomorphisms of left R-modules where $I, J \in F(R)$, α and β are called equivalent if they agree on an ideal $C \in F(R)$, $C \subseteq I \cap J$. This is an equivalence relation and the equivalence class of a map $I \xrightarrow{\alpha} R$ is denoted by $[I \xrightarrow{\alpha} R]$. The set $Q(R)$ of all equivalence classes becomes a unitary ring with the following definitions:

$$[I \xrightarrow{\alpha} R] + [J \xrightarrow{\beta} R] = [I \cap J \xrightarrow{\alpha+\beta} R] \ ,$$
$$[I \xrightarrow{\alpha} R]\,[J \xrightarrow{\beta} R] = [JI \xrightarrow{\alpha\circ\beta} R] \ .$$

The map $r \mapsto [R \xrightarrow{\cdot r} R]$ embeds R as a subring of $Q(R)$. The centre of $Q(R)$ will be called the extended centroid of the ring R, which is denoted by $C(R)$. If R is simple, $C(R)$ is a field and it is the usual terminology. If R is simple with an identity, $C(R)$ concides with its centre $Z(R)$. If R is a simple ring (maybe without identity) and its centroid is the field k, we call R a central simple algebra. The following lemma is needed.

Lemma 4. $\mathrm{End}({}_{A\#H^{*\mathrm{rat}}}A) \cong {A_1}^\circ$ *as an algebra, where* ${A_1}^\circ$ *is the opposite algebra of* A_1.

Proof: Suppose $\phi \in \mathrm{End}({}_{A\#H^{*\mathrm{rat}}}A)$, Since $H^{*\mathrm{rat}}$ is dense in H^*, there exists a $g \in H^{*\mathrm{rat}}$ such that $g \cdot 1_A = 1_A$, $g \cdot \phi(1) = \phi(1)$. For any $f \in H^*$, also there exists a $g_0 \in H^{*\mathrm{rat}}$such that $g_0 \cdot 1_A = f \cdot 1_A$, $g_0 \cdot \phi(1) = f \cdot \phi(1)$. Then the proof is similar to [CFM, Lemma 0.3]. ∎

Recall that A/A_1 is said to be right H-Galois if the map

$$\gamma\colon \quad A \otimes_{A_1} A \longrightarrow A \otimes_k H, \quad a \otimes b \mapsto \sum_{(b)} a\, b_0 \otimes b_1$$

is bijective.

Theorem 3. *Suppose* H *is a co-Frobenius Hopf algebra and* A *an* H*-comodule algebra. If* A/A_1 *is right* H*-Galois, then*

1) $C(A_1) \cong C(A \# H^{*\mathrm{rat}})$.

2) *If* A_1 *is central simple, then so is* $A \# H^{*\mathrm{rat}}$.

3) *If* A_1 *is a divisible ring, then* $A \# H^{*\mathrm{rat}}$ *is a dense ring of linear transformations of the vector space* A *over* A_1.

Proof: Note that

$$\beta\colon\ H \longrightarrow H^{*\mathrm{rat}}\,, \qquad h \mapsto (t \leftharpoondown h)$$

is bijective. Since A/A_1 is right H-Galois, γ is bijective. So

$$[\,,\,]\colon\quad A \otimes_{A_1} A \xrightarrow{\gamma} A \otimes_k H \xrightarrow{i\otimes\beta} A \otimes H^{*\mathrm{rat}}$$

is surjective, that is $[A, A] = A \# H^{*\mathrm{rat}}$. μ_1 is $A \# H^{*\mathrm{rat}}$-faithful. Indeed, let $x \in A \# H^{*\mathrm{rat}}$, there exist $f, g \in H^{*\mathrm{rat}}$ such that $fx = xg = x$. If $(A \cdot x, A) = 0$, then $[A, A]\, x[A, A] = 0$. Hence $(A \# H^{*\mathrm{rat}})\, x(A \# H^{*\mathrm{rat}}) = 0$. In particular, $x = fxg = 0$. It follows that $C(A_1) \cong C(A \# H^{*\mathrm{rat}})$ [NW3, Theorem 1].

2) Let $H(R)$ denote the heart of a subdirectly irreducible ring R. If A_1 is central simple, then $H(A_1) = A_1$ and $A \# H^{*\mathrm{rat}}$ is prime and subdirectly irreducible. Note that

$$H(A \# H^{*\mathrm{rat}}) = [AH(A_1), A] = [AA_1, A] = [A, A] = A \# H^{*\mathrm{rat}}\ .$$

On the other hand, $C(A \# H^{*\mathrm{rat}}) \cong C(A_1)$ by 1). Hence $A \# H^{*\mathrm{rat}}$ is central simple.

3) Let $S(R)$ denote the socle of a ring R. If A_1 is a divisible ring, then $A \# H^{*\mathrm{rat}} = S(A \# H^{*\mathrm{rat}})$ is primitive. Let $0 \neq I$ be a left $A \# H^{*\mathrm{rat}}$-submodule of A, I is a left H^*-stable ideal. Since $A \# H^{*\mathrm{rat}}$ is prime, $0 \neq t \cdot I$ and $t \cdot I = A_1$. It is easy to check that $A = I$ and A is an irreducible left $A \# H^{*\mathrm{rat}}$-module. A is also a faithful left $A \# H^{*\mathrm{rat}}$-module by Lemma 3. Hence $A \# H^{*\mathrm{rat}}$ is a dense ring of linear transformations of A over A_1 by Lemma 4. ∎

Corollary 2 [CC2, Theorems 3.7–3.9]**.** *$H \# H^{*\mathrm{rat}}$ is central simple and it is a dense ring of finite-rank linear transformations of H over k.*

Proof: H is a right H-comodule algebra (the structure map is Δ)

$$B = H^{CoH} = \Big\{h \in H \mid \Delta(h) = h \otimes 1\Big\} = k \cdot 1_H\ .$$

Since

$$\gamma\colon\ H \otimes_B H \to H \otimes_k H\,, \qquad g \otimes h \to \sum_{(h)} gh_1 \otimes h_2\ ,$$

is bijective (its inverse is $\tau\colon\ g \otimes h \to \sum_{(h)} g\, S(h_1) \otimes h_2$), H/B is H-Galois. By Theorem 3, $H \# H^{*\mathrm{rat}}$ is central simple and it is a dense ring of linear transformations of H over k. It is easy to see that any element in $H \# H^{*\mathrm{rat}}$ acts on H is finite-rank. This completes the proof. ∎

Note added in proof. From the referee's report, the authors know that Proposition 1 and Lemma 4 are developed by Beattie, Dăscălescu and Raianu in [BDR] independently. The authors wish to thank the referee for some valuable comments and helps.

REFERENCES

[B] Beattie, M. – Duality theorems for rings with actions or coactions, *J. Algebra,* 115 (1988), 303–312.

[BDR] Beattie, M., Dăscălescu, S. and Raianu, S. – Galois extensions for coFrobenius Hopf algebras, *J. Algebra,* to appear.

[CC1] Chen, H. and Cai, C. – Hopf algebra coactions, *Comm. in Algebra,* 22(1) (1994), 253–267.

[CC2] Cai, C. and Chen, H. – Coactions, smash products and Hopf modules, *J. Algebra,* 167 (1994), 85–99.

[CFM] Cohen, M., Fischman, D. and Montgomery, S. – Hopf Galois extensions, smash products and Morita context, *J. Algebra,* 133 (1990), 351–372.

[CM] Cohen, M. and Montgomery, S. – Group graded rings, smash product and group actions, *Trans. A. M. Soc.,* 282 (1984), 237–258.

[D] Doi, Y. – Homological coalgebras, *J. Math. Soc. Japan,* 33 (1981), 31–50.

[M] Montgomery, S. – *Hopf Algebras and Their Actions on Rings*, CBMS Lectures in Math., vol. 82, AMS, Providence, RI, 1993.

[NW1] Nicholson, W.K. and Watters, J.F. – Normal classes of prime rings determined by modules, *Proc. Amer. Math. Soc.,* 83 (1981), 27–30.

[NW2] Nicholson, W.K. and Watters, J.F. – Normal radicals and normal classes of rings, *J. Algebra,* 59 (1979), 5–15.

[NW3] Nicholson, W.K. and Watters, J.F. – Central closures and Morita contexts, *Comm. Algebra,* 12 (1984), 103–123.

[Y] Yang, S. – Smash product, Morita context rings and extended centroids, *Northeast, Math. J.,* 10 (1994), 241–248.

[YC] Yang, S. and Chen, J. – Hopf actions on factor rings and induced Hopf actions, *Comm. in Algebra,* 25 (1997), 1103–1110.

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